Landau fluid models of collisionless magnetohydrodynamics

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A closed set of fluid moment equations including models of kinetic Landau damping is developed which describes the evolution of collisionless plasmas in the magnetohydrodynamic parameter regime. The model is fully electromagnetic and describes the dynamics of both compressional and shear Alfvén waves, as well as ion acoustic waves. The model allows for separate parallel and perpendicular pressures $p_\parallel$ and $p_\perp$, and, unlike previous models such as the Chew–Goldberger–Low theory, correctly predicts the instability threshold for the mirror instability. Both a simple 3+1 moment model and a more accurate 4+2 moment model are developed, and both could be useful for numerical simulations of astrophysical and fusion plasmas. © *1997 American Institute of Physics.* [S1070-664X(97)03311-9]

I. INTRODUCTION

The dynamics of collisionless plasmas are of great interest both in astrophysics and in laboratory fusion research. However, such plasmas are often studied using models which implicitly assume high collisionality and which ignore important kinetic effects such as parallel Landau damping. In particular, models based on Ideal magnetohydrodynamics (MHD) assume collisional equilibration on a fast time scale and are not in general applicable to collisionless plasmas. The Chew–Goldberger–Low (CGL) theory[1] relaxes the high collisionality assumption, but assumes an adiabaticity condition which is rarely met, and neglects parallel Landau damping, which can be important in the collisionless regime. Hence results from CGL theory are not always reliable, as evidenced by the well known factor of six error in the CGL prediction of the stability threshold for the mirror instability.[2,3] Simplified models such as Ideal MHD and CGL are often employed despite their limitations because of the qualitative insights they provide and the difficulty of working directly with a kinetic formulation. There are some particle simulations of collisionless MHD phenomena,[4–7] but there are also many fluid MHD simulations which could benefit from being extended into lower collisionality regimes.

In this paper we will develop a relatively simple description of collisionless plasma dynamics which includes parallel Landau damping. We wish to construct a model which is valid over a wide parameter regime and can later be narrowed and simplified for particular cases. As a starting point we will employ Kulsrud's formulation of collisionless MHD.[3,8,9] Kulsrud's formulation requires solving a kinetic equation for the perturbed pressures $p_\parallel$ and $p_\perp$, or introducing further assumptions such as adiabaticity to evaluate the pressures. We shall take moments of Kulsrud's kinetic equation, and close the moment hierarchy with Landau closures analogous to those derived by Hammett, Perkins, and Dorland,[10–12] generalized to allow anisotropic pressures and magnetic perturbations. This yields a fairly simple set of moment equations with desirable nonlinear conservation properties, and a linear response function very similar to the kinetic response of a collisionless bi-Maxwellian plasma.

We shall refer to the model as Landau MHD, because the model incorporates the effects of parallel Landau damping, and it is valid within the collisionless MHD regime. It is useful to consider the Landau MHD model as an extension of CGL theory which incorporates Landau damping, and can incorporate collisional effects as well.

One of the limitations of the Landau MHD model we present is that it is derived only in the standard ordering of ideal MHD, $\epsilon \sim \omega/\Omega_c \sim k\rho$, where the plasma varies on frequency scales ω small compared to the gyrofrequency Ω_c, and varies on spatial scales $1/k$ long compared to the gyroradius ρ. Thus it covers phenomenon related to compressional and shear Alfvén waves and instabilities, ion acoustic waves, and ion and electron kinetic effects such as Landau damping. However, it does not include drift-waves or other micro-instabilities (which have been the focus of other Landau-fluid work) because they result from finite-Larmor/gyro radius (FLR) effects which vanish in the usual MHD ordering. Also, though collisional effects on the ion and electron heat fluxes and on the pressure tensor can be kept in our model, there is no resistive component to the ideal Ohm's law. This is because the parallel current $\Sigma_s n_s e_s u_{\parallel_s} = 0$ to lowest order in the $1/e$ expansion of Kulsrud's collisionless MHD, and collisions would alter the Ohm's law only at higher order in the $\epsilon \sim \omega/\Omega \sim k\rho$ expansions. Thus the plasma is still an ideal electrical conductor in our model and the magnetic field lines are frozen into the plasma.

Alternative orderings are possible to bring in FLR or resistive effects. One approach would be to take fluid moments of the electromagnetic gyrokinetic equation,[13,14] which allows $k_\perp \rho \sim 1$, and work out the appropriate closures. Another approach, taken by Chang and Callen,[15,16] in effect carries Kulsrud's expansion to higher order in FLR, by using $k_\perp \rho \sim k_\parallel / k_\perp \sim \Delta$ with $\Delta^2 \sim \epsilon \sim \omega/\Omega_c$. This "extended-MHD" ordering orders the compressional Alfvén wave out of the equations, but retains the slower Shear Alfvén and ion acoustic waves, and includes resistive effects in the Ohm's law as well as drift-wave instabilities with moderate $k_\perp \rho$

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$\sim\epsilon^{1/2}$. Chang and Callen use an alternative derivation of Landau-fluid closures which is actually linearly exact (employing the full Z functions). It reduces to our formulation in the appropriate limits.[11] Their approach advances 3 moments (density, parallel flow, and temperature) for each species with linear closures for the heat flux and stress tensor, while here we advance up to 6 moments (4 parallel and 2 perpendicular moments) for each species. These six moment equations retain additional nonlinear effects, and simplify some of the manipulations of the stress tensor by keeping separate $p_\perp$ and $p_\parallel$ (which is also essential to study the mirror instability that Kulsrud used to point out problems with the CGL theory). They can be reduced to simpler systems with fewer moments in various limits. Future work could try to extend our methods to the electromagnetic gyrokinetic equation or merge with the methods of Chang and Callen for the extended-MHD ordering.

There are previous authors who have tried some forms of Landau closures in MHD equations. Bondeson and Ward[17] used viscous and pressure-damped models of Landau damping in studying wall stabilization of external MHD modes in advanced tokamak designs. An important feature of this work was the use of Lagrangian variables so that the $|k_\parallel|$ operator involved in Landau-fluid closures would (at least linearly) effectively operate along perturbed magnetic field lines, which Finn and Gerwin[18] showed was important to do. However, the Bondeson and Ward model was a relatively low-order Landau-fluid model and was not entirely consistent, assuming high collisionality in the derivation of the initial 1-fluid equations and low collisionality elsewhere. A recent paper by Medvedev and Diamond[19] has incorporated Hammett–Perkins type closures into a set of two fluid equations, used to describe large amplitude shear Alfvén and magnetosonic waves in interplanetary plasmas. The Medvedev and Diamond equations assume isotropic pressure, and are valid only in a limited parameter regime ($\beta \approx 1$). The Landau MHD model presented here should provide an extension of this previous work, useful for the study of resistive wall stabilization, as well as for general problems of MHD mode growth and saturation in both laboratory and astrophysical plasmas.

The organization of this paper is as follows. In Section II we summarize Kulsrud's collisionless MHD formulation. In Section III, a moment hierarchy based on Kulsrud's kinetic equation is derived and discussed. In Sections IV and V closures for ''4+2'' and ''3+1'' models are derived, following Hammett and Perkins,[10] and Dorland.[12] In Section VI we investigate collisional effects, including the reduction of the model to an appropriate limit of the Braginskii equations. In Section VII we discuss practical nonlinear implementation of the closure terms. In Section VIII, the Landau MHD formulation is applied to analyze the mirror instability, and in Section IX we offer concluding remarks.

II. COLLISIONLESS MHD

As a starting point, we employ the collisionless MHD model described by Kulsrud,[3] based on earlier work by Kruskal and Oberman[8] and by Rosenbluth and Rostoker.[9] This formulation begins with the Vlasov–Maxwell system of equations, and asymptotically expands in ρ_c/L, the smallness of the gyroradius relative to macroscopic scale lengths. This is accomplished by the formal expansion of the distribution function f, the magnetic field B, and the electric field E in the inverse charge $1/e$. This is equivalent to taking all relevant frequencies in the problem to be very small compared to the cyclotron frequency, Ω_c, and the plasma frequency, ω_p.

In this ordering, the Vlasov equation reduces to a condition on the zeroth-order parallel (relative to the magnetic field) electric field $E_{\parallel_0}=0$, and the following kinetic equation for the zeroth-order distribution function of each species $f_{0_s}(v_\parallel,\mu,\mathbf{r},t)$:

$$\frac{\partial f_{0_s}}{\partial t}+(v_\parallel\hat{\mathbf{b}}+\mathbf{v}_E)\cdot\nabla f_{0_s}+\left(-\hat{\mathbf{b}}\cdot\frac{D\mathbf{v}_E}{Dt}-\mu\hat{\mathbf{b}}\cdot\nabla B+\frac{e_s}{m_s}E_\parallel\right)\times\frac{\partial f_{0_s}}{\partial v_\parallel}=0, \tag{1}$$

where e_s is the charge on species s, $\hat{\mathbf{b}}$ is a unit vector in the magnetic field direction $\hat{\mathbf{b}}=\mathbf{B}/B$, $\mathbf{v}_E\doteq c(\mathbf{E}\times\mathbf{B})/B^2$, $\mu\doteq v_\perp^2/2B$, and $D/Dt\doteq\partial/\partial t+(v_\parallel\hat{\mathbf{b}}+\mathbf{v}_E)\cdot\nabla$.

Combining moments of this kinetic equation with Maxwell's equations and taking the usual low Alfvén speed limit $v_A^2\ll c^2$ yields Kulsrud's set of collisionless MHD equations:

$$\frac{\partial\rho}{\partial t}+\nabla\cdot(\rho\mathbf{U})=0, \tag{2}$$

$$\rho\left(\frac{\partial\mathbf{U}}{\partial t}+\mathbf{U}\cdot\nabla\mathbf{U}\right)=\frac{(\nabla\times\mathbf{B})\times\mathbf{B}}{4\pi}-\nabla\cdot\mathbf{P}, \tag{3}$$

$$\frac{\partial\mathbf{B}}{\partial t}=\nabla\times(\mathbf{U}\times\mathbf{B}), \tag{4}$$

$$\mathbf{P}=p_\perp\mathbf{I}+(p_\perp-p_\parallel)\hat{\mathbf{b}}\hat{\mathbf{b}}, \tag{5}$$

$$p_\perp=\sum_s\frac{m_s}{2}\int f_{0_s}v_\perp^2d^3v, \tag{6}$$

$$p_\parallel=\sum_s m_s\int f_{0_s}(v_\parallel-\mathbf{U}\cdot\hat{\mathbf{b}})^2d^3v, \tag{7}$$

$$\sum_s e_s\int f_{0_s}\,d^3v=0, \tag{8}$$

where ρ is the total mass density, $\mathbf{U}=\mathbf{v}_E+u_\parallel\hat{\mathbf{b}}$ is the fluid velocity, and $\mathbf{P}$ is the pressure tensor.

The above set of equations is exact to zeroth order in the expansion parameter, but the kinetic equation itself, Eq. (1), must be used to evaluate $p_\parallel$ and $p_\perp$ to close the system. Because Eq. (1) is difficult to solve directly, this system is rarely employed without further simplification.

One such simplification is the introduction of the double adiabatic law (also known as CGL theory[3,1]). In the CGL model, Eq. (1) is replaced by two equations of state which determine $p_\perp$ and $p_\parallel$:

$$\frac{d}{dt}\left(\frac{p_\perp}{\rho B}\right)=0, \tag{9}$$

$$\frac{d}{dt}\left(\frac{p_\parallel B^2}{\rho^3}\right)=0, \tag{10}$$

where the total derivative is defined by $d/dt \doteq \partial/\partial t+(u_\parallel \hat{\mathbf{b}}+\mathbf{v}_E)\cdot\boldsymbol{\nabla}$.

These equations of state are equivalent to setting the heat flow tensor **Q** to zero. This assumption that both electron and ion heat flow are negligible is strictly valid only when the mode phase velocity ($\omega/k_\parallel$) is much greater than the electron and ion thermal speeds, a criterion rarely satisfied for Alfvén waves and never satisfied for sound waves. Furthermore, the simple truncation of the moment hierarchy implied by this assumption eliminates Landau damping from the problem, leaving the system with no damping at all, which can lead to unphysical behavior. However, CGL theory is often employed, even when it is invalid, because of its simple, Lagrangian form. Of course this can lead to incorrect results, as in the well known case of the mirror instability.

III. THE MOMENT HIERARCHY

We wish to develop a formulation which maintains much of the simplicity of the CGL model, while increasing its range of applicability and including models of kinetic Landau damping. This will be accomplished by first taking moments of Eq. (1) and, in the next section, closing the hierarchy using Landau closures analogous to those developed for the electrostatic case by Hammett and Perkins.[10]

Multiplying Eq. (1) by B and adding Eq. (4) multiplied by f_s, leads to a kinetic equation in the phase space conserving form:

$$\frac{\partial}{\partial t}f_sB+\boldsymbol{\nabla}\cdot[f_sB(v_\parallel\hat{\mathbf{b}}+\mathbf{v}_E)]+\frac{\partial}{\partial v_\parallel}\left[f_sB\left(-\hat{\mathbf{b}}\cdot\frac{D\mathbf{v}_E}{Dt}-\mu\hat{\mathbf{b}}\cdot\boldsymbol{\nabla}B+\frac{e_s}{m_s}E_\parallel\right)\right]=BC(f_s). \tag{11}$$

The subscript zero on f_s has been suppressed. All calculations involve only the zeroth-order distribution function in the original expansion in $1/e$, though a subsidiary ordering will be introduced to derive the Landau closures.

Note the addition of a collision operator to the right hand side of the kinetic equation to allow for generalization to regimes where collisions play an important role. Here a simple BGK collision operator[20] is employed:

$$C(f_j)=-\sum_k \nu_{jk}(f_j-F_{Mjk}), \tag{12}$$

where ν_{jk} is the effective collision rate of species j with species k. These collisions cause f_j to relax to a shifted Maxwellian with the effective temperature of species j and the mass velocity of species k,

$$F_{Mjk}=\frac{n_j}{(2\pi T_j/m_j)^{3/2}}\exp\left[-\frac{m_j(v_\parallel-u_{\parallel k})^2}{2T_j}-\frac{m_j\mu B}{T_j}\right], \tag{13}$$

where $T_j=(T_{\parallel j}+2T_{\perp j})/3$. The BGK collision operator in this form conserves mass, momentum, and energy.

Defining the velocity space moments as follows:

$$n_s=\int f_s d^3v,\quad n_su_{\parallel_s}=\int f_sv_\parallel d^3v$$

$$p_{\parallel_s}=m\int f_s(v_\parallel-u_\parallel)^2d^3v,\quad p_{\perp_s}=m\int f_s\mu Bd^3v$$

$$q_{\parallel_s}=m\int f_s(v_\parallel-u_\parallel)^3d^3v,\quad q_{\perp_s}=m\int f_s\mu B(v_\parallel-u_\parallel)d^3v$$

$$r_{\parallel,\parallel_s}=m\int f_s(v_\parallel-u_\parallel)^4d^3v,\quad r_{\parallel,\perp_s}=m\int f_s\mu B(v_\parallel-u_\parallel)^2d^3v$$

$$r_{\perp,\perp_s}=m\int f_s\mu^2B^2d^3v,$$

Poisson's equation and Ampere's law reduce, to lowest order in $1/e$, to the conditions $\Sigma_s n_s e_s=0$ and $\Sigma_s n_s e_s u_{\parallel_s}=0$. Specializing to the case of one species of $Z=1$ ions implies $n=n_e=n_i$ and $u_\parallel=u_{\parallel_i}=u_{\parallel_e}$. The usual definitions for total higher moments $p_\parallel=\Sigma_s p_{\parallel_s}$, $p_\perp=\Sigma_s p_{\perp_s}$, $q_\parallel=\Sigma_s q_{\parallel_s}$, etc. are employed. Note that, because $u_{\parallel_i}=u_{\parallel_e}$, the collision term serves primarily to isotropize the distribution. Taking integrals of the form $\int dv_\parallel\, d\mu\, v_\parallel^j\mu^k\ldots$ of Eq. (11) then leads to the following set of exact moment equations:

$$\frac{\partial n}{\partial t}+\boldsymbol{\nabla}\cdot(n\mathbf{U})=0, \tag{14}$$

$$\frac{\partial u_\parallel}{\partial t}+\mathbf{U}\cdot\boldsymbol{\nabla}u_\parallel+\hat{\mathbf{b}}\cdot\left(\frac{\partial\mathbf{v}_E}{\partial t}+\mathbf{U}\cdot\boldsymbol{\nabla}\mathbf{v}_E\right)+\frac{1}{nm_s}\boldsymbol{\nabla}\cdot(\hat{\mathbf{b}}p_{\parallel_s})-\frac{p_{\perp_s}}{nm_s}\boldsymbol{\nabla}\cdot\hat{\mathbf{b}}-\frac{e_s}{m_s}E_\parallel=0, \tag{15}$$

$$\frac{\partial p_{\parallel_s}}{\partial t}+\boldsymbol{\nabla}\cdot(\mathbf{U}p_{\parallel_s})+\boldsymbol{\nabla}\cdot(\hat{\mathbf{b}}q_{\parallel_s})+2p_{\parallel_s}\hat{\mathbf{b}}\cdot\boldsymbol{\nabla}\mathbf{U}\cdot\hat{\mathbf{b}}-2q_{\perp_s}\boldsymbol{\nabla}\cdot\hat{\mathbf{b}}=-\tfrac{2}{3}\nu_s(p_{\parallel_s}-p_{\perp_s}), \tag{16}$$

$$\frac{\partial p_{\perp_s}}{\partial t}+\boldsymbol{\nabla}\cdot(\mathbf{U}p_{\perp_s})+\boldsymbol{\nabla}\cdot(\hat{\mathbf{b}}q_{\perp_s})+p_{\perp_s}\boldsymbol{\nabla}\cdot\mathbf{U}-p_{\perp_s}\hat{\mathbf{b}}\cdot\boldsymbol{\nabla}\mathbf{U}\cdot\hat{\mathbf{b}}+q_{\perp_s}\boldsymbol{\nabla}\cdot\hat{\mathbf{b}}=-\tfrac{1}{3}\nu_s(p_{\perp_s}-p_{\parallel_s}), \tag{17}$$

$$\frac{\partial q_{\parallel_s}}{\partial t}+\boldsymbol{\nabla}\cdot(\mathbf{U}q_{\parallel_s})+\boldsymbol{\nabla}\cdot(\hat{\mathbf{b}}r_{\parallel,\parallel_s})+3q_{\parallel_s}\hat{\mathbf{b}}\cdot\boldsymbol{\nabla}\mathbf{U}\cdot\hat{\mathbf{b}}-\frac{3p_{\parallel_s}}{nm_s}\hat{\mathbf{b}}\cdot\boldsymbol{\nabla}p_{\parallel_s}+3\left(\frac{p_{\perp_s}p_{\parallel_s}}{nm_s}-\frac{p_{\parallel_s}^2}{nm_s}-r_{\parallel,\perp_s}\right)\boldsymbol{\nabla}\cdot\hat{\mathbf{b}}=-\nu_sq_{\parallel_s}, \tag{18}$$

$$\frac{\partial q_{\perp_s}}{\partial t}+\boldsymbol{\nabla}\cdot(\mathbf{U}q_{\perp_s})+\boldsymbol{\nabla}\cdot(\hat{\mathbf{b}}r_{\parallel,\perp_s})+q_{\perp_s}\boldsymbol{\nabla}\cdot(u_\parallel\hat{\mathbf{b}})-\frac{p_{\perp_s}}{nm_s}\hat{\mathbf{b}}\cdot\boldsymbol{\nabla}p_{\parallel_s}+\left(\frac{p_{\perp_s}^2}{nm_s}-\frac{p_{\perp_s}p_{\parallel_s}}{nm_s}-r_{\perp,\perp_s}+r_{\parallel,\perp_s}\right)\boldsymbol{\nabla}\cdot\hat{\mathbf{b}}=-\nu_sq_{\perp_s}, \tag{19}$$

where $\rho=n(m_e+m_i)$, $\mathbf{U}=\mathbf{v}_E+u_\parallel\hat{\mathbf{b}}$, and $\nu_i=\nu_{ii}+\nu_{ie}$ and $\nu_e=\nu_{ee}+\nu_{ei}$.

Using the condition $u_{\parallel_i}=u_{\parallel_e}$ to solve for $E_\parallel$ [as given in Kulsrud's Eq. (49)], it is straightforward to show that the first two moment equations, Eqs. (14) and (15) are equivalent to Eq. (2), and the parallel component of Eq. (3), that is,

$$\frac{\partial u_\parallel}{\partial t}+\mathbf{U}\cdot\nabla u_\parallel+\hat{\mathbf{b}}\cdot\left(\frac{\partial \mathbf{v}_E}{\partial t}+\mathbf{U}\cdot\nabla\mathbf{v}_E\right)+\frac{1}{\rho}[\hat{\mathbf{b}}\cdot\nabla p_\parallel+(p_\parallel-p_\perp)\nabla\cdot\hat{\mathbf{b}}]=0. \tag{20}$$

A. Conservation properties

Just as in the electrostatic case,[10] the moment hierarchy has favorable conservation properties. Each moment equation acts as a conservation relation, provided the hierarchy is closed by approximating the highest moments, without inserting additional terms such as viscosity.

Momentum is conserved by any closure which keeps Eqs. (2) and (3) and closes for pressure or higher moments. Combining Eqs. (2) and (3) yields

$$\frac{\partial(\rho\mathbf{U})}{\partial t}=-\nabla\cdot\left[\rho\mathbf{U}\mathbf{U}+\left(\frac{B^2}{8\pi}\mathbf{I}-\frac{\mathbf{B}\mathbf{B}}{4\pi}\right)+\mathbf{P}\right]. \tag{21}$$

Similarly, energy is conserved by any closure which uses approximations only for the heat flow moments $q_{\parallel_s}$ and $q_{\perp_s}$, or higher moments. To demonstrate this, define the kinetic+thermal+magnetic energy density $\mathscr{E}=\rho U^2/2+B^2/8\pi+p_\perp+p_\parallel/2$. Combining Eqs. (2), (3), (4), (16), and (17) yields:

$$\frac{\partial\mathscr{E}}{\partial t}=-\nabla\cdot\left[\left(\frac{1}{2}\rho U^2+p_\perp+\frac{1}{2}p_\parallel\right)\mathbf{U}\right]-\nabla\cdot\left[\frac{\mathbf{B}\times(\mathbf{U}\times\mathbf{B})}{4\pi}\right]-\nabla\cdot(\mathbf{U}\cdot\mathbf{P})-\nabla\cdot\mathbf{q}, \tag{22}$$

where $\mathbf{q}\equiv(q_\perp+q_\parallel/2)\hat{\mathbf{b}}$. Integrating over volume, we can take the left hand side as the rate of change of the energy inside a volume, and the right hand side as the flow of energy across the surface. We note that Kulsrud's equations (66) and (67)[3] (not employed elsewhere in the paper) appear to be in error.

IV. THE 4+2 MODEL

A closure for the moment hierarchy must now be derived to produce a complete model. In general, a model which evolves more moments will be more accurate, though more complex and more computationally intensive to implement. A 4+2 moment model, that is a model which evolves four parallel moments $(n,u_\parallel,p_{\parallel_s},q_{\parallel_s})$ and two perpendicular moments $(p_{\perp_s},q_{\perp_s})$, will be developed first. The 4+2 model will truncate the moment hierarchy with Eqs. (18) and (19), and will require closures for $r_{\parallel,\parallel_s}$ and $r_{\parallel,\perp_s}$. Simpler models, such as a 3+1 moment model, can be derived as the low frequency limit of the 4+2 model, following a procedure developed by Dorland.[12]

A closure for the 4+2 model will be derived following the procedure laid out by Hammett and Dorland.[10,12] This procedure, derived for electrostatic perturbations, must be extended for use with general electromagnetic perturbations in two dimensions (parallel and perpendicular). The collisionless case $(\nu\ll\omega)$ will be considered first, and collisional effects will be investigated in Sec. VI. The closure should conserve mass, momentum, and energy, while providing a linear response which closely matches that expected from kinetic theory.

A. Linear response from kinetic theory

We first use the guiding center kinetic equation, Eq. (1), to derive the kinetic linear response. We wish to linearize around a zeroth-order distribution which allows the decoupling of electron and ion pressures as well as the decoupling of parallel and perpendicular pressures that one expects in a collisionless plasma. To accomplish this we choose a bi-Maxwellian distribution with separate equilibrium parallel and perpendicular temperatures $T_{\parallel_{0s}}$ and $T_{\perp_{0s}}$. Since the plasma is collisionless, it is not expected to be exactly Maxwellian, even for a particular species in a particular direction. However, we wish only to calculate a linear response which we can approximate with our Landau closure. The linear response thus needs to provide the correct general form of the linear Landau damping, allowing for independent variation of species pressures, and of the parallel and perpendicular pressures. Hence the bi-Maxwellian is a convenient choice.

We introduce a subsidiary ordering in which the zeroth-order distribution is bi-Maxwellian with no zeroth-order flows or gradients, $f_s=F_{Ms}+f_{1s}$, where

$$F_{Ms}=\frac{n_0}{(2\pi/m_s)^{3/2}T_{\perp_{0s}}T_{\parallel_{0s}}^{1/2}}\exp\left[-\frac{m_sB_0\mu}{T_{\perp_{0s}}}-\frac{m_sv_\parallel^2}{2T_{\parallel_{0s}}}\right]. \tag{23}$$

The moments $(n=n_0+n_1,\ \mathbf{U}=\mathbf{U}_1$, etc.), the magnetic field $(\mathbf{B}=\mathbf{B}_0+\mathbf{B}_1)$, and the parallel electric field $(E_\parallel=E_{\parallel_1})$ are similarly linearized, with the zeroth-order part uniform. Note again that this is a subsidiary ordering. All terms are zeroth order with respect to the initial ordering in $1/e$.

Equation (1) is then linearized and Fourier analyzed to find f_{1s}. Defining $\hat{\mathbf{z}}$ as the unit vector in the parallel direction $\mathbf{B}_0=B_0\hat{\mathbf{z}}$, and defining the wave vector $\mathbf{k}=k_z\hat{\mathbf{z}}+k_x\hat{\mathbf{x}}$,

$$f_{1s}=\left(-\frac{v_\perp^2}{2}\frac{ik_zB_1}{B_0}+\frac{e_s}{m_s}E_\parallel\right)\frac{m_sv_\parallel}{T_{\parallel_{0s}}(-i\omega+ik_zv_\parallel)}f_{0s}. \tag{24}$$

Taking moments, keeping in mind that $\int d^3v=2\pi\int(B_0+B_1)d\mu dv_\parallel$, yields

$$n_{1s}=-\frac{in_0}{k_zT_{\parallel_{0s}}}e_sE_\parallel\mathscr{R}(\zeta_s)+\frac{B_1n_0}{B_0}\left[1-\frac{T_{\perp_{0s}}}{T_{\parallel_{0s}}}\mathscr{R}(\zeta_s)\right], \tag{25}$$

$$p_{\parallel_{1s}}=-\frac{ip_{\parallel_{0s}}}{k_zT_{\parallel_{0s}}}e_sE_\parallel[1+2\zeta_s^2\mathscr{R}(\zeta_s)]+\frac{B_1p_{\parallel_{0s}}}{B_0}\left[1-\frac{T_{\perp_{0s}}}{T_{\parallel_{0s}}}(1+2\zeta_s^2\mathscr{R}(\zeta_s))\right], \tag{26}$$

$$p_{\perp 1s}=-\frac{ip_{\perp 0s}}{k_z T_{\parallel 0s}}e_s E_\parallel \mathscr{R}(\zeta_s)+\frac{2B_1 p_{\perp 0s}}{B_0}\left[1-\frac{T_{\perp 0s}}{T_{\parallel 0s}}\mathscr{R}(\zeta_s)\right], \tag{27}$$

where $\zeta_s=\omega/\sqrt{2}|k_z|v_{t_{\parallel_s}}$ is the normalized frequency, and $\mathscr{R}(\zeta_s)=1+\zeta_s Z(\zeta_s)$ is the electrostatic response function. The usual plasma dispersion function is defined [for Im(ζ) >0] by $Z(\zeta)=(1/\sqrt{\pi})\int dt\, \exp(-t^2)/(t-\zeta)$, and the thermal velocities are defined to be $v_{t_{\parallel_s}}=\sqrt{T_{\parallel 0s}/m_s}$ and $v_{t_{\perp_s}}=\sqrt{T_{\perp 0s}/m_s}$.

Note that it is possible to solve for $E_\parallel$ using quasineutrality, and to solve for B_1 using Eq. (4). However, we find it most convenient and physically enlightening to leave the response functions in the above form for matching to the moment model.

B. The 4+2 Landau closure

We now choose a closure for our 4+2 hierarchy which will closely match the linear response calculated in the previous section. As noted we require closures for both $r_{\parallel,\parallel_s}$ and $r_{\parallel,\perp_s}$. Additional terms such as viscosity would violate energy conservation[10,11] and so will not be employed in the 4+2 equations.

The linearized moment equations in the collisionless ($\nu=0$) limit are, omitting the subscript on perturbed moments and defining $\nabla_\parallel \doteq \hat{\mathbf{b}}_0\cdot\nabla$,

$$\frac{\partial n}{\partial t}+n_0\nabla\cdot\mathbf{U}=0, \tag{28}$$

$$\frac{\partial u_\parallel}{\partial t}+\frac{1}{n_0 m_s}\nabla_\parallel p_{\parallel_s}+\frac{(p_{\perp 0s}-p_{\parallel 0s})}{n_0 m_s}\frac{\nabla_\parallel B_1}{B_0}-\frac{e_s}{m_s}E_\parallel=0, \tag{29}$$

$$\frac{\partial p_{\parallel_s}}{\partial t}+p_{\parallel 0s}\nabla\cdot\mathbf{v}_E+\nabla_\parallel q_{\parallel_s}+3p_{\parallel 0s}\nabla_\parallel u_\parallel=0, \tag{30}$$

$$\frac{\partial p_{\perp_s}}{\partial t}+2p_{\perp 0s}\nabla\cdot\mathbf{v}_E+\nabla_\parallel q_{\perp_s}+p_{\perp 0s}\nabla_\parallel u_\parallel=0, \tag{31}$$

$$\frac{\partial q_{\parallel_s}}{\partial t}+\nabla_\parallel r_{\parallel,\parallel_s}-\frac{3p_{\parallel 0s}}{n_0 m_s}\nabla_\parallel p_{\parallel_s}+\left(-r_{\parallel,\parallel 0s}+3r_{\parallel,\perp 0s}+\frac{3p_{\parallel 0s}^2}{n_0 m_s}-\frac{3p_{\parallel 0s}p_{\perp 0s}}{n_0 m_s}\right)\frac{\nabla_\parallel B_1}{B_0}=0, \tag{32}$$

$$\frac{\partial q_{\perp_s}}{\partial t}+\nabla_\parallel r_{\parallel,\perp_s}-\frac{p_{\perp 0s}}{n_0 m_s}\nabla_\parallel p_{\parallel_s}+\left(r_{\perp,\perp 0s}-2r_{\parallel,\perp 0s}-\frac{p_{\perp 0s}^2}{n_0 m_s}+\frac{p_{\perp 0s}p_{\parallel 0s}}{n_0 m_s}\right)\frac{\nabla_\parallel B_1}{B_0}=0. \tag{33}$$

The bi-Maxwellian values $r_{\parallel,\parallel 0s}=3p_{\parallel 0s}/n_0 m_s$, $r_{\parallel,\perp 0s}=p_{\parallel 0s}p_{\perp 0s}/n_0 m_s$, and $r_{\perp,\perp 0s}=2p_{\perp 0s}/n_0 m_s$ are easily calculated. Fourier transforming into $(\mathbf{k},t)$ space, and using the linearized Equation (4), $\mathbf{k}\cdot\mathbf{v}_E=\omega B_1/B_0$, yields a simple set of equations for each moment in terms of the other moments and the perturbed magnetic field.

The system is closed by writing the highest moments ($r_{\parallel,\parallel_s}$ and $r_{\parallel,\perp_s}$) as a linear sum of the lower moments, with coefficients that are in general functions of $\mathbf{k}$ and the equilibrium quantities. Generalized linear response functions can then be derived. The closure coefficients are determined by comparison with linear kinetic theory in the high and low frequency limits.

Guided by previous work,[10,12] we choose closures with a bi-Maxwellian part and an additional term which models phase mixing. We first try a simple generalization of the 4+2 closure derived by Dorland[12] for the electrostatic case, modified for the case of a bi-Maxwellian equilibrium distribution:

$$r_{\parallel,\parallel_s}=3v_{t\parallel_s}^2(2p_{\parallel_s}-T_{\parallel 0s}n)+\beta_\parallel n_0 v_{t\parallel_s}^2 T_{\parallel_s}-\sqrt{2}D_\parallel v_{t\parallel_s}\frac{ik_\parallel q_{\parallel_s}}{|k_\parallel|}, \tag{34}$$

$$r_{\parallel,\perp_s}=v_{t\perp_s}^2 p_{\parallel_s}+v_{t\parallel_s}^2 p_{\perp_s}-v_{t\parallel_s}^2 T_{\perp 0s}n-\sqrt{2}D_\perp v_{t_{\parallel_s}}\frac{ik_\parallel q_{\perp_s}}{|k_\parallel|}. \tag{35}$$

The coefficients $\beta_\parallel$, $D_\parallel$, and $D_\perp$ are determined by matching the perturbed density and perpendicular pressure to the kinetic results in the adiabatic ($|\zeta|\ll 1$) and fluid ($|\zeta|\gg 1$) limits. It is possible to match the density response through order ζ^2 for small $|\zeta|$ and through order $1/\zeta^5$ for large $|\zeta|$. The $p_\perp$ response can be matched through order ζ for small $|\zeta|$ and through order $1/\zeta^2$ for large $|\zeta|$. This yields $\beta_\parallel=(32-9\pi)/(3\pi-8)$, $D_\parallel=2\sqrt{\pi}/(3\pi-8)$, and $D_\perp=\sqrt{\pi}/2$ (the same result as in the earlier electrostatic derivation[12]).

The density response is then

$$n_{1s}=-\frac{in_0}{k_z T_{\parallel 0s}}e_s E_\parallel \mathscr{R}_4(\zeta_s)+\frac{B_1 n_0}{B_0}\left[1-\frac{T_{\perp 0s}}{T_{\parallel 0s}}\mathscr{R}_4(\zeta_s)\right], \tag{36}$$

where $\mathscr{R}_4(\zeta_s)$ is a four-pole model of the electrostatic response function $\mathscr{R}(\zeta_s)$:

$$\mathscr{R}_4(\zeta_s)=\frac{4-2i\sqrt{\pi}\zeta_s+(8-3\pi)\zeta_s^2}{4-6i\sqrt{\pi}\zeta_s+(16-9\pi)\zeta_s^2+4i\sqrt{\pi}\zeta_s^3+(6\pi-16)\zeta_s^4}. \tag{37}$$

The linear kinetic response functions for the 4 parallel moments $n,u_\parallel,p_{\parallel_s},q_{\parallel_s}$ are all modeled equally well, with $\mathscr{R}_4(\zeta_s)$ replacing $\mathscr{R}(\zeta_s)$ in the expressions for each. The 4+2 density response is compared to linear kinetic response in Figs. 1 and 2. Note that in the figures, the quasineutrality relation $n_{1i}=n_{1e}$ has been used to eliminate $E_\parallel$ from the expressions for the response functions.

In the $p_{\perp_s}$ response, $\mathscr{R}(\zeta_s)$ is modeled partially by the four-pole function $\mathscr{R}_4(\zeta_s)$ and partially by the two-pole function $\mathscr{R}_2(\zeta_s)=1/(1-i\sqrt{\pi}\zeta_s-2\zeta_s^2)$, yielding

$$p_{\perp_s}=-\frac{ip_{\perp_{0s}}}{k_z T_{\|_{0s}}}e_s E_{\|}\mathscr{R}_4(\zeta_s)+\frac{2B_1 p_{\perp_{0s}}}{B_0}\times\left[1-\frac{T_{\perp_{0s}}}{T_{\|_{0s}}}\left(\frac{\mathscr{R}_4(\zeta_s)}{2}+\frac{\mathscr{R}_2(\zeta_s)}{2}\right)\right]. \quad (38)$$

As shown in Figs. 3 and 4, the $p_{\perp_s}$ response is not matched as closely as the parallel moment response for large ζ_s, but the fit is still quite good.

Note that we could have chosen a more general form for the $r_{\|,\|_s}$ and $r_{\|,\perp_s}$ closures, involving all lower moments and the perturbed magnetic field. However, upon matching the linear kinetic response in the $|\zeta|\ll 1$ and $|\zeta|\gg 1$ limits, these general closures will reduce to the closure given here.

The complete 4+2 system of equations is Eqs. (2) through (5), plus Eqs. (16) through (19) closed by the inverse Fourier transform of Eqs. (34) and (35). The system can be solved numerically in k-space where the closure functions are more easily evaluated.

V. THE 3+1 MODEL

For many applications, a simpler, less computationally intensive model will prove adequate. The simplest model which evolves $p_\|$ and $p_\perp$ involves truncating the hierarchy with Eqs. (16) and (17), using closure approximations for $q_\|$ and $q_\perp$. We refer to such a model as a ''3+1 model'' because it evolves 3 parallel moments (n, $u_\|$, $p_\|$) and 1 perpendicular moment ($p_\perp$). Note that the CGL model is a 3+1 model which invokes the simple closure $q_\|=q_\perp=0$.

The 3+1 closures can be derived following the procedure laid out in the previous section, by writing $q_\|$ and $q_\perp$ as a sum of the lower moments and B_1, and solving for coefficients by matching with the linear kinetic density and perpendicular pressure response. However, the 3+1 closures for both $q_{\|_s}$ and $q_{\perp_s}$ can be more simply derived as the $\zeta_s\rightarrow 0$ limit of the 4+2 model, following the moment reduction scheme outlined by Dorland.[12] Parker and Carati[21] showed how to extend this scheme to an arbitrary number of moments, and used it to show some similarities to renormalization methods.

Substituting the 4+2 closures into Eqs. (32) and (33), in $(\mathbf{k},t)$ space, and taking the limit $|\zeta_s|\ll 1$ yields

$$q_{\|_s}=-n_0\sqrt{\frac{8}{\pi}}\,v_{t_{\|_s}}\frac{ik_\| T_{\|_s}}{|k_\||}, \quad (39)$$

$$q_{\perp_s}=-n_0\sqrt{\frac{2}{\pi}}\,v_{t_{\|_s}}\frac{ik_\| T_{\perp_s}}{|k_\||}+n_0\sqrt{\frac{2}{\pi}}\,v_{t_{\|_s}}T_{\perp_{0s}}\times\left(1-\frac{T_{\perp_{0s}}}{T_{\|_{0s}}}\right)\frac{ik_\| B_1}{|k_\||B_0}. \quad (40)$$

Note the term proportional to B_1 in the $q_\perp$ closure. This term is not found in the electrostatic case, where $B_1=0$, and it also vanishes for isotropic equilibrium pressures. This term is needed to properly conserve μ linearly in the presence of magnetic field compression and anisotropic pressure.

Substituting the closures, Eqs. (39)–(40), into the 3+1 equations yields the density response:

$$n_{1s}=-\frac{in_0}{k_z T_{\|_{0s}}}e_s E_\|\mathscr{R}_3(\zeta_s)+\frac{B_1 n_0}{B_0}\left[1-\frac{T_{\perp_{0s}}}{T_{\|_{0s}}}\mathscr{R}_3(\zeta_s)\right], \quad (41)$$

and the perpendicular pressure response:

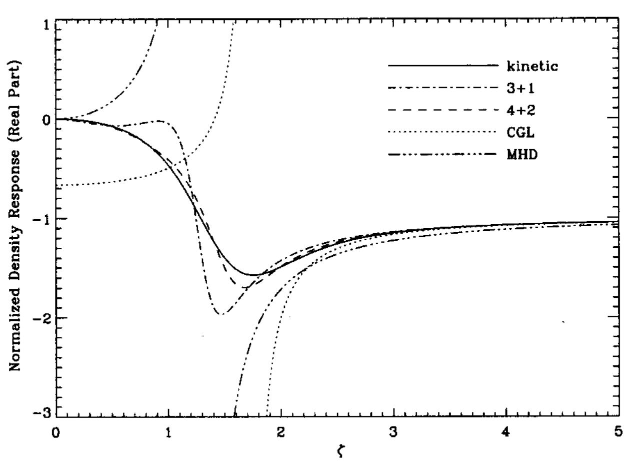

FIG. 1. The real part of the normalized linear density response $(n_1/ik_x\xi_x n_0)$, versus real normalized frequency $(\zeta_i=\omega/\sqrt{2}|k_\||v_{T_{\|_i}})$. The 3+1 and 4+2 moment Landau MHD models are compared with linear kinetic theory. Predictions of CGL theory and ideal MHD theory are also shown. Parameters chosen are $Z=1$, $T_{\perp_0}/T_{\|_0}=1$, $T_{\perp_{0i}}=T_{\perp_{0e}}$, $T_{\|_{0i}}=T_{\|_{0e}}$, and $\sqrt{m_i/m_e}=40$.

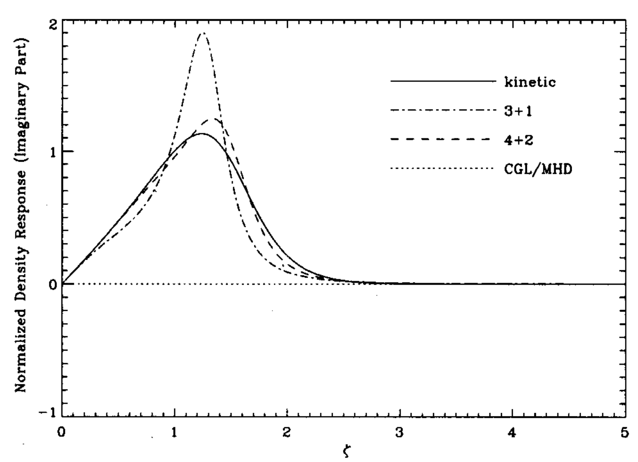

FIG. 2. The imaginary part of the normalized linear density response $(n_1/ik_x\xi_x n_0)$, versus real normalized frequency $(\zeta_i=\omega/\sqrt{2}|k_\||v_{T_{\|_i}})$. The 3+1 and 4+2 moment Landau MHD models are compared with linear kinetic theory. Both CGL theory and Ideal MHD theory predict no imaginary density response. Parameters are identical to those in Fig. 1.

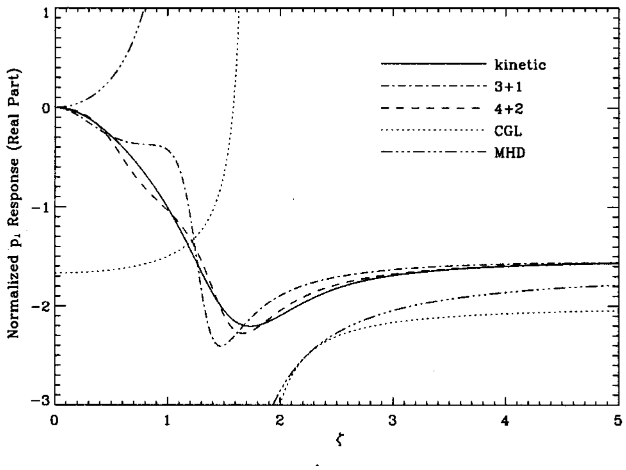

FIG. 3. The real part of the normalized linear total perpendicular pressure response $(p_{\perp_1}/ik_x\xi_x p_{\perp_0})$, versus real normalized frequency ($\zeta_i = \omega/\sqrt{2}|k_\|| v_{T_{\|_i}}$). The 3+1 and 4+2 moment Landau MHD models are compared with kinetic theory. Predictions of CGL theory and ideal MHD theory are also shown. Note the significant variation in the real $p_\perp$ response between the 3+1 model and the kinetic model, even for large ζ. Parameters are identical to those in Fig. 1.

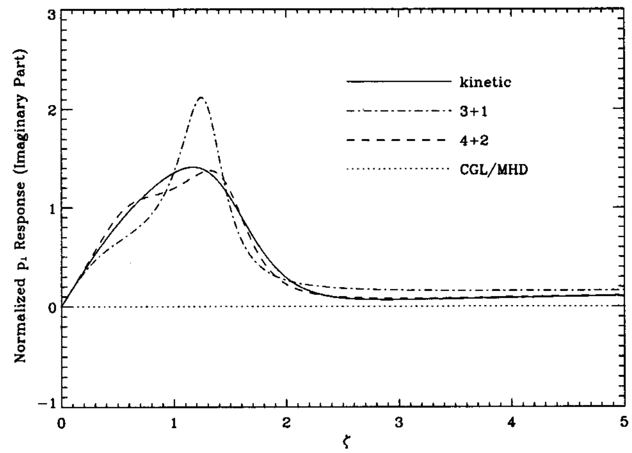

FIG. 4. The imaginary part of the normalized linear total perpendicular pressure response $(p_{\perp_1}/ik_x\xi_x p_{\perp_0})$, versus real normalized frequency ($\zeta_i = \omega/\sqrt{2}|k_\|| v_{T_{\|_i}}$). The 3+1 and 4+2 moment Landau MHD models are compared with kinetic theory. Both CGL theory and ideal MHD theory predict no imaginary pressure response. Parameters are identical to those in Fig. 1.

$$p_{\perp_s} = -\frac{ip_{\perp_{0s}}}{k_z T_{\|_{0s}}} e_s E_\| \mathscr{R}_3(\zeta_s) + \frac{2B_1 p_{\perp_{0s}}}{B_0} \times \left[1 - \frac{T_{\perp_{0s}}}{T_{\|_{0s}}}\left(\frac{\mathscr{R}_3(\zeta_s)}{2} + \frac{\mathscr{R}_1(\zeta_s)}{2}\right)\right], \quad (42)$$

where $\mathscr{R}_3(\zeta_s)$ is a three-pole model of the electrostatic response function:

$$\mathscr{R}_3(\zeta_s) = \frac{2 - i\sqrt{\pi}\zeta_s}{2 - 3i\sqrt{\pi}\zeta_s - 4\zeta_s^2 + 2i\sqrt{\pi}\zeta_s^3}, \quad (43)$$

and $\mathscr{R}_1(\zeta_s)$ is a one-pole model of $\mathscr{R}(\zeta_s)$, $\mathscr{R}_1(\zeta_s) = 1/(1 - i\sqrt{\pi}\zeta_s)$. The 3+1 density and $p_\perp$ responses are plotted in Figs. 1 through 4. Of course the response functions, particularly for $p_\perp$, do not fit the kinetic results as well as for the 4+2 model. However, the qualitative behavior is correct, and the behavior in both limits ($\zeta_s \ll 1$) and ($\zeta_s \gg 1$) is accurate.

The complete 3+1 system of equations is Eqs. (2) through (5), plus Eqs. (16) and (17) closed by the inverse Fourier transform of Eqs. (39) and (40). This set is significantly simpler than the 4+2 equations, while still conserving particles, momentum, and energy, and providing a reasonable model of the linear kinetic response.

Further moment reduction to 3+0, 2+1, 2+0, and even 1+0 models is possible. These simpler models can be useful in certain cases where conservation of thermal energy is not important. However, the 3+1 and 4+2 models allow a separate evolution of $p_\|$ and $p_\perp$, which is often important in describing collisionless modes.

VI. COLLISIONAL EFFECTS

The 3+1 and 4+2 Landau fluid collisionless MHD models have been derived for the completely collisionless case, where the collision rate is very small compared to a typical mode frequency ($\nu \ll \omega$). However, it is possible to introduce some collisional effects into the models using a collision operator such as the BGK operator introduced in Sec. III. It is then possible to examine regimes with a wide range of collisionality, provided that $\nu \ll \Omega_c$, as required by the initial ordering. The accuracy with which collisional effects are modeled will of course be limited by the accuracy of the initial collision operator employed. Furthermore, the modeling of certain collisional effects, such as momentum transfer and resistive tearing of magnetic field lines, is hampered by the use of only the lowest-order collisionless MHD expansion in inverse charge.

The moment hierarchy previously derived [Eqs. (14) through (19)] already includes the collision terms arising from a simple BGK collision operator. However, the form of the equations is quite different from the forms normally used in MHD. We will attack this discrepancy by rewriting Eqs. (16) through (19), and showing that they reduce approximately to Braginskii's transport equations[22] in the limit $\omega, |k| v_{t_s} \ll \nu_s \ll \Omega_c$ (ω is a typical mode frequency, and k is a typical wave number).

First define an average pressure, $p_s = (p_{\|_s} + 2p_{\perp_s})/3$, a differential pressure $\delta p_s = p_{\|_s} - p_{\perp_s}$, and a heat flow $q_s = q_{\|_s}/2 + q_{\perp_s}$. We can then divide the pressure tensor, $\mathbf{P_s}$, into an isotropic part and an anisotropic part labeled $\mathbf{\Pi_s}$. That is $\mathbf{P}_s = p_s\mathbf{I} + \mathbf{\Pi_s} = p_s\mathbf{I} + (-\delta p_s \mathbf{I} + 2\delta p_s \hat{\mathbf{b}}\hat{\mathbf{b}})/3$. Combining Eqs. (16) through (19) then yields

$$\frac{dp_s}{dt} + \frac{5}{3} p_s \nabla \cdot \mathbf{U} = -\frac{2}{3}\nabla \cdot (\hat{\mathbf{b}} q_s) - \frac{2}{3}\mathbf{\Pi_s}:\nabla\mathbf{U}, \quad (44)$$

$$\frac{d\,\delta p_s}{dt} + \frac{5}{3}\delta p_s \nabla \cdot \mathbf{U} + \mathbf{\Pi_s}:\nabla\mathbf{U} + 3p_s \hat{\mathbf{b}}\cdot\nabla\mathbf{U}\cdot\hat{\mathbf{b}} - p_s \nabla\cdot\mathbf{U} - 3q_{\perp_s}\nabla\cdot\mathbf{U} + \nabla\cdot[\hat{\mathbf{b}}(q_{\|_s} - q_{\perp_s})] = -\nu_s \delta p_s, \quad (45)$$

$$\frac{\partial q_s}{\partial t}+\nabla\cdot\left[\hat{\mathbf{b}}\left(\frac{r_{\parallel,\parallel_s}}{2}+r_{\parallel,\perp_s}\right)\right]+\frac{3}{2}q_{\parallel_s}\hat{\mathbf{b}}\cdot\nabla\mathbf{U}\cdot\hat{\mathbf{b}}$$

$$-\frac{\frac{3}{2}p_{\parallel_s}+p_{\perp_s}}{nm_s}\hat{\mathbf{b}}\cdot\nabla p_{\perp_s}+q_{\perp_s}\nabla\cdot(u_\parallel\hat{\mathbf{b}})+\left(\frac{p_{\perp_s}^{\;2}}{nm_s}+\frac{p_{\perp_s}p_{\parallel_s}}{2nm_s}\right.$$

$$\left.-\frac{3p_{\perp_s}^{\;2}}{2nm_s}-\frac{r_{\parallel,\perp_s}}{2}-r_{\perp,\perp_s}\right)\nabla\cdot\hat{\mathbf{b}}=-\nu_s q_s. \tag{46}$$

A. The high collisionality limit

In the limit of high collisionality ($\nu\gg\omega$), the above three equations yield an approximation to the Braginskii transport equations,[22] with the condition $\nu\ll\Omega_c$, as required by the initial ordering.

Formally expanding all moments in the collision time ($1/\nu$), it is apparent from Eqs. (16)–(19) that $q_{\parallel_{0s}}=q_{\perp_{0s}}=\delta p_{0s}=0$. Equation (45) then reduces, to lowest order, to

$$\delta p_{1s}=-\frac{p_{0s}}{\nu_s}(3\hat{\mathbf{b}}\cdot\nabla\mathbf{U}\cdot\hat{\mathbf{b}}-\nabla\cdot\mathbf{U}).$$

If ν_s from the original BGK collision operator is taken to be the reciprocal of Braginskii's collision time ($\nu_s=1/\tau_{s_{\mathrm{Brag}}}$), the resulting expression for $\mathbf{\Pi_s}=(-\delta p_s\mathbf{I}+2\,\delta p_s\hat{\mathbf{b}}\hat{\mathbf{b}})/3$ matches Braginskii's result to within an order unity constant (0.96 for $Z=1$ ions, and 0.73 for electrons).

Similarly, a heat flux nearly matching Braginskii's can be derived in the same limit. To lowest order, Eq. (46) becomes

$$\nabla\cdot\left[\hat{\mathbf{b}}\left(\frac{r_{\parallel,\parallel_{0s}}}{2}+r_{\parallel,\perp_{0s}}\right)\right]-\frac{5}{2}\frac{p_{0s}}{n_0m_s}\hat{\mathbf{b}}\cdot\nabla p_{0s}$$

$$+\left(-\frac{r_{\parallel,\perp_{0s}}}{2}-r_{\perp,\perp_{0s}}\right)\nabla\cdot\hat{\mathbf{b}}=-\nu_s q_{1s}. \tag{47}$$

In this collisional limit, the r_0 moments will take on their Maxwellian values ($r_{\parallel,\parallel_{0s}}=3p_0^2/m_sn_0, r_{\parallel,\perp_{0s}}=p_0^2/m_sn_0$, $r_{\perp,\perp_{0s}}=2p_0^2/m_sn_0$). Substituting yields

$$q_{1s}=-\frac{5}{2}\frac{p_0}{\nu_sm_s}\nabla_\parallel T_{0s},$$

which matches the Braginskii heat fluxes to within factors of order unity.

To match Braginskii's results more precisely, one could replace the simple BGK collision operator used here with a more precise Landau or Fokker–Planck operator. This should allow reproduction of the collisional energy flow between species (Q_s) as well as the above heat flow and anisotropic pressure terms. However, modeling momentum exchange terms is problematic because the initial formal expansion in $1/e$ used to derive the collisionless MHD equations implies $u_{\parallel_i}=u_{\parallel_e}$. The effects of resistive momentum exchange thus require going to higher order in the ideal MHD ordering, or using an alternative ordering procedure.

B. Collisionally modified 3+1 closure

Collisional effects have not been considered in the derivation of the Landau closures themselves. In principle, it is possible to rederive the linear kinetic response functions with collision terms, and choose Landau closures which match this collisional linear response. However, a simpler procedure appears to be adequate.

This alternate approach,[23,24] is to derive a collisionless closure for a many moment model (here the 4+2 model), and then reduce the number of moments by taking the low frequency limit of the highest moment equations, with the collisional terms included. This will incorporate some collisional effects into the lower moment closure (here it will include the collisional effects described by the $q_\parallel$ and $q_\perp$ equations into the 3+1 model). The modified 3+1 closures resulting from this procedure are

$$q_{\parallel_s}=-8n_0v_{t\parallel_s}^2\frac{ik_\parallel T_{\parallel_s}}{(\sqrt{8\pi}|k_\parallel|v_{t_{\parallel_s}}+(3\pi-8)\nu_s)}, \tag{48}$$

$$q_{\perp_s}=-\frac{n_0v_{t_{\parallel_s}}^2ik_\parallel T_{\perp_s}}{\left(\sqrt{\frac{\pi}{2}}\,|k_\parallel|v_{t_{\parallel_s}}+\nu_s\right)}$$

$$+\left(1-\frac{T_{\perp 0s}}{T_{\parallel_{0s}}}\right)\frac{n_0v_{t\parallel_s}^2T_{\perp 0s}ik_\parallel B_1}{B_0\left(\sqrt{\frac{\pi}{2}}\,|k_\parallel|v_{t_{\parallel_s}}+\nu_s\right)}. \tag{49}$$

These closures allow a smooth transition from the collisionless regime where Landau damping is important, to the collisional regime where Landau damping vanishes.

Hence some collisional effects can be included within the Landau collisionless MHD model, and the model can be extended for use in the marginally collisional regime ($\nu\sim\omega$) as well as the collisionless regime ($\nu\ll\omega$). However, the accurate modeling of some collisional effects, particularly those associated with momentum exchange, is made difficult by the use of the collisionless MHD ordering. A model based on Braginskii or resistive MHD is more appropriate for use in the highly collisional regime ($\nu\gg\omega$).

VII. NONLINEAR IMPLEMENTATION OF THE CLOSURE TERMS

The closures for both the 4+2 and 3+1 models employ terms containing $|k_\parallel|/k_\parallel$. Numerical evaluation of these terms in k-space is straightforward for electrostatic problems (such as ion temperature gradient/drift-wave turbulence), since only a simple Fourier transform along the equilibrium magnetic field direction is required. But as pointed out by Finn and Gerwin,[18] Landau damping must be evaluated along perturbed field lines, i.e. Landau damping involves particles mixing due to their free-streaming along the total (equilibrium+fluctuating) magnetic field, and so $k_\parallel$ involves Fourier transforms along these perturbed magnetic field lines. Conceptually, a parallel heat flux is driven by a parallel temperature gradient: $q_\parallel\propto\nabla_\parallel T_\parallel=\hat{\mathbf{b}}\cdot\nabla T_\parallel$. Linearizing this yields $q_{\parallel_1} \propto \hat{\mathbf{b}}_0 \cdot \nabla T_{\parallel_1} + \hat{\mathbf{b}}_1 \cdot \nabla T_{\parallel_0}$. We see that considering only the Fourier transform of $\nabla T_{\parallel_1}$ in the $\hat{\mathbf{b}}_0$ direction would not be sufficient even linearly. In fact, in the ideal MHD limit where the magnetic field is frozen into the fluid, if the temperature is initially uniform along a magnetic field line it will always remain uniform along a field line if the plasma motion is incompressible, so that the perpendicular gradient term will exactly cancel the parallel gradient term: $q_{\parallel_1} \propto \hat{\mathbf{b}}_0 \cdot \nabla T_{\parallel_1} + \hat{\mathbf{b}}_1 \cdot \nabla T_{\parallel_0} = 0$. To account for this, Bondeson and Ward[17] employed Lagrangian variables and applied a Landau damping model only to the component of the temperature fluctuations driven by compression. Alternatively, one could use the higher-order $4+2$ moment equations which involve $|k_\parallel|$ operating on a higher moment like $q_\parallel$. Upon linearizing $\nabla_\parallel q_\parallel = \hat{\mathbf{b}}_0 \cdot \nabla q_{\parallel_1} + \hat{\mathbf{b}}_1 \cdot \nabla q_{\parallel_0}$, we often have only to consider the first term since $q_{\parallel_0}$ is zero for many types of equilibria.

However, the situation is more complicated for nonlinear electromagnetic calculations. Then the nonlinear term $\hat{\mathbf{b}}_1 \cdot \nabla T_{\parallel_1}$ can not formally be neglected compared to $\hat{\mathbf{b}}_0 \cdot \nabla T_{\parallel_1}$. To be rigorous, the transformation between the k-space closure and its real space equivalent must be made along the perturbed field lines. One way to do this would be with a Lagrangian coordinate system which moved with the magnetic field and had one coordinate aligned with the magnetic field. Then the standard fast Fourier transform (FFT) algorithm along this coordinate could be used to evaluate the $|k_\parallel|$ closures. Alternatively, if the simulation uses a fixed Eulerian grid, then at every time step where $|k_\parallel| T_\parallel$ is to be evaluated, one would need to map $T_\parallel$ from the simulation grid to a field-line-following coordinate system, carry out the FFT, and then map the result back to the simulation grid.

One can avoid FFT's by working directly with the real-space form of the the closures. This is somewhat more expensive computationally, since it involves convolutions in one direction [$\mathcal{O}(N^4)$ operations, where N is the number of grid points in each direction] rather than the faster FFT algorithm [$\mathcal{O}(N^3 \log N)$ operations]. But because the convolutions are done in only one direction instead of a three-dimensional (3-D) convolution [$\mathcal{O}(N^6)$ operations], this may be acceptable.

For example, the real-space form of the collisionless $3+1$ moment closure for $q_\parallel$, Eq. (39), is the convolution[10]

$$q_{\parallel_s}(z) = -n_0 \left(\frac{2}{\pi}\right)^{3/2} v_{t_{\parallel_s}} \int_0^\infty dz' \, \frac{T_{\parallel_s}(z+z') - T_{\parallel_s}(z-z')}{z'}, \tag{50}$$

where the integration is performed along the perturbed field line. Evaluation of this integral (or its discrete analogue) in principle requires evaluation of the parallel temperature fluctuation at an infinite number of points along the field. In practice the integral can be cut off at a reasonable parallel correlation length.[23] Truncating the integral at $z' = L$ means that the Landau damping is applied primarily to modes with $k_\parallel > 1/L$, while modes with $k_\parallel \ll 1/L$ will experience relatively little damping due to the Landau resonances. This approximation is probably adequate in cases where the Landau-damping is only important for the high-$k_\parallel$ component of the fluctuation spectrum, and convergence can be tested by varying L.

When collisions are important, the collisional form of the $q_\parallel$ closure, Eq. (48), should be used. The real space form of this closure is then

$$q_{\parallel_s} = -n_0 \left(\frac{2}{\pi}\right)^{3/2} v_{t_{\parallel_s}} \int_0^\infty d\hat{z}' \, g(\hat{z}') [T_{\parallel_s}(\hat{z}+\hat{z}') - T_{\parallel_s}(\hat{z}-\hat{z}')], \tag{51}$$

$$g(\hat{z}) = \int_0^\infty d\hat{k} \, \frac{\hat{k}}{\hat{k}+1} \sin(\hat{k}\hat{z}),$$

where $\hat{k} \doteq k L_\parallel$ and $\hat{z} \doteq z / L_\parallel$ have been normalized to the parallel collisional mean free path,

$$L_\parallel \doteq \frac{\sqrt{8\pi}}{3\pi - 8} \frac{v_{t_{\parallel_s}}}{\nu_s}.$$

For small $\hat{z}$ Eq. (51) behaves just as Eq. (50), but for large $\hat{z}$, $g(\hat{z})$ falls off rapidly, as $1/\hat{z}^3$, and the closure integral may be quite accurately truncated after a few mean free paths.

Equation (50) includes nonlinear magnetic effects if the integral is evaluated along perturbed magnetic field lines, but it still assumes that density and temperature vary weakly along a field line so that constant equilibrium values of n_0 and v_t can be used. There are various possible extensions of this closure which could be proposed to model cases with stronger parallel nonlinearities (for example, see Sec. 3.4 of Smith's thesis[25]). The relative advantages or accuracy of various possibilities has not yet been studied, but one reasonable nonlinear model is

$$q_{\parallel_s}(z) = -\left(\frac{2}{\pi}\right)^{3/2} \int_0^\infty dz' \, \frac{n(z+z')(T_{\parallel s}^{3/2}(z+z') - T_{\parallel s0}^{3/2}) - n(z-z')(T_{\parallel s}^{3/2}(z-z') - T_{\parallel s0}^{3/2})}{m_s^{1/2} z'}. \tag{52}$$

This has the physically reasonable property of weighting the convolution integral by the density, so that particles streaming from low density regions contribute less to the heat flux. This model (or some variant thereof) might be useful to model the heat flux on field lines which intersect solid materials (where the plasma density goes to zero), such as in the edge of fusion devices. A possible choice for $T_{\parallel_0}$ is $\int dz \, n(z) T_\parallel(z) / \int dz \, n(z)$.

VIII. AN EXAMPLE: THE MIRROR INSTABILITY

To demonstrate the usefulness of our model, and the fundamental importance of kinetic effects in simple collisionless MHD problems, we will investigate the magnetic mirror instability. Kulsrud[3] cites this example to demonstrate the use of his guiding-center kinetic theory and to expose the limitations of simple fluid theories such as CGL.[1] We will show here that our Landau-fluid models recover the exact instability threshold for the mirror mode, and provide a good model of the mode's linear growth rate above the threshold.

Consider a strongly-magnetized, homogeneous plasma consisting of electrons and singly charged ions. Take the magnetic field to be uniform in the $\hat{\mathbf{z}}$ direction, $\mathbf{B_0}=B_0\hat{\mathbf{z}}$. The equilibrium distribution is taken to be an anisotropic bi-Maxwellian with unequal parallel and perpendicular temperatures. For simplicity, take the electron and ion temperatures to be equal in each direction, $T_{\parallel_{0i}}=T_{\parallel_{0e}}=T_{\parallel_0}$ and $T_{\perp_{0i}}=T_{\perp_{0e}}=T_{\perp_0}$. Define the $\hat{\mathbf{x}}$ direction by writing the wave vector $\mathbf{k}=k_x\hat{\mathbf{x}}+k_z\hat{\mathbf{z}}$, and define a ''plasma displacement'' vector ξ by $\mathbf{U}=-i\omega\xi$.

Linearizing and Fourier transforming Eqs. (2) through (5) then yields the following equations of motion:

$$-\rho_0\omega^2\xi_x=-ik_xp_\perp+k_z^2(p_{\parallel_0}-p_{\perp_0})\xi_x-(k_x^2+k_z^2)\times(B_0^2/4\pi)\xi_x, \tag{53}$$

$$-\rho_0\omega^2\xi_z=-ik_zp_\parallel+k_xk_z(p_{\parallel_0}-p_{\perp_0})\xi_x, \tag{54}$$

where the subscript on the perturbed pressures is again suppressed. Expressions for the perturbed pressures $p_\parallel$ and $p_\perp$ are needed to close this system and solve for the instability growth rate. We will close the system in four different ways: first with linear kinetic theory, then using CGL theory, then with the $3+1$ Landau MHD model, and finally with the $4+2$ Landau MHD model, in order to compare the instability thresholds and linear growth rates determined by each.

To calculate a kinetic result, we proceed exactly as in Eqs. (24) through (27). Using quasineutrality to solve for $E_\parallel$, and using Eq. (4) for $B_1=-ik_x\xi_xB_0$, yields

$$eE_\parallel=k_xk_z\xi_xT_{\perp_0}\frac{\mathscr{R}(\zeta_i)-\mathscr{R}(\zeta_e)}{\mathscr{R}(\zeta_i)+\mathscr{R}(\zeta_e)}. \tag{55}$$

This leads to the following expressions for the perturbed pressures:

$$p_\perp=2ik_x\xi_xp_{\perp_0}\left[\frac{T_{\perp_0}}{T_{\parallel_0}}\left(\frac{\mathscr{R}(\zeta_i)+\mathscr{R}(\zeta_e)}{4}+\frac{\mathscr{R}(\zeta_i)\mathscr{R}(\zeta_e)}{\mathscr{R}(\zeta_i)+\mathscr{R}(\zeta_e)}\right)-1\right], \tag{56}$$

$$p_\parallel=ik_x\xi_xp_{\parallel_0}\left[\frac{T_{\perp_0}}{T_{\parallel_0}}\left(1+\frac{2\mathscr{R}(\zeta_i)\mathscr{R}(\zeta_e)(\zeta_i^2+\zeta_e^2)}{\mathscr{R}(\zeta_i)+\mathscr{R}(\zeta_e)}\right)-1\right]. \tag{57}$$

Substituting for $p_\perp$ in Eq. (53) leads to the dispersion relation

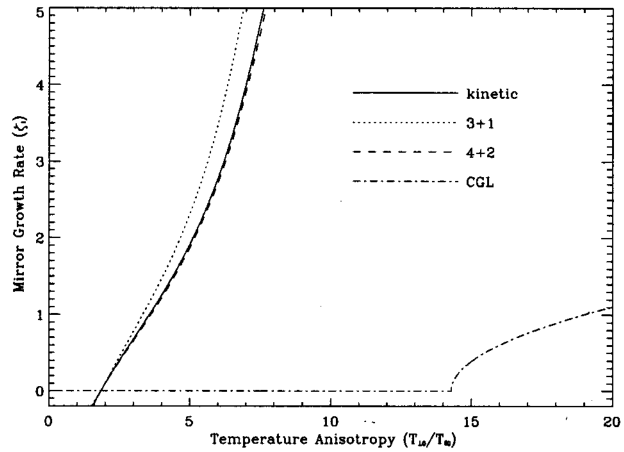

FIG. 5. The linear growth rate of the mirror instability ($k_\perp^2 \gg k_\parallel^2$) as predicted by kinetic theory, $3+1$ and $4+2$ Landau MHD models, and CGL theory (ideal MHD cannot predict the mirror growth rate as it posits an isotropic pressure). The normalized growth rate $[\zeta_i=\mathrm{Im}(\omega)/\sqrt{2}|k_\parallel|v_{T_{\parallel i}}]$ is plotted versus the temperature anisotropy ($T_{\perp_0}/T_{\parallel_0}$) at constant $\beta=\{(2/3)p_{\perp_0}+(1/3)p_{\parallel_0}\}/(B_0^2/8\pi)$. The parameters chosen are $Z=1$, $T_{\perp_{0i}}=T_{\perp_{0e}}$, $T_{\parallel_{0i}}=T_{\parallel_{0e}}$, $\beta=1$, and $\sqrt{m_i/m_e}=40$.

$$\zeta_i^2+\zeta_e^2=2\frac{k_x^2}{k_z^2}\left(-\frac{T_{\perp 0}^2}{T_{\parallel 0}^2}\mathscr{A}_k(\zeta)+\frac{T_{\perp_0}}{T_{\parallel_0}}+\frac{B_0^2}{8\pi p_{\parallel_0}}\right)+\left(\frac{T_{\perp_0}}{T_{\parallel_0}}-1+\frac{B_0^2}{4\pi p_{\parallel_0}}\right), \tag{58}$$

where the function $\mathscr{A}_k(\zeta)$ is defined by $\mathscr{A}_k(\zeta)=\{\mathscr{R}(\zeta_i)^2+6\mathscr{R}(\zeta_i)\mathscr{R}(\zeta_e)+\mathscr{R}(\zeta_e)^2\}/\{4(\mathscr{R}(\zeta_i)+\mathscr{R}(\zeta_e))\}$. For parallel propagation ($|k_z|\gg|k_x|$), the above reduces to the dispersion relation for the ''firehose'' instability, and the kinetic effects drop out within our ordering (note that a different ordering can be used to analyze these much smaller kinetic effects for limited parameter regimes—see Medvedev and Diamond[19]). All of the models considered will reproduce the firehose linear growth rate exactly. In the opposite limit ($|k_x|\gg|k_z|$), the dispersion relation becomes

$$\zeta_i^2+\zeta_e^2=2\frac{k_x^2}{k_z^2}\left(-\frac{T_{\perp 0}^2}{T_{\parallel 0}^2}\mathscr{A}_k(\zeta)+\frac{T_{\perp_0}}{T_{\parallel_0}}+\frac{B_0^2}{8\pi p_{\parallel_0}}\right). \tag{59}$$

This relation has an infinite number of roots, due to the presence of plasma Z-functions. The magnetic mirror instability is the root for which the real part of the frequency goes to zero. Taking the limit $\zeta\to 0$ leads to the instability criterion for the mirror mode, $p_{\perp 0}^2/p_{\parallel_0}>p_{\perp_0}+B_0^2/8\pi$. The linear mirror growth rate versus the degree of anisotropy $T_{\perp_0}/T_{\parallel_0}$ is plotted in Fig. 5 for a fixed mass ratio at fixed total plasma beta, $\beta=((2/3)p_{\perp_0}+(1/3)p_{\parallel_0})/(B_0^2/8\pi)$.

The Chew–Goldberger–Low[1] theory can also be used to investigate the mirror instability. CGL's simple truncation of the moment hierarchy with $q_\parallel=q_\perp=0$ leads to the following linearized expressions for the two perturbed pressures:

$$p_\parallel=-ip_{\parallel_0}(k_x\xi_x+3k_z\xi_z), \tag{60}$$

$$p_\perp = -ip_{\perp_0}(2k_x\xi_x + k_z\xi_z). \tag{61}$$

Plugging these into the equations of motion leads to the following dispersion relation:

$$\zeta_i^2+\zeta_e^2 = 2\frac{k_x^2}{k_z^2}\left(-\frac{T_{\perp 0}^{\ 2}}{T_{\parallel 0}^2}\frac{1}{6-2(\zeta_i^2+\zeta_e^2)}+\frac{T_{\perp_0}}{T_{\parallel_0}}+\frac{B_0^2}{8\pi p_{\parallel_0}}\right) + \left(\frac{T_{\perp_0}}{T_{\parallel_0}}-1+\frac{B_0^2}{4\pi p_{\parallel_0}}\right). \tag{62}$$

In the $|k_z|\gg|k_x|$ limit, CGL theory correctly predicts the instability threshold for the firehose instability. However, in the opposite limit $|k_x|\gg|k_z|$, CGL's description of the mirror mode is drastically in error. CGL predicts the mirror mode goes unstable for $p_{\perp 0}^{\ 2}/6p_{\parallel_0} > p_{\perp_0} + B_0^2/8\pi$, a factor of 6 error from kinetic theory, as noted by Kulsrud.[3] The linear growth rate is plotted in Fig. 5.

The $3+1$ Landau MHD model does markedly better in modeling the mirror mode. The $3+1$ dispersion relation is derived using quasineutrality and Eq. (41) to solve for $E_\parallel$, and using $B_1 = -ik_x\xi_x B_0$ to find

$$eE_\parallel = k_x k_z \xi_x T_{\perp_0}\frac{\mathscr{R}_3(\zeta_i)-\mathscr{R}_3(\zeta_e)}{\mathscr{R}_3(\zeta_i)+\mathscr{R}_3(\zeta_e)}. \tag{63}$$

Plugging this into the $3+1$ model expressions for the perturbed pressures worked out in Sec. IV B, and summing the 2 species pressures yields

$$p_\perp = 2ik_x\xi_x p_{\perp_0}\left[\frac{T_{\perp_0}}{T_{\parallel_0}}\left(\frac{\mathscr{R}_1(\zeta_i)+\mathscr{R}_1(\zeta_e)}{4} + \frac{\mathscr{R}_3(\zeta_i)\mathscr{R}_3(\zeta_e)}{\mathscr{R}_3(\zeta_i)+\mathscr{R}_3(\zeta_e)}\right)-1\right], \tag{64}$$

$$p_\parallel = ik_x\xi_x p_{\parallel_0}\left[\frac{T_{\perp_0}}{T_{\parallel_0}}\left(1+\frac{2\mathscr{R}_3(\zeta_i)\mathscr{R}_3(\zeta_e)(\zeta_i^2+\zeta_e^2)}{\mathscr{R}_3(\zeta_i)+\mathscr{R}_3(\zeta_e)}\right)-1\right]. \tag{65}$$

Substituting these results into the equations of motion leads to the following dispersion relation:

$$\zeta_i^2+\zeta_e^2 = 2\frac{k_x^2}{k_z^2}\left(-\frac{T_{\perp 0}^{\ 2}}{T_{\parallel 0}^2}\mathscr{A}_3(\zeta)+\frac{T_{\perp_0}}{T_{\parallel_0}}+\frac{B_0^2}{8\pi p_{\parallel_0}}\right) + \left(\frac{T_{\perp_0}}{T_{\parallel_0}}-1+\frac{B_0^2}{4\pi p_{\parallel_0}}\right), \tag{66}$$

where $\mathscr{A}_3(\zeta) \equiv (\mathscr{R}_1(\zeta_i)+\mathscr{R}_1(\zeta_e))/4 + \mathscr{R}_3(\zeta_i)\mathscr{R}_3(\zeta_e)/(\mathscr{R}_3(\zeta_i)+\mathscr{R}_3(\zeta_e))$. As expected, the $3+1$ results are identical to the kinetic results, except that the electrostatic response function $\mathscr{R}(\zeta_s)$ is replaced everywhere by either a three-pole or a one-pole model ($\mathscr{R}_3(\zeta_s)$ or $\mathscr{R}_1(\zeta_s)$). In the limit $|k_z|\gg|k_x|$, the $3+1$ model recovers the linear kinetic firehose dispersion relation. Taking the opposite limit $|k_x|\gg|k_z|$, leads to the mirror mode dispersion relation. Again the small frequency limit ($\zeta\to 0$) is taken to investigate the mirror mode. Unlike CGL, the $3+1$ model recovers the correct stability threshold for the mirror instability ($p_{\perp 0}^{\ 2}/p_{\parallel_0} > p_{\perp_0} + B_0^2/8\pi$). The mirror mode linear growth rate predicted by the $3+1$ model is compared to the other models in Fig. 5.

The $4+2$ model provides a yet more accurate model of the linear mirror mode growth rate. The calculation of the dispersion relation is completely analogous to that for the $3+1$ model, and all of the results are identical to those given in the previous paragraph, with the simple substitutions $\mathscr{R}_3(\zeta_s)\to\mathscr{R}_4(\zeta_s)$ and $\mathscr{R}_1(\zeta_s)\to\mathscr{R}_2(\zeta_s)$. Again the instability threshold for the mirror mode matches the kinetic result exactly, and the linear growth rates are compared in Fig. 5.

IX. DISCUSSION

A fluid description of plasma dynamics in the collisionless MHD regime, including models of kinetic effects such as phase mixing and Landau damping, has been developed. This ''Landau MHD'' model is based on Kulsrud's formulation of collisionless MHD,[3,8,9] and it is enhanced through the use of Landau closures analogous to those developed by Hammett and Perkins.[10] The model is a significant improvement over previous models, such as CGL theory,[1] because it includes accurate models of linear kinetic effects, while maintaining desirable nonlinear conservation properties and a fairly simple form in k-space. The model describes all waves which appear within the collisionless MHD ordering, including shear and compressional Alfvén waves, as well as ion acoustic waves. The effects of collisions have also been considered, through the use of a simple BGK collision operator. It has been shown that, in the high collisionality limit ($\omega\ll\nu\ll\Omega_c$), the model reproduces Braginskii's stress tensor and thermal conductivities approximately.

Both a $3+1$ moment Landau MHD model and a more accurate but more cumbersome $4+2$ moment model have been developed. Both have been derived for fairly general conditions, making no assumptions about adiabaticity or plasma beta, and including models of both ion and electron Landau damping. Collisional effects have been included in the moment equations through the use of a BGK collision operator, and a collisionally modified version of the $3+1$ closure has been derived. One species of $Z=1$ ions is assumed, but the generalization to multiple ion species is possible. The model can be easily reduced to account for further restrictions on adiabaticity, e.g., by replacing the full electron moment hierarchy with a simple adiabatic electron response when appropriate. Additional simplifications are easily made for isotropic pressures ($T_{\parallel_0} = T_{\perp_0}$), or electrostatic perturbations ($\mathbf{B_1} = 0$), etc. For nearly incompressible modes, a different ordering which eliminates the compressional Alfvén time scale is possible, as outlined by Medvedev and Diamond.[19]

Some of the limitations of our model are imposed by the use of a general collisionless MHD ordering together with a gyroaveraged kinetic equation. This ordering eliminates all finite Larmor radius (FLR) effects ($k_\perp\rho\to 0$), including the curvature and ∇B drifts. To bring FLR effects into the problem, it is necessary to introduce an additional ordering which removes the compressional Alfvén time scale.

Another complication is the evaluation of the $|k_\parallel|/k_\parallel$ terms found in the Landau closures. As pointed out by Finn and Gerwin,[18] the Landau damping must be evaluated along perturbed field lines. Hence, for nonlinear calculations, transforming the closure to real space requires an integral along the perturbed field line. The numerical evaluation of these nonlinear closures may be burdensome in some cases, as discussed in Sec. VII.

It is anticipated that the model will be useful for nonlinear numerical simulations. Some of the caveats involved in using Landau closures in nonlinear simulations have been extensively discussed in the gyrofluid literature,[11,12,23,24,26–30] but these caveats are an area of ongoing research. There are some regimes where certain nonlinear kinetic effects are not well modeled by Landau-fluid closures.[30] But we generally believe[12,24,27,28] these closures will be adequate for stronger turbulence regimes where rapid decorrelation is occurring and the velocity space details of the distribution function are not critically important.

It is hoped that the model will prove useful for simulating both laboratory and astrophysical plasmas in the collisionless MHD regime. The model should be able to predict the onset and structure of instabilities, as well as the heat and particle transport caused by the instabilities.

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