Adaptive Mesh Refinement MHD

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Outline

- MHD equations and numerical method
  - *Unsplit upwinding method*
- div(B) issues
  - *Projection method*
- Semi-implicit MHD code – Progress.
- Results
  - *Plane wave propagation*
  - *Rotor problem*
  - *Magnetic reconnection*
- Conclusion and future work

Electromagnetic Coupling (courtesy T. Gombosi, Univ. of Michigan)

- Weakly coupled formulation
  - *Hydrodynamic quantities in conservative form, electrodynamic terms in source term*
  - *Hydrodynamic conservation & jump conditions*
  - *One characteristic wave speed (ion-acoustic)*

$$\frac{\partial}{\partial t}\begin{pmatrix} r \\ r\mathbf{u} \\ \frac{1}{2} r u^2 + \frac{1}{g-1}p \end{pmatrix} + \left\{\nabla\cdot\begin{pmatrix} r\mathbf{u} \\ r\mathbf{uu}+\mathbf{I}p \\ \left[\frac{1}{2} r u^2+\frac{g}{g-1}p\right]\mathbf{u} \end{pmatrix}^T\right\}^T = \begin{pmatrix} 0 \\ \boxed{\frac{1}{m_0}\mathbf{j}\times\mathbf{B}} \\ 0 \end{pmatrix}$$

$$\boxed{\frac{\partial \mathbf{B}}{\partial \mathbf{t}} = \nabla\times(\mathbf{u}\times\mathbf{B})}$$

$$\mathbf{j} = \frac{1}{m_0}\nabla\times\mathbf{B}$$

- Tightly coupled formulation
  - *Fully conservative form*
  - *MHD conservation and jump conditions*
  - *Three characteristic wave speeds (slow, Alfvén, fast)*
  - *One degenerate eigenvalue/eigenvector*

$$\frac{\partial}{\partial t}\begin{pmatrix} r \\ r\mathbf{u} \\ \frac{1}{2} r u^2 + \frac{1}{g-1}p \boxed{+\frac{1}{2m_0}B^2} \\ \mathbf{B} \end{pmatrix} + \left\{\nabla\cdot\begin{pmatrix} r\mathbf{u} \\ r\mathbf{uu}+\left(p+\boxed{\frac{1}{2m_0}B^2\right)\mathbf{I}-\frac{1}{m_0}\mathbf{BB}} \\ \left[\frac{1}{2} r u^2+\frac{g}{g-1}p\boxed{+\frac{1}{2m_0}B^2\right]\mathbf{u}-\frac{1}{m_0}(\mathbf{u}\cdot\mathbf{B})\mathbf{B}} \\ \mathbf{uB}-\mathbf{Bu} \end{pmatrix}^T\right\}^T = \begin{pmatrix} 0 \\ \mathbf{0} \\ 0 \\ \mathbf{0} \end{pmatrix}$$

Single-fluid resistive MHD Equations

- Equations in conservation form

$$\frac{\partial U}{\partial t} + \boxed{\frac{\partial F_j(U)}{\partial x_j}} = \boxed{\frac{\partial \tilde{F}_j(U)}{\partial x_j}}$$

Parabolic

Hyperbolic

$$U = \{\rho, \rho u_i, B_i, e\}^T$$

$$F_j(U) = \left\{ \begin{array}{c} \rho u_j \\ \rho u_i u_j + p\delta_{ij} + \frac{1}{2} B_k B_k \delta_{ij} - B_i B_j \\ u_j B_i - B_j u_i \\ \left(e + p + \frac{1}{2} B_k B_k\right) u_j - B_i u_i B_j \end{array} \right\}$$

$$\tilde{F}_j(U) = \left\{ \begin{array}{c} 0 \\ Re^{-1} \tau_{ij} \\ S^{-1} \eta \left(\frac{\partial B_i}{\partial x_j} + \frac{\partial B_j}{\partial x_i} \right) \\ S^{-1} \eta \left(\frac{1}{2} \frac{\partial B_i B_i}{\partial x_j} - B_i \frac{\partial B_j}{\partial x_i} \right) + Re^{-1} \tau_{ij} u_i + Pe^{-1} \kappa \frac{\partial T}{\partial x_j} \end{array} \right\}$$

Reynolds no.

Lundquist no.

Peclet no.

$$\tau_{ij} = \rho \nu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3} \delta_{ij} \frac{\partial u_k}{\partial x_k} \right)$$

$$e = \frac{p}{\gamma - 1} + \frac{1}{2} \rho u_i u_i + \frac{1}{2} B_i B_i$$

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Numerical Method

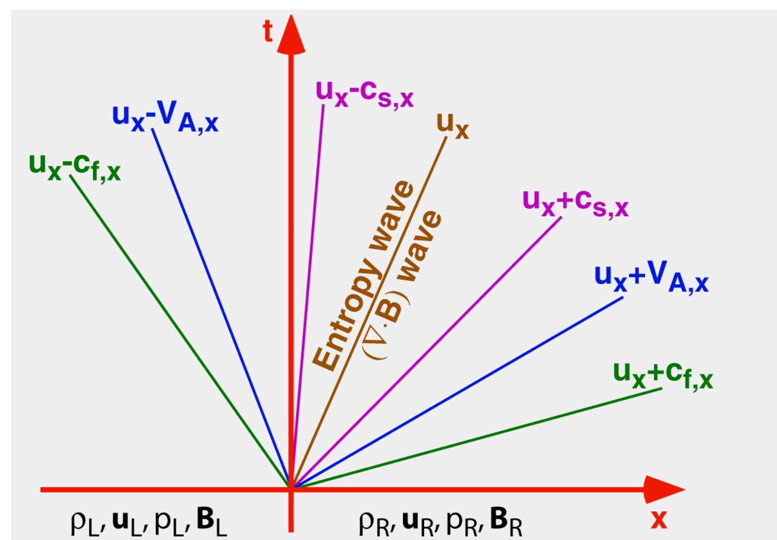
- MHD Equations written in symmetrizable near-conservative form (Godunov, Numerical Methods for Mechanics of Continuum Media, 1, 1972, Powell et al., J. Comput. Phys., vol 154, 1999).
  - *Deviation from total conservative form is of the order of $\nabla\cdot$**B** truncation errors*

$$\frac{\partial}{\partial t}\begin{pmatrix} r \\ r\mathbf{u} \\ \frac{1}{2} r u^2+\frac{1}{g-1}p+\frac{1}{2m_0}B^2 \\ \mathbf{B}\end{pmatrix} + \left\{\nabla\cdot\begin{pmatrix} r\mathbf{u}\\ r\mathbf{uu}+\left(p+\frac{1}{2m_0}B^2\right)\mathbf{I}-\frac{1}{m_0}\mathbf{BB} \\ \left[\frac{1}{2} ru^2+\frac{g}{g-1}p+\frac{1}{2m_0}B^2\right]\mathbf{u}-\frac{1}{m_0}(\mathbf{u}\cdot\mathbf{B})\mathbf{B} \\ \mathbf{uB}-\mathbf{Bu}\end{pmatrix}^T\right\}^T = -(\nabla\cdot\mathbf{B})\begin{pmatrix}0\\ \frac{1}{m_0}\mathbf{B}\\ \frac{1}{m_0}(\mathbf{u}\cdot\mathbf{B})\\ \mathbf{u}\end{pmatrix}$$

- The symmetrizable MHD equations lead to the 8-wave method.
  - *The fluid velocity advects both the entropy and div(**B**)*
- Finite volume approach. Hyperbolic fluxes determined using the unsplit upwinding method (Colella, J. Comput. Phys., Vol 87, 1990)
  - *Predictor-corrector.*
  - *Fluxes obtained by solving Riemann problem*
  - *Good phase error properties due to corner coupling terms*

$$U_i^{n+1} = U_i^n - \frac{\Delta t}{h}\sum_{d=0}^{\mathbf{D}-1}\left(F^{n+\frac{1}{2}}_{i+\frac{1}{2}e^d} - F^{n+\frac{1}{2}}_{i-\frac{1}{2}e^d}\right)$$

$$F^{n+\frac{1}{2}}_{i+\frac{1}{2}e^d} = R\left(W^{n+\frac{1}{2}}_{i,+,d}, W^{n+\frac{1}{2}}_{i+e^d,-,d}, d\right)$$



The **r¢ B=0** Problem

- Conservation of Ñ×**B** =0:
  - *Analytically: if Ñ×**B** =0 at t=0 than it remains zero at all times*
  - *Numerically: In upwinding schemes the curl and div operators do not commute*
- Purposes to control Ñ×**B** numerically:
  - *To improve accuracy*
  - *To improve robustness*
  - *To avoid unphysical effects (Parallel Lorentz force)*
- 8-wave formulation: **r¢** B = O(h^{α}) (Powell et al, JCP 1999; Brackbill and Barnes, JCP 1980)
- Constrained Transport (Balsara & Spicer JCP 1999, Dai & Woodward JCP 1998, Evans & Hawley Astro. J. 1988)
- Constrained Transport/Central Difference (Toth JCP 2000)
- Projection Method
- Vector Potential (Claim: CT/CD schemes can be cast as an “underlying” vector potential. Evans and Hawley, Astro. J. 1988)
- Require ad-hoc corrections to total energy
- May lead to numerical instability (e.g. negative pressure – ad-hoc fix based on switching between total energy and entropy formulation by Balsara)

r¢ B=0 by Projection

- Compute the estimates to the fluxes $F^{n+1/2}_{i+1/2,j}$ using the unsplit formulation
- Use face-centered values of B to compute **r¢** B.
  Solve the Poisson equation **r**$^2\phi$ = **r ¢** B
- Correct B at faces: B=B-**r**ƒ
- Correct the fluxes $F^{n+1/2}_{i+1/2,j}$ with projected values of B
- Update conservative variables using the fluxes
  - *The non-conservative source term S(U) ɑ **r¢** B has been algebraically removed*
- On uniform Cartesian grids, projection provides the smallest correction to remove the divergence of B. (Toth, JCP 2000)
- Does the nature of the equations change?
  - *Hyperbolicity implies finite signal speed*
  - *Do corrections to B via **r**^{2}ƒ=**r¢** B violate hyperbolicity?*
- Conservation implies that single isolated monopoles cannot occur. Numerical evidence suggests these occur in pairs which are spatially close.
  - *Corrections to B behave as ɑ $1/r^2$ in 2D and $1/r^3$ in 3D*
- Projection does not alter the order of accuracy of the upwinding scheme and is consistent

Adaptive Mesh Refinement with Chombo

- ***Chombo*** is a collection of C++ libraries for implementing block-structured adaptive mesh refinement (AMR) finite difference calculations (http://www.seesar.lbl.gov/ANAG/chombo)
  - *(Chombo is an AMR developer's toolkit)*
- Mixed language model
  - *C++ for higher-level data structures*
  - *FORTRAN for regular single grid calculations*
  - *C++ abstractions map to high-level mathematical description of AMR algorithm components*
- Reusable components. Component design based on mathematical abstractions to classes
- Based on public-domain standards
  - *MPI, HDF5*
- Chombovis: visualization package based on VTK, HDF5
- Layered hierarchical properly nested meshes
- Adaptivity in both space and time

Unsplit + Projection AMR Implementation

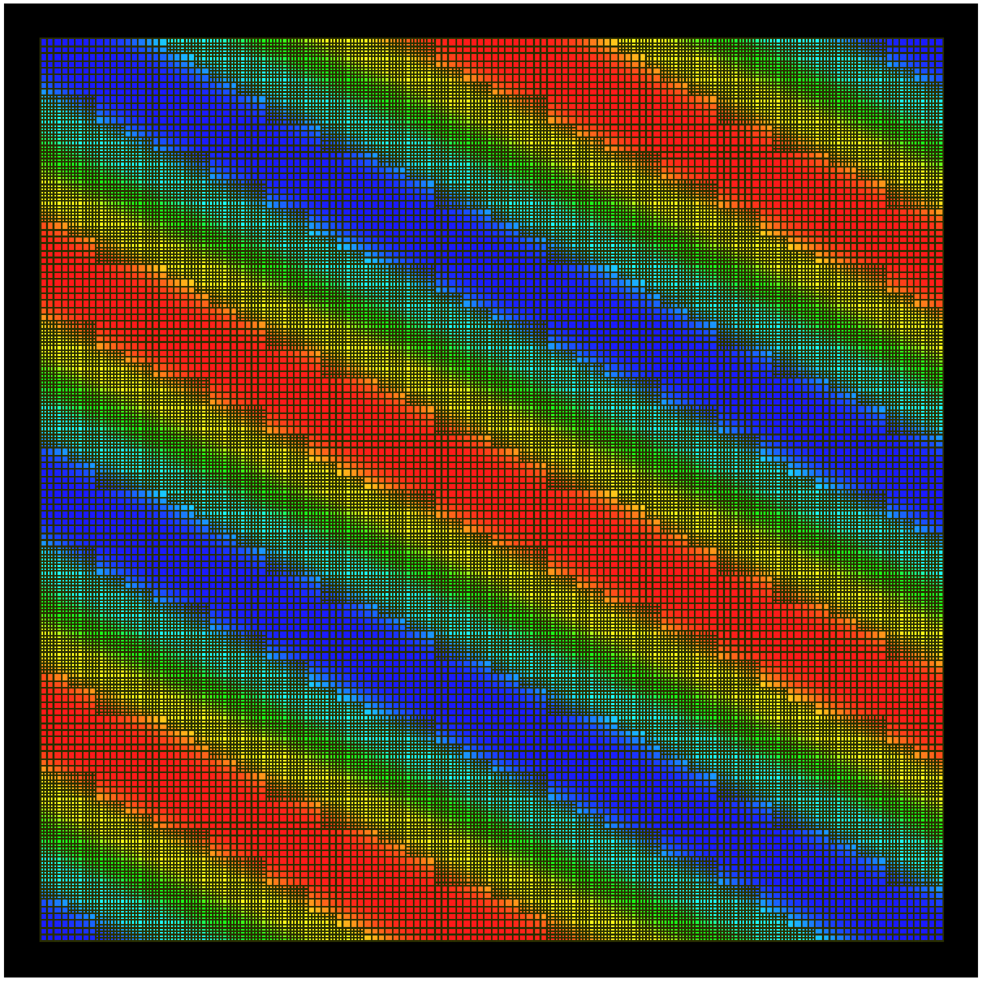
- Implemented the Unsplit method using CHOMBO
- Solenoidal B is achieved via projection, solving the elliptic equation $\nabla^2\phi=\nabla\cdot B$
  - *Solved using Multgrid on each level (union of rectangular meshes)*
  - *Coarser level provides Dirichlet boundary condition for ϕ*
    - *Requires **O(h^3)** interpolation of coarser mesh ϕ on boundary of fine level*
  - *a "bottom smoother" (conjugate gradient solver) is invoked when mesh cannot be coarsened*
  - *Physical boundary conditions are Neumann **$d\phi/dn=0$ (Reflecting) or Dirichlet***
- Multigrid convergence is sensitive to block size
- Flux corrections at coarse-fine boundaries to maintain conservation
  - *A consequence of this step: **$\nabla\cdot$** $B=0$ is violated on coarse meshes in cells adjacent to fine meshes.*
- Code is parallel
- Second order accurate in space and time

Treatment of parabolic flux terms

- Approach 1: Explicit
  - *Computed at time step 'n'*
  - *Magnetic reconnection results use this approach.*
- Approach 2: Implicit treatment
  - *Implicit Runge Kutta, TGA Approach (Twizell, Gumel, Arigu, Advances in Comp. Math. 6(3):333-352, 1996)*
  - *Implemented for resistive terms in magnetic field equations*
    - *Work for constant h*
  - *Viscous and conductivity terms require non-constant coefficient Helmholtz solvers (Work in progress)*
- Quadratic interpolation ($O(h^3)$) at coarse-fine boundaries
  - *Corner terms required and obtained by linear interpolation*
- Flux-refluxing step requires implicit solution on all levels synchronized at the current time step.
  - *Backward Euler used for this step*

Code verification – Plane wave propagation

- A plane wave is initialized oblique to the mesh
  Initial conditions for l-th characteristic wave
  W(x) = $\mathbf{W}_0$**(x)** + ε exp(i **k¢ x**)
  r_l
- Plane wave chosen to correspond to Alfven velocity or fast magnetosonic sound speed
- Low β (=0.01)
- Poisson solve converged in 8 iterations to a max residual of 10^{-14}
- 3D Wave example



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Code verification – Weak rotor problem

- Ideal MH
  problem
  uniform
  Comput. P
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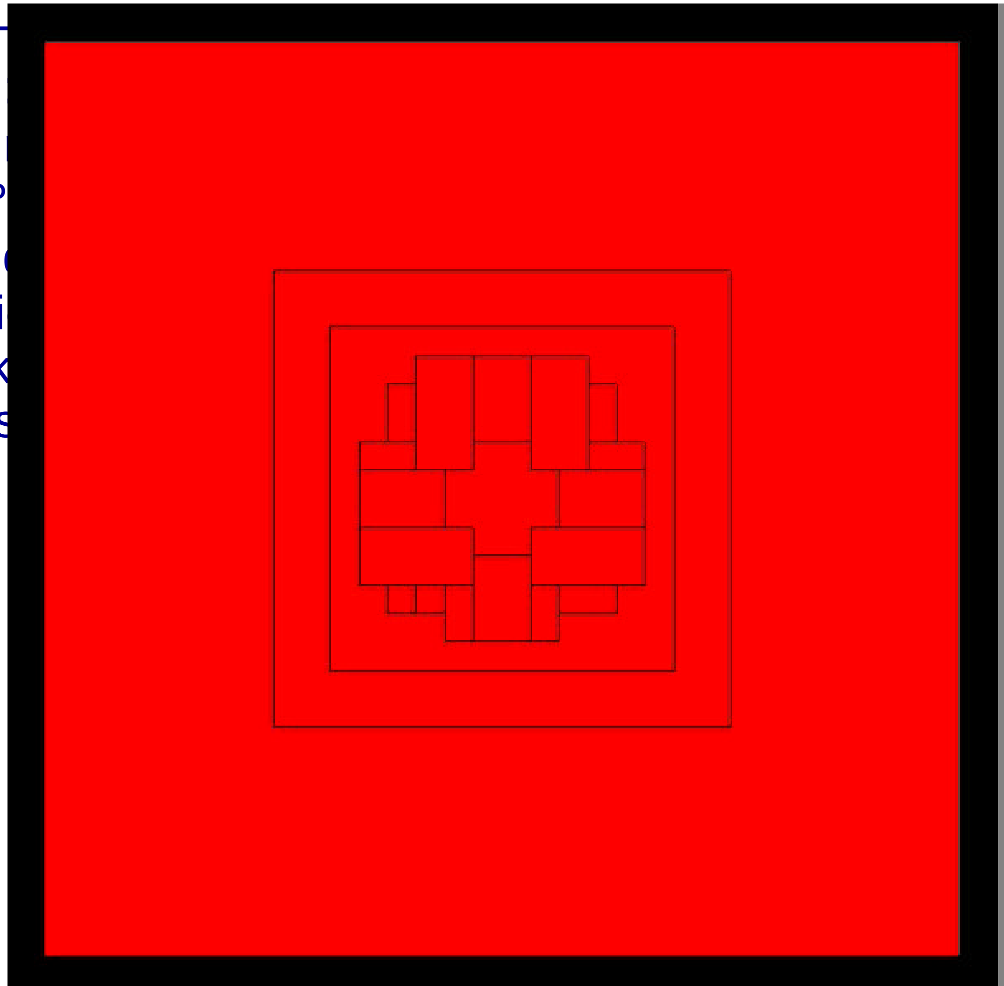
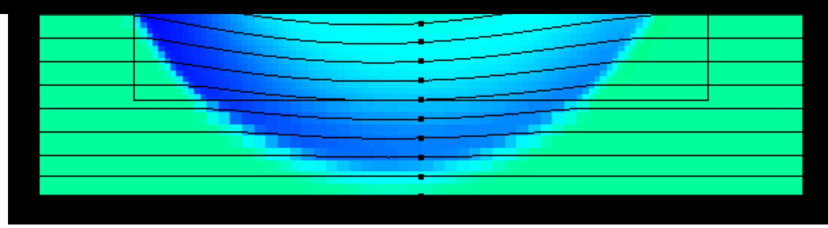
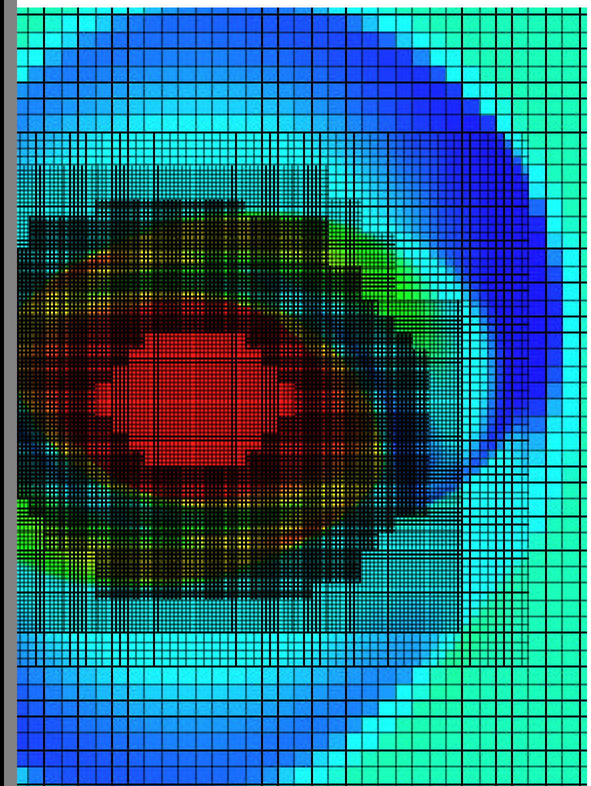
  progress

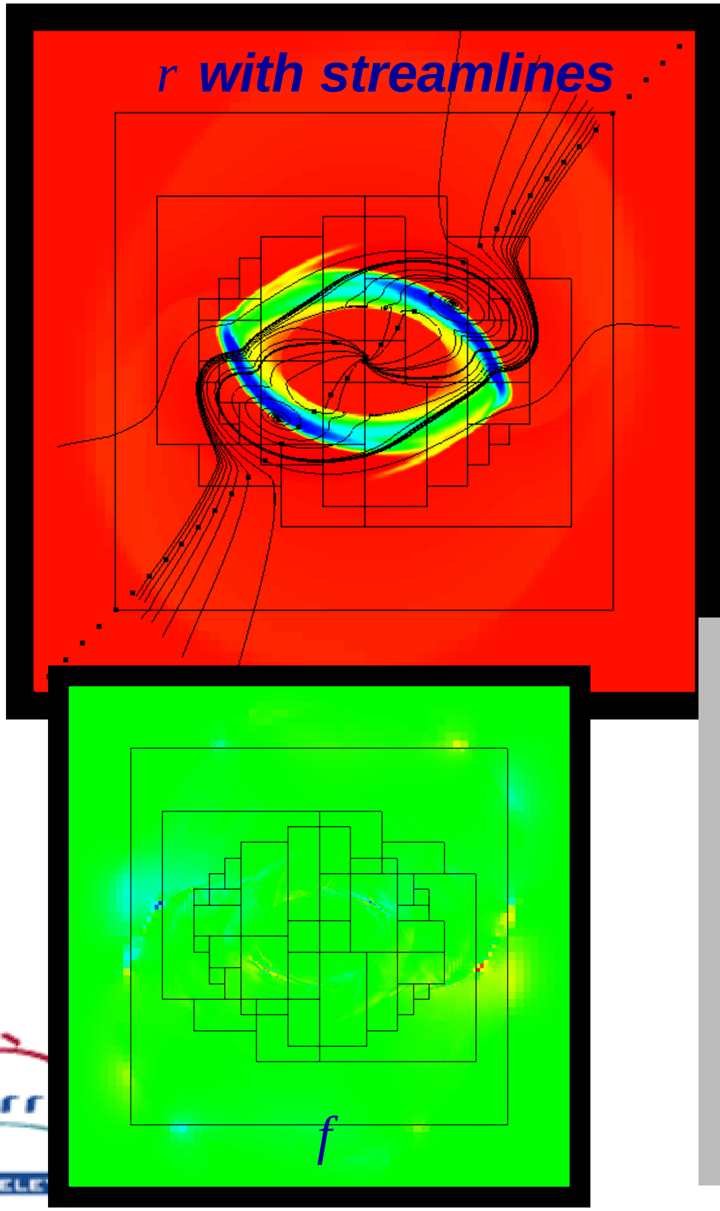


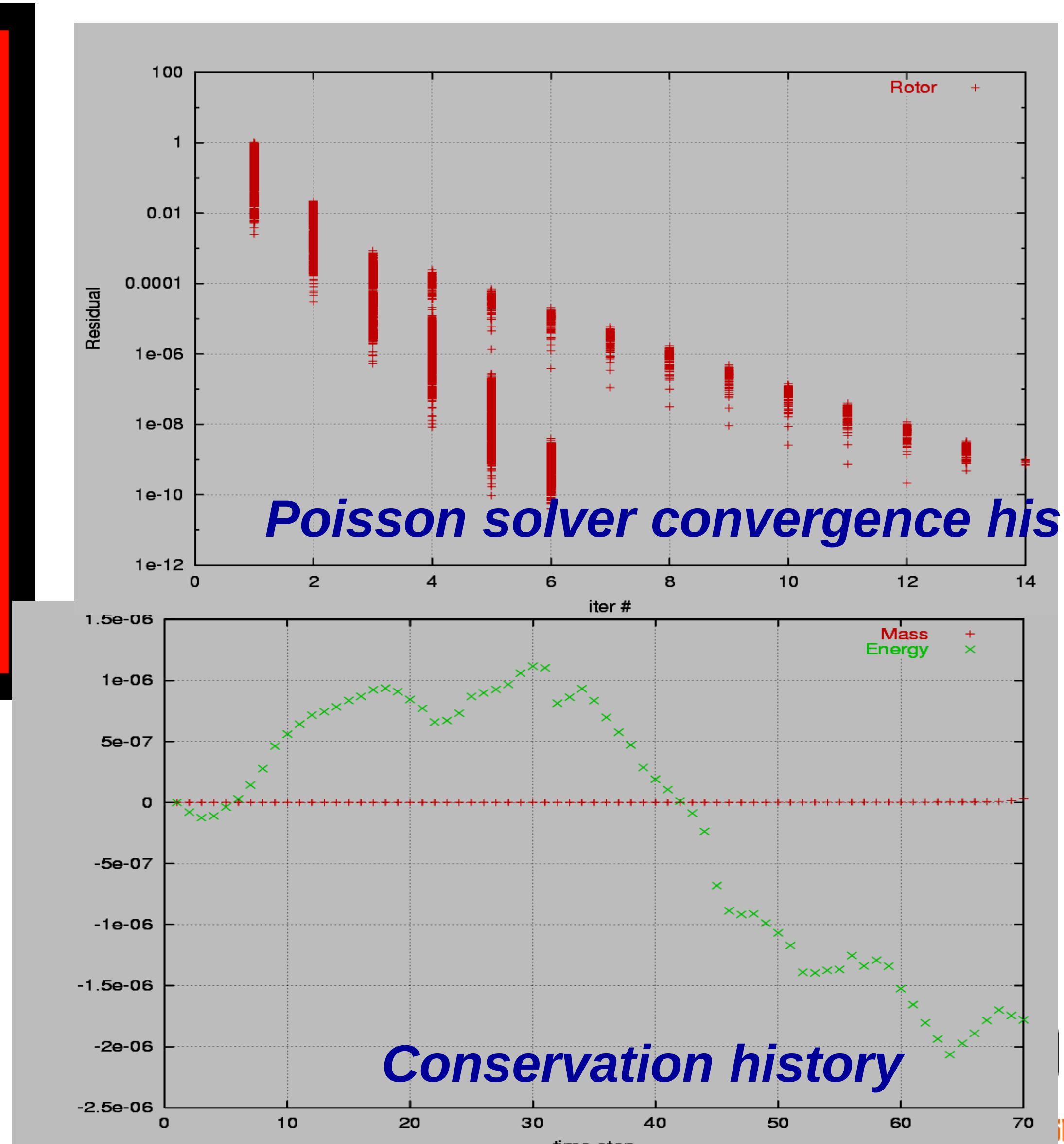


h B streamlines

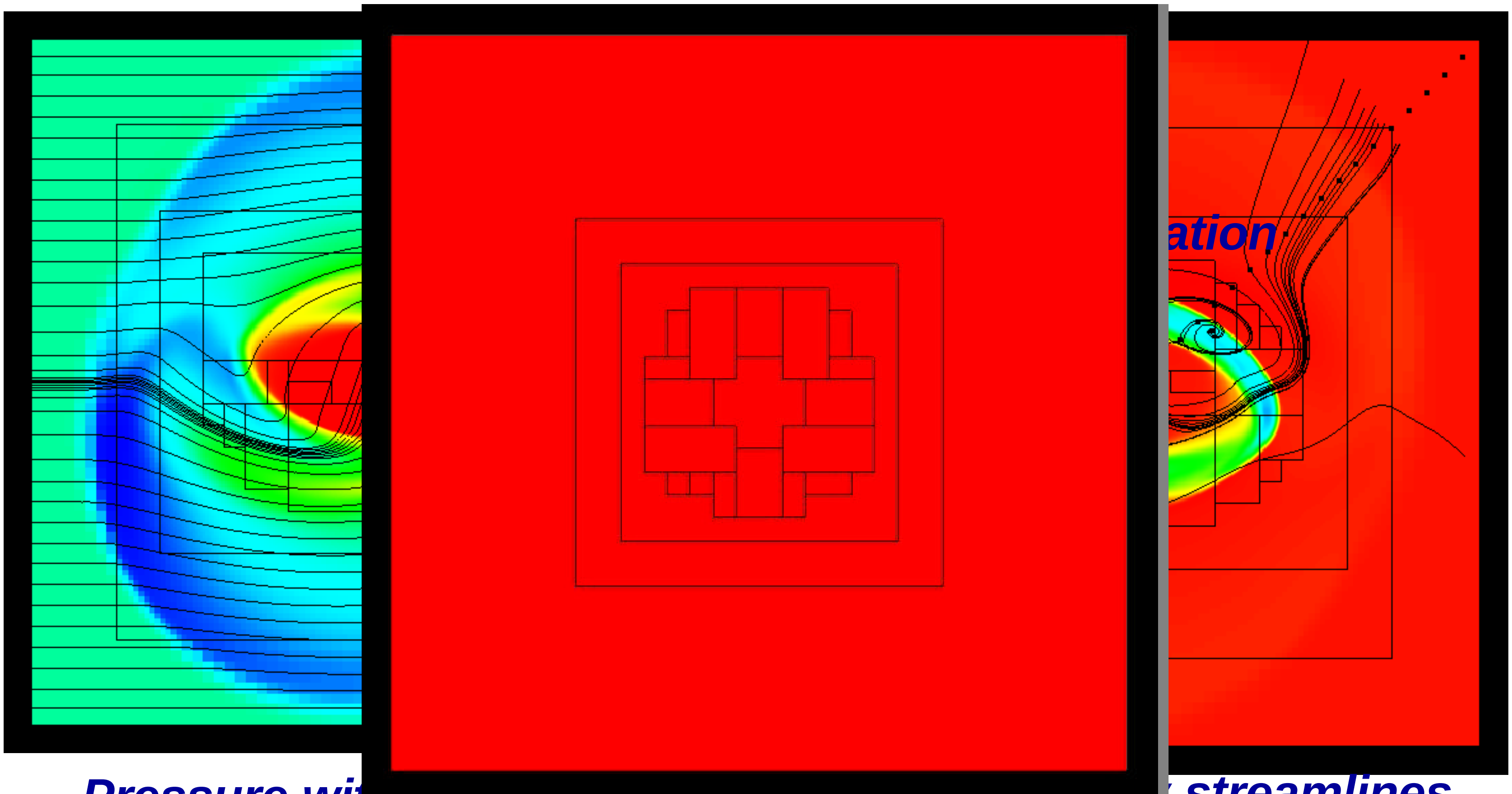


Code verification – Weak rotor problem





Weak rotor – Resistive MHD (Implicit η)



Pressure with B field lines

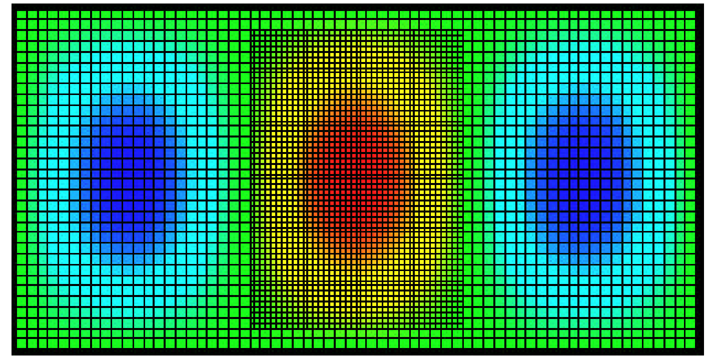
with velocity streamlines

Magnetic Reconnection: IC and BCs

- Initial conditions on domain [-1:1]x[0:1]

$$
\begin{aligned}
\rho(x,y,0) &= 1 \\
u_i(x,y,0) &= 0 \\
p(x,y,0) &= 0.2 \\
\psi(x,y,0) &= -\cos k_x x \sin k_y y \\
B_z(x,y,0) &= -(k_x^2 + k_y^2)^{\frac{1}{2}} \cos k_x x \sin k_y y
\end{aligned}
$$

$$k_x = \frac{3\pi}{2}, \quad k_y = 2\pi$$

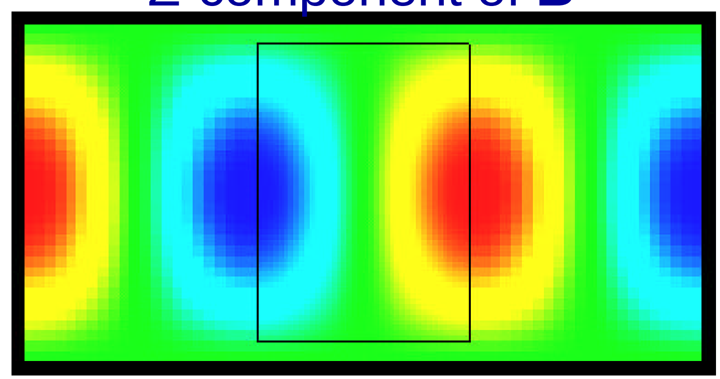


Z-component of **B**

- Boundary conditions
  No mass flux, (open L/R boundaries)
  T/B Perfectly conducting walls

$$\vec{B} \cdot \hat{n} = 0 \qquad \vec{E} \cdot \hat{t} = 0$$

  Dirichlet Temperature BC



Y-component of **B**

- Other parameters: Re=10^3, Pe= 10^3
  Dimensionless conductivity and viscosity set to unity
- Resitivity variation to annihilate middle island

$$\eta = \eta^- + (\eta^+ - \eta^-)\left[1 - \exp(-177.69\psi^2)\right] \times max(0, \ -sign(\psi))$$

$$\eta^- = 1$$

$$\eta^+ = 0.1/S$$

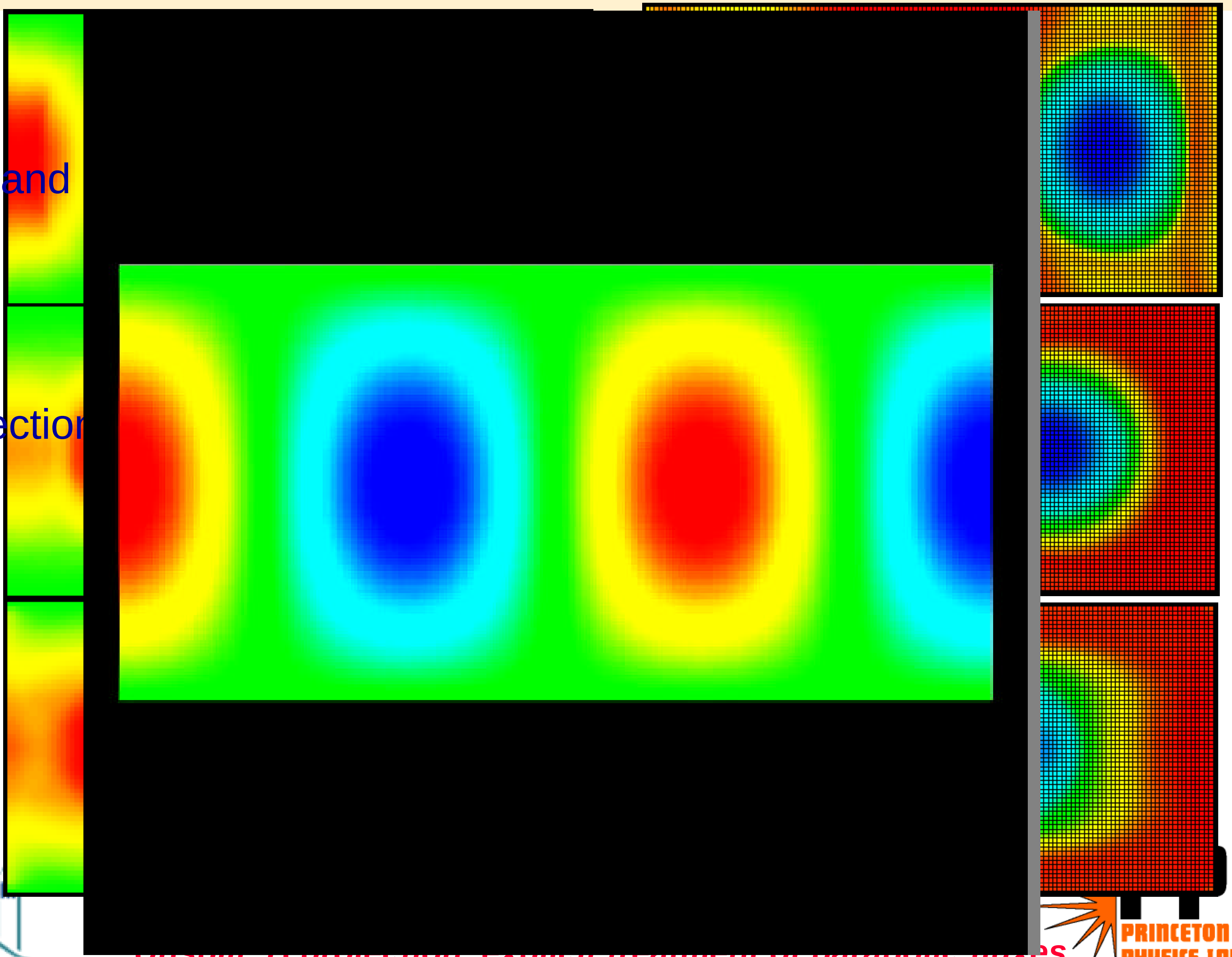
J. Breslau, PhD thesis, Princeton University

Reconnection S= 10^3

Stage 1
Middle island
decays

Stage 2
Reconnection

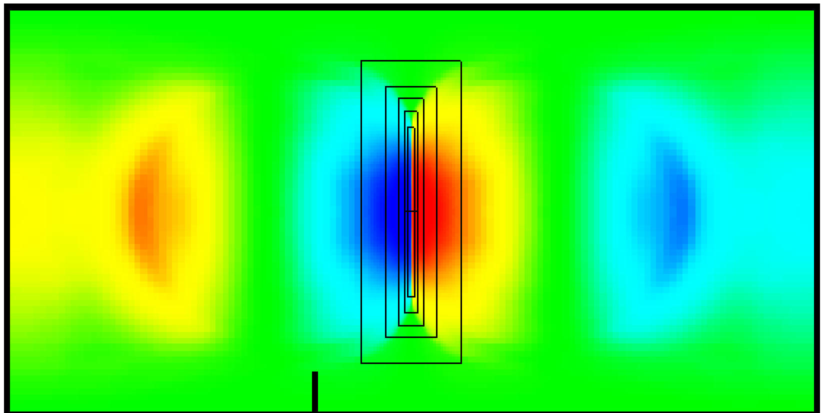
Stage 3
Decay

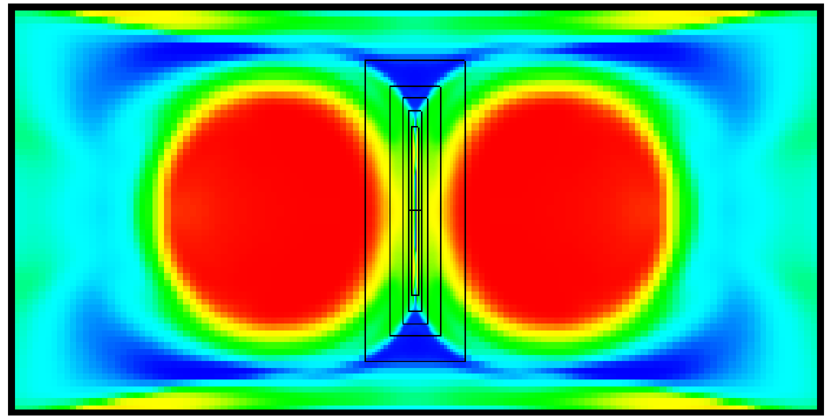


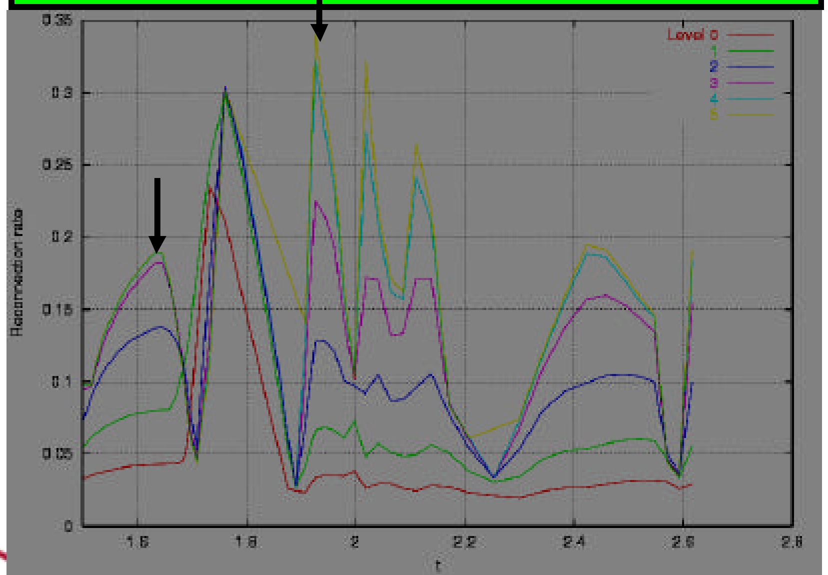
Unsplit, B projection, explicit treatment of parabolic fluxes

Reconnection S= 10^4

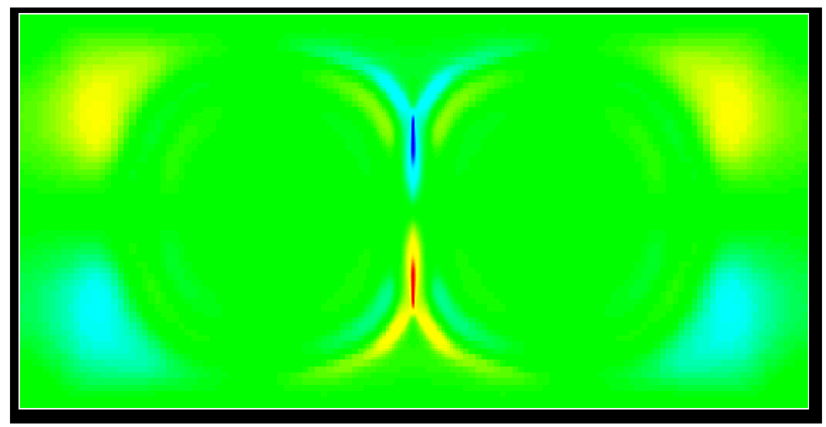
6 Level AMR run. Effective unigrid: 4096x2048.







Thin O($\eta^{1/2}$) high pressure region

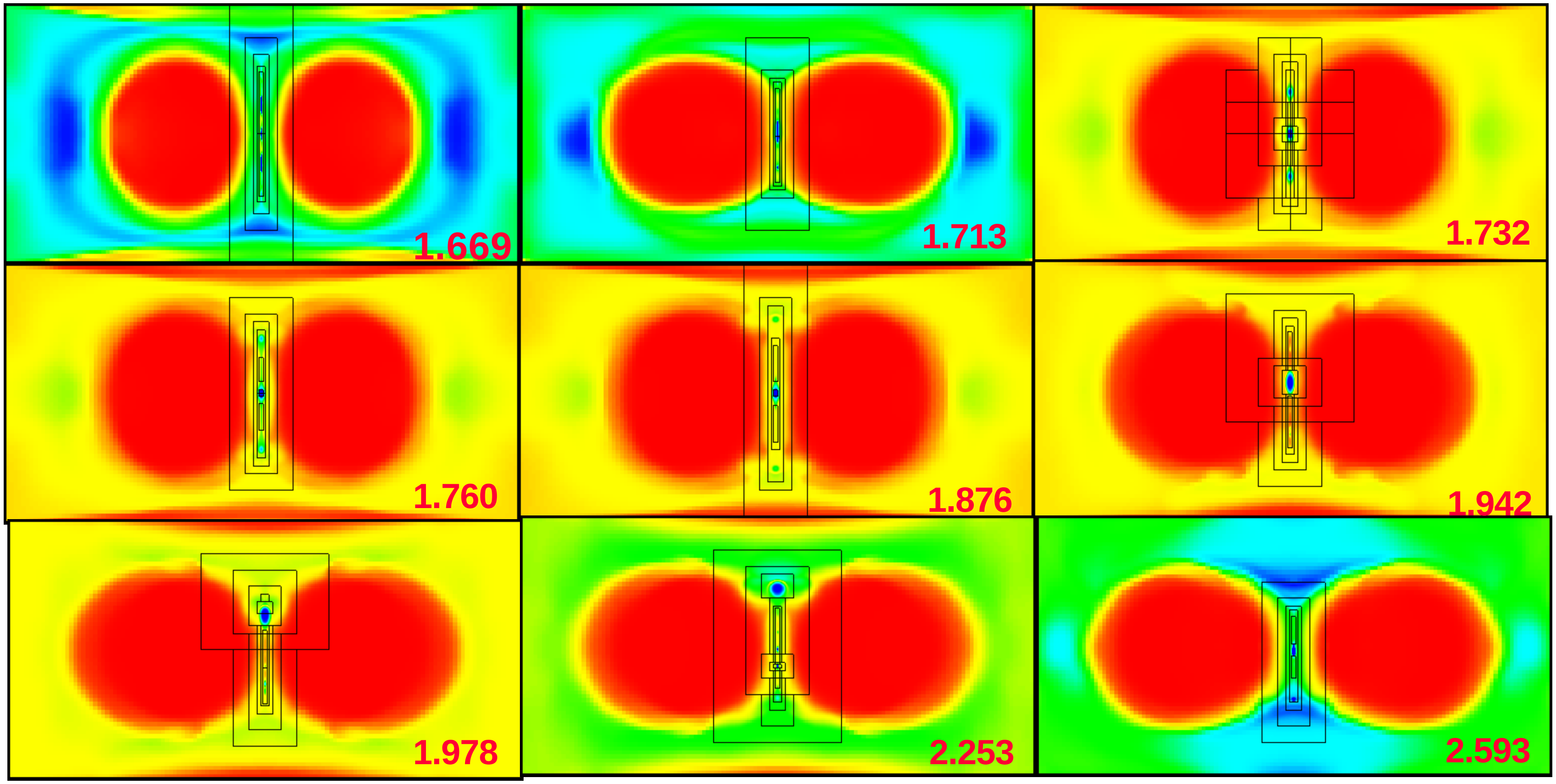


Super-Alfvenic jets v/a > 4

Reconnection rates show computation is well-resolved

Pressure: Reconnection S= 10^4

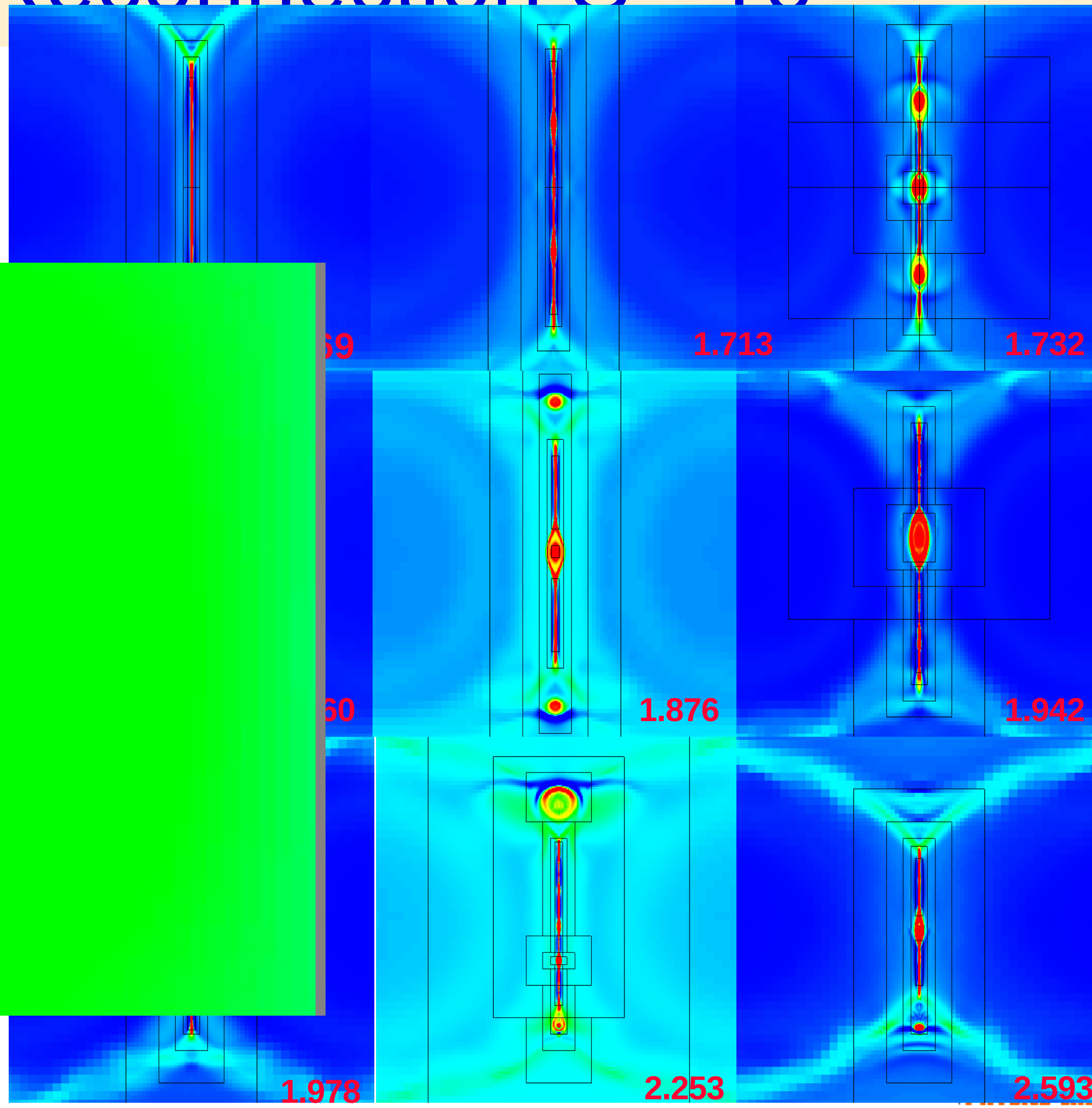
High pressure region shows "patches".



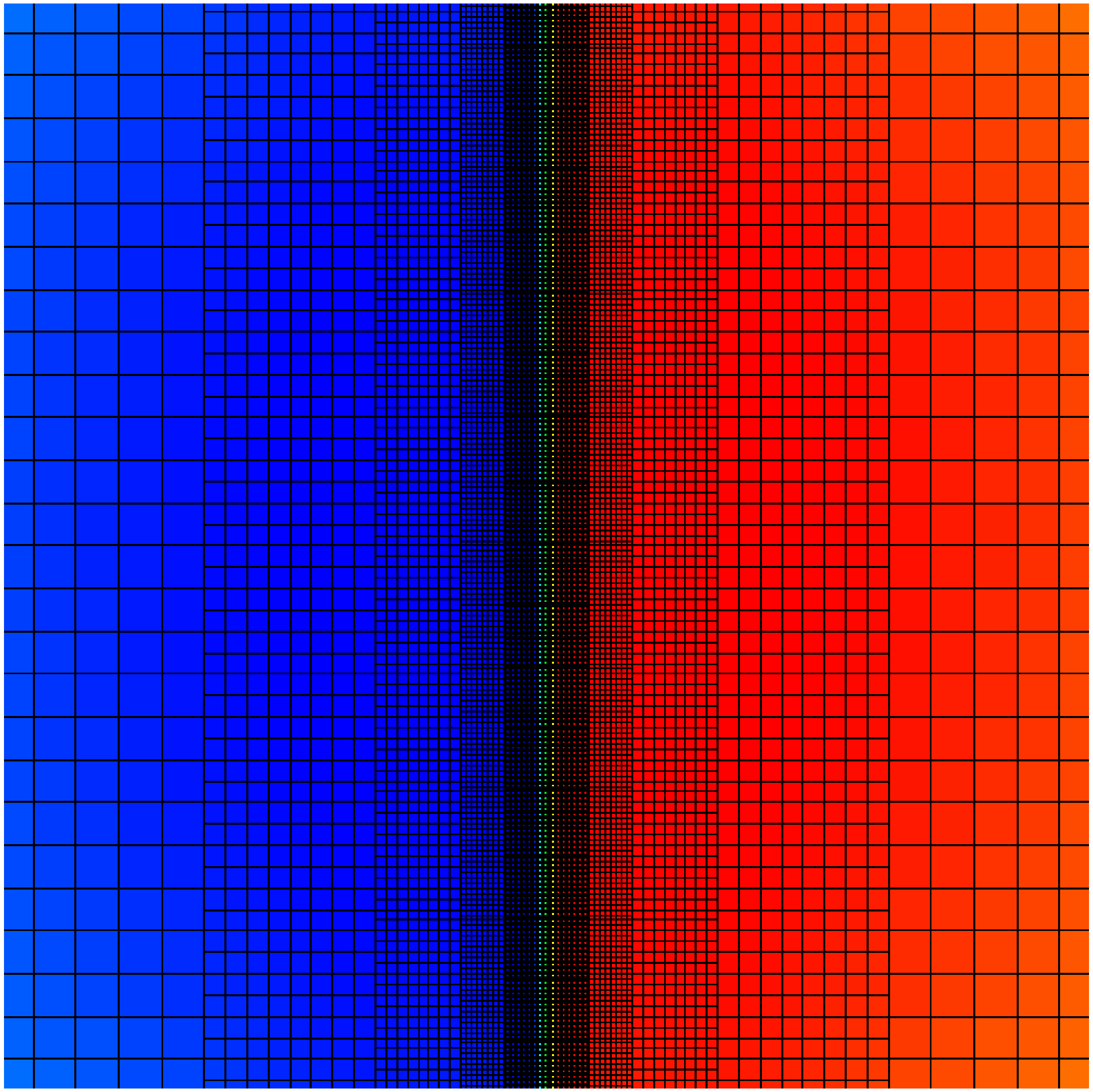
Current: Reconnection S= 10^4

Time sequence of current (J_z)

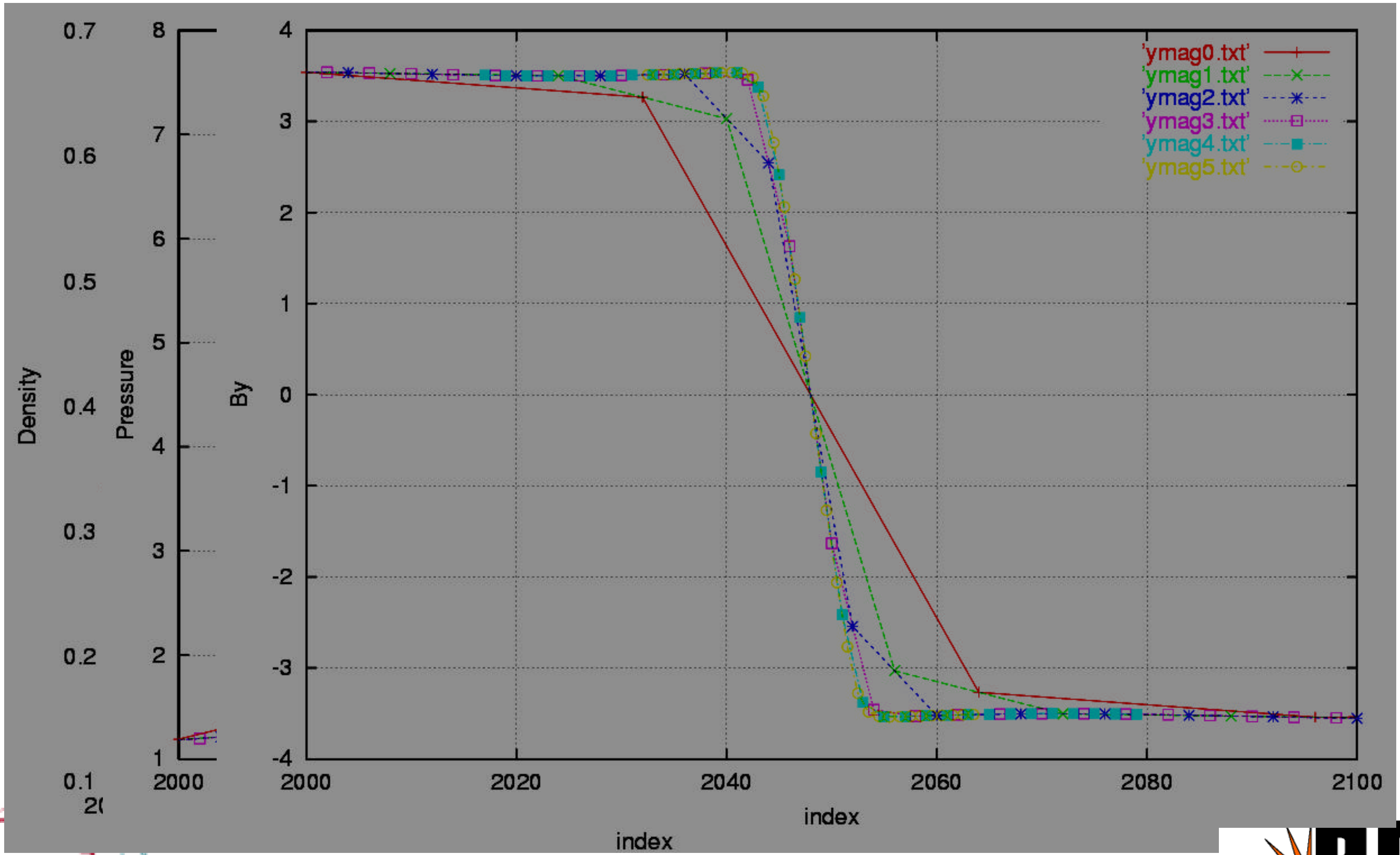
Thin current layer



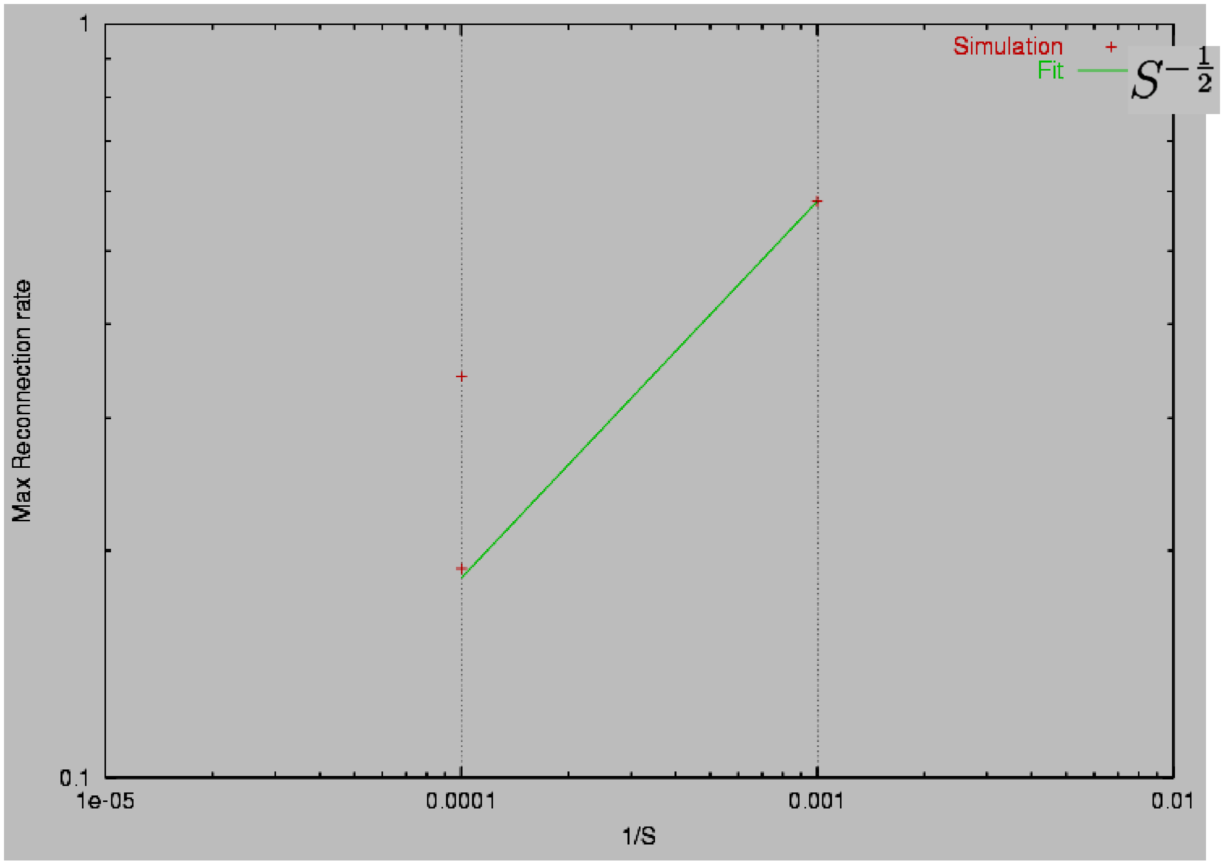
Reconnection layer details S= 10^4



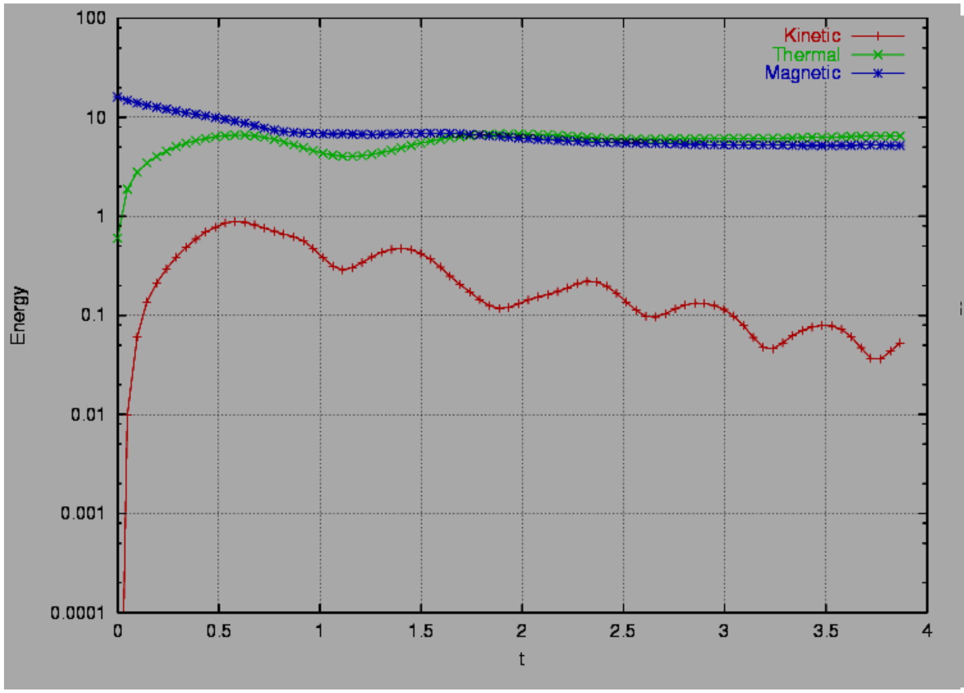
Reconnection S= 10^4 – Profiles in Reconnection Layer

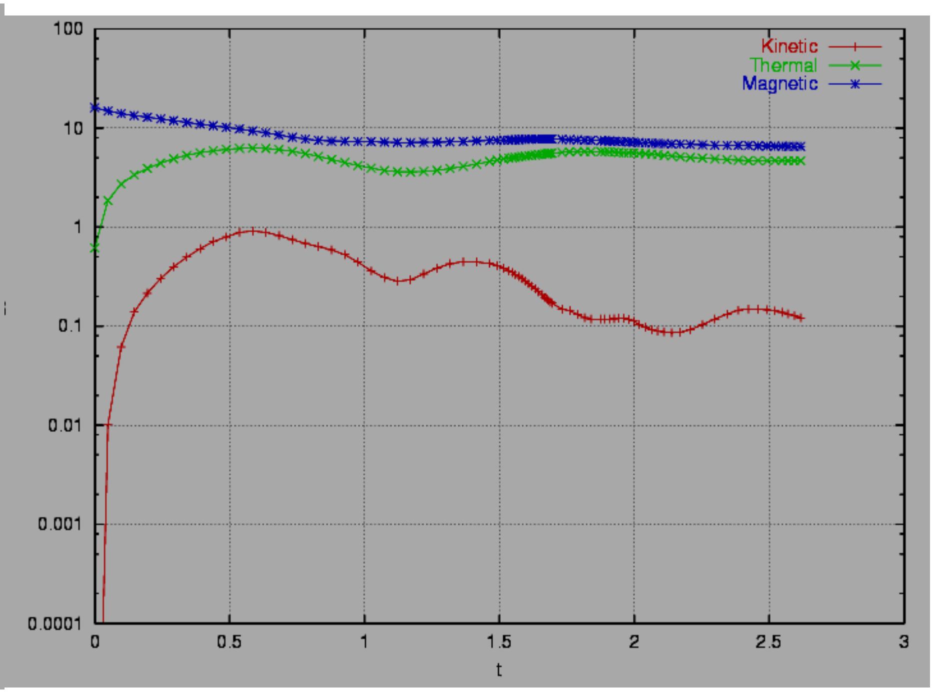


Results:Max $\dot{\psi}$ scaling with S



Reconnection Energy Budget





S=10³

S=10⁴

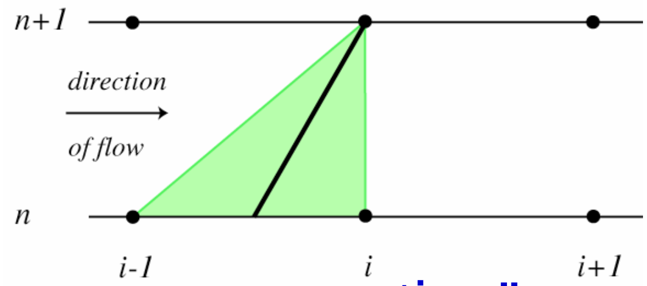
Observations and Conclusion

- Observations
  - *Thin current layer well resolved*
  - *For S=10^4*
    - *“patchy” reconnection*
    - *Current layer is unstable*
    - *Asymmetric evolution after current layer becomes unstable induced by perturbations at mesh level.*
  - *Reconnection not a smooth process – bouncing*
- A conservative solenoidal B AMR MHD code was developed
  - *Unsplit upwinding method for hyperbolic fluxes*
  - **r¢** *B=0 achieved via projection*
- This preliminary study indicates that AMR is a viable approach to *efficiently* resolve the near-singular current sheet in high Lundquist magnetic reconnection

Future Work

- High resolution parallel 2D magnetic reconnection runs.
- Implicit treatment of viscous/conductivity terms
- Two-fluid MHD with Hall effect
- Tokamak geometry
- Implicit treatment of fast wave
- 3D magnetic reconnection
- Pellet injection AMR simulations (of importance to ITER and other fusion reactors)

Numerical Method: Upwind Differencing



- The "one-way wave equation" propagating to the right:

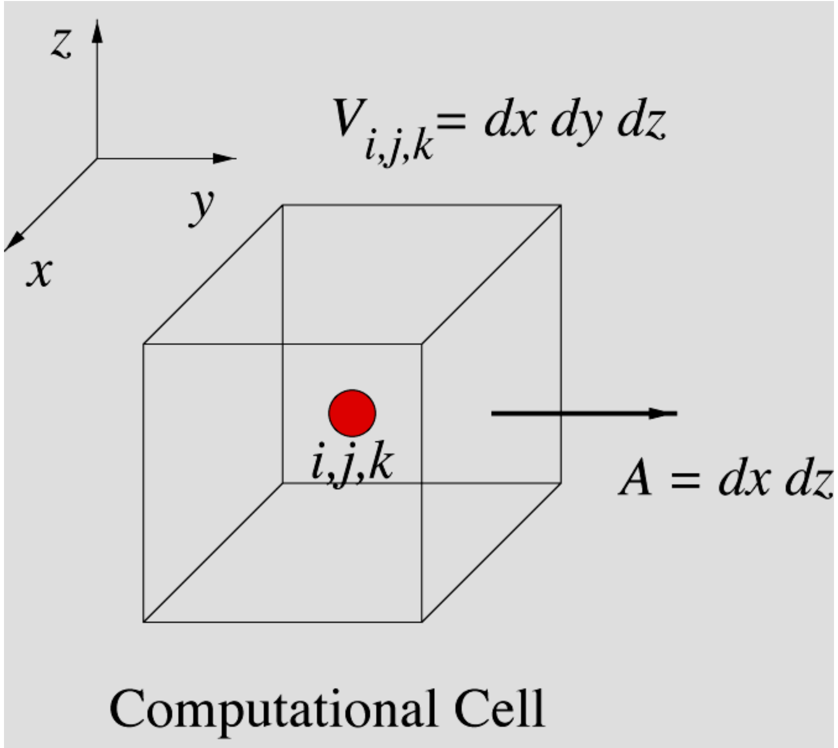
$$\frac{\partial y}{\partial t} + a\frac{\partial y}{\partial x} = 0 \qquad a > 0$$

- When the wave equation is discretized "upwind" (i.e. using data at the old time level that comes from the left the wave equations becomes:

$$y_i^{n+1} = y_i^n + \frac{a\Delta t}{\Delta x}\left(y_{i-1}^n - y_i^n\right) = (1-n)y_i^n + ny_{i-1}^n \qquad n \le 1$$

- Advantages:
  - *Physical: The numerical scheme "knows" where the information is coming from*
  - *Robustness: The new value is a linear interpolation between two old values and therefore no new extrema are introduced*

Numerical Method: Finite Volume Approach



- Conservative (divergence) form of conservation laws:

$$\frac{dU}{dt} + \nabla \cdot F = S$$

- Volume integral for computational cell:

$$\frac{dU_{i,j,k}}{dt} = -\sum_{faces} A \cdot F + S_{i,j,k}$$

- Fluxes of mass, momentum, energy and magnetic field entering from one cell to another through cell interfaces are the essence of finite volume schemes. This is a Riemann problem.

Symmetrizable MHD Equations

- The symmetrizable MHD equations can be written in a near-conservative form (Godunov, Numerical Methods for Mechanics of Continuum Media, 1, 1972, Powell et al., J. Comput. Phys., vol 154, 1999):

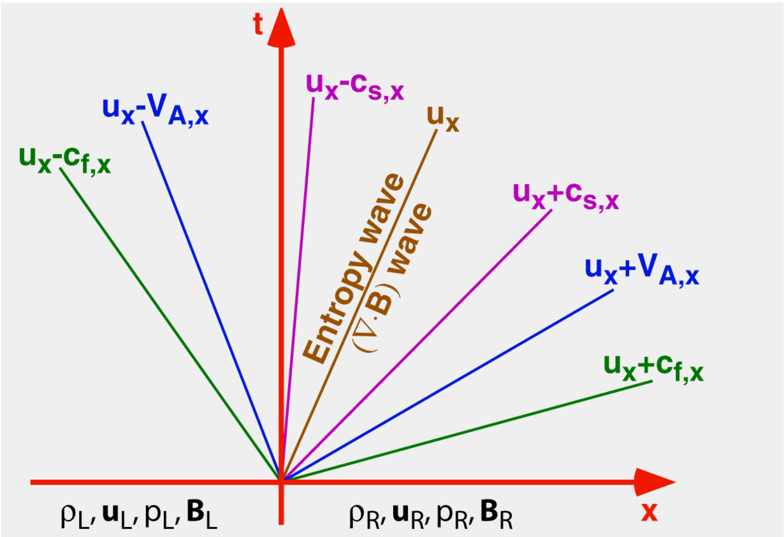
$$\frac{\partial}{\partial t}\begin{pmatrix} r \\ r\mathbf{u} \\ \frac{1}{2} r u^2 + \frac{1}{g-1}p + \frac{1}{2m_0}B^2 \\ \mathbf{B}\end{pmatrix} + \left\{\nabla\cdot\begin{pmatrix} r\mathbf{u} \\ r\mathbf{uu} + \left(p+\frac{1}{2m_0}B^2\right)\mathbf{I} - \frac{1}{m_0}\mathbf{BB} \\ \left[\frac{1}{2} r u^2 + \frac{g}{g-1}p + \frac{1}{2m_0}B^2\right]\mathbf{u} - \frac{1}{m_0}(\mathbf{u}\cdot\mathbf{B})\mathbf{B} \\ \mathbf{uB}-\mathbf{Bu}\end{pmatrix}^T\right\}^T = -(\nabla\cdot\mathbf{B})\begin{pmatrix}0\\ \frac{1}{m_0}\mathbf{B} \\ \frac{1}{m_0}(\mathbf{u}\cdot\mathbf{B}) \\ \mathbf{u}\end{pmatrix}$$

- Deviation from total conservative form is of the order of Ñ×**B** truncation errors
- The symmetrizable MHD equations lead to the 8-wave method. The eigenvalues are

$$\lambda = \{u, u, u+c_a, u-c_a, u+c_f, u-c_f, u+c_s, u-c_s\}$$

  – *The fluid velocity advects both the entropy and div(**B**) in the 8-wave formulation*

Numerical method: Riemann Solver



- Discontinuous initial condition
  - *Interaction between two states*
  - *Transport of mass, momentum, energy and magnetic flux through the interface due to waves propagating in the two media*
- Riemann solver calculates interface fluxes from left and right states

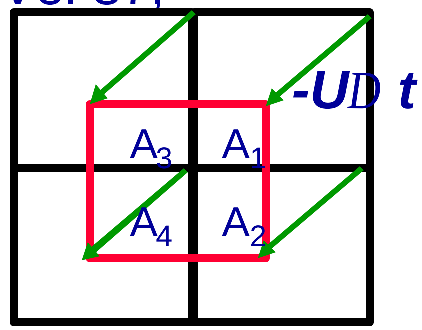
- The eigenvalues and eigenvectors of the Jacobian, d**F**/d**U** are at the heart of the Riemann solver:

$$\mathbf{F}_{LR} = \frac{1}{2}(\mathbf{F}_L + \mathbf{F}_R) - \frac{1}{2}\sum_{k=1}^{8} R_k |l_k| L_k (\mathbf{U}_R - \mathbf{U}_L)$$

  - *Each wave is treated in an upwind manner*
  - *The interface flux function is constructed from the individual upwind waves*
  - *For each wave the artificial dissipation (necessary for stability) is proportional to the corresponding wave speed*

Unsplit method – Basic concept

- Original idea by P. Colella (Colella, J. Comput. Phys., Vol 87, 1990)
- Consider a two dimensional scalar advection equation



$$\rho_t + u \cdot \nabla \rho = 0$$

- Tracing back characteristics at t+Δ t

$$\rho^{n+1}_{i,j} = A_1 \rho^n_{i,j} + A_2 \rho^n_{i,j-1} + A_3 \rho^n_{i-1,j} + A_4 \rho^n_{i-1,j-1}$$

- Expressed as predictor-corrector

$$\begin{aligned}
\rho^{n+1}_{i,j} &= \rho^n_{i,j} - \frac{u\Delta t}{\Delta x}(\rho^{n+1/2}_{i+1/2,j} - \rho^{n+1/2}_{i-1/2,j}) - \frac{v\Delta t}{\Delta y}(\rho^{n+1/2}_{i,j+1/2} - \rho^{n+1/2}_{i,j-1/2}) \\
\rho^{n+1/2}_{i+1/2,j} &= \rho^n_{i,j} + \frac{\delta t}{2}\frac{\partial \rho}{\partial t} + \frac{\delta x}{2}\frac{\partial \rho}{\partial x} \\
&= \rho^n_{i,j} + \underbrace{\left(\frac{\Delta x}{2} - \frac{u\Delta t}{2}\right)\frac{\partial \rho}{\partial x}}_{\mathrm{I}} \underbrace{- \frac{v\Delta t}{2}\frac{\partial \rho}{\partial y}}_{\mathrm{II}}
\end{aligned}$$

Corner coupling

- Second order in space and time
- Accounts for information propagating across corners of zone

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Unsplit method: Hyperbolic conservation laws

- Hyperbolic conservation laws

$$\frac{\partial U}{\partial t} + \sum_{d=0}^{D-1} \frac{\partial F^d}{\partial x^d} = S$$

- Expressed in “primitive” variables

$$\frac{\partial W}{\partial t} + \sum_{d=0}^{\mathbf{D}-1} A^d(W)\,\frac{\partial W^d}{\partial x^d} = S'$$

$$A^d = \nabla_U W \cdot \nabla_U F^d \cdot \nabla_W U$$

$$S' = \nabla_U W \cdot S$$

- Require a second order estimate of fluxes

$$F^{n+\frac{1}{2}}_{\boldsymbol{i}+\frac{1}{2}\boldsymbol{e}^d} \approx F^d(\boldsymbol{x}_0 + (\boldsymbol{i} + \tfrac{1}{2}\boldsymbol{e}^d)h, t^n + \tfrac{1}{2}\Delta t)$$

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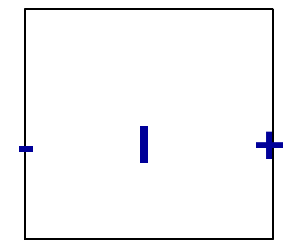
Unsplit method: Hyperbolic conservation laws

- Compute the effect of normal derivative terms and source term on the extrapolation in space and time from cell centers to cell faces

$$\left.\begin{aligned}
W_{\boldsymbol{i},\pm,d} &= W^n_{\boldsymbol{i}} + \frac{1}{2}(\pm I - \frac{\Delta t}{h} A^d_{\boldsymbol{i}}) P_\pm(\Delta^d W_{\boldsymbol{i}}) \\
A^d_{\boldsymbol{i}} &= A^d(W_{\boldsymbol{i}}) \\
P_\pm(W) &= \sum_{\pm\lambda_k > 0} (l_k \cdot W) r_k \\
W_{\boldsymbol{i},\pm,d} &= W_{\boldsymbol{i},\pm,d} + \frac{\Delta t}{2} \nabla_U W \cdot S^n_{\boldsymbol{i}}
\end{aligned}\right\} \mathrm{I}$$



- Compute estimates of F^d for computing 1D Flux derivatives $\partial F^d / \partial x^d$

$$F^{\mathrm{1D}}_{\boldsymbol{i}+\frac{1}{2}\boldsymbol{e}^d} = R(W_{\boldsymbol{i},+,d}, W_{\boldsymbol{i}+\boldsymbol{e}^d,-,d}, d)$$

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Unsplit method: Hyperbolic conservation laws

- Compute final correction to $W_{I,§,d}$ due to final transverse derivatives

$$\left.\begin{array}{c} W^{n+\frac{1}{2}}_{\boldsymbol{i},\pm,d} = W_{\boldsymbol{i},\pm,d} - \dfrac{\Delta t}{2h} \nabla_U W \cdot (F^{\text{1D}}_{\boldsymbol{i}+\frac{1}{2}\boldsymbol{e}^{d_1}} - F^{\text{1D}}_{\boldsymbol{i}-\frac{1}{2}\boldsymbol{e}^{d_1}}) \\ d \neq d_1, \; 0 \leq d, d_1 < \mathbf{D} \end{array}\right\} \text{II}$$

- Compute final estimate of fluxes

$$F^{n+\frac{1}{2}}_{\boldsymbol{i}+\frac{1}{2}\boldsymbol{e}^d} = R(W^{n+\frac{1}{2}}_{\boldsymbol{i},+,d}, W^{n+\frac{1}{2}}_{\boldsymbol{i}+\boldsymbol{e}^d,-,d}, d)$$

- Update the conserved quantities

$$U^{n+1}_{\boldsymbol{i}} = U^n_{\boldsymbol{i}} - \frac{\Delta t}{h} \sum_{d=0}^{\mathbf{D}-1} (F^{n+\frac{1}{2}}_{\boldsymbol{i}+\frac{1}{2}\boldsymbol{e}^d} - F^{n+\frac{1}{2}}_{\boldsymbol{i}-\frac{1}{2}\boldsymbol{e}^d})$$

- Procedure described for D=2. For D=3, we need additional corrections to account for (1,1,1) diagonal coupling

  D=2 requires 4 Rieman solves per time step

  D=3 requires 12 Riemann solves per time step

The **r¢ B=0** Problem

- Conservation of Ñ×**B** =0:
  - *Analytically: if Ñ×**B** =0 at t=0 than it remains zero at all times*
  - *Numerically: In upwinding schemes the curl and div operators do not commute*
- Approaches:
  - *Purist: Maxwell’s equations demand Ñ×**B** =0 exactly, so Ñ×**B** must be zero numerically*
  - *Modeler: There is truncation error in components of B, so what is special in a particular discretized form of Ñ×**B**?*
- Purposes to control Ñ×**B** numerically:
  - *To improve accuracy*
  - *To improve robustness*
  - *To avoid unphysical effects (Parallel Lorentz force)*

Approaches to address the **r¢ B**=0 constraint

- 8-wave formulation: **r¢** B = O(h^{α}) (Powell et al, JCP 1999; Brackbill and Barnes, JCP 1980)
- Constrained Transport (Balsara & Spicer JCP 1999, Dai & Woodward JCP 1998, Evans & Hawley Astro. J. 1988)
  – *Field Interpolated/Flux Interpolated Constrained Transport*
  – *Require a staggered representation of B*
  – *Satisfy **r¢** B=0 at cell centers using face values of B*
- Constrained Transport/Central Difference (Toth JCP 2000)
  – *Flux Interpolated/Field Interpolated*
  – *Satisfy **r¢** B=0 at cell centers using cell centered B*
- Projection Method
- Vector Potential (Claim: CT/CD schemes can be cast as an “underlying” vector potential. Evans and Hawley, Astro. J. 1988)
- Require ad-hoc corrections to total energy
- May lead to numerical instability (e.g. negative pressure – ad-hoc fix based on switching between total energy and entropy formulation by Balsara)

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r¢ B=0 using a Vector Potential

- The original 8-wave formulation proved numerically unstable for a 2D reconnection problem (Samtaney et al, Sherwood 2002)
- Stability was achieved with a combination of the generalized upwinding (8-wave formulation by Powell et al. JCP vol 154, 284-309, 1999) and vector potential in 2D
- Vector potential evolved using central differences
- At end of each stage in time integration replace x and y components of **B** using vector potential
  - *Central difference approximation of div(**B**) is zero*
- Issues with vector potential + upwinding approach
  - *Ad-hoc*
  - *Loss of accuracy*
  - *3D requires a gauge condition ➔ Elliptic problem*
  - *Non conservative (but 2D Magnetic reconnection does not exhibit discontinuities)*

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r¢ B=0 by Projection

- Compute the estimates to the fluxes $F^{n+1/2}_{i+1/2,j}$ using the unsplit formulation
- Use face-centered values of B to compute **r¢** B. Solve the Poisson equation **r**$^2\phi$ = **r ¢** B
- Correct B at faces: B=B-**r**f
- Correct the fluxes $F^{n+1/2}_{i+1/2,j}$ with projected values of B
- Update conservative variables using the fluxes
  - *The non-conservative source term S(U) a* ***r¢*** *B has been algebraically removed*
- On uniform Cartesian grids, projection provides the smallest correction to remove the divergence of B. (Toth, JCP 2000)

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r¢ B=0 by Projection

- Does the nature of the equations change?
  - *Hyperbolicity implies finite signal speed*
  - *Do corrections to B via **r**^{2}f=**r¢** B violate hyperbolicity?*
- Conservation implies that single isolated monopoles cannot occur. Numerical evidence suggests these occur in pairs which are spatially close.
  - *Corrections to B behave as a 1/r^2 in 2D and 1/r^3 in 3D*
- Projection does not alter the order of accuracy of the upwinding scheme and is consistent

AMR Implementation

- CHOMBO framework used for adaptive mesh refinement
- Implemented the unsplit method for hyperbolic fluxes
- Parabolic fluxes treated explicitly
  - *Quadratic interpolation ($O(h^3)$) at coarse-fine boundaries*
- Solenoidal B is achieved via projection, solving the elliptic equation **r**$^2\phi$=**r¢** B
  - *Solved using Multgrid on each level (union of rectangular meshes)*
  - *Coarser level provides Dirichlet boundary condition for ƒ*
    - *Requires **$O(h^3)$** interpolation of coarser mesh ƒ on boundary of fine level*
  - *a "bottom smoother" (conjugate gradient solver) is invoked when mesh cannot be coarsened*
  - *Physical boundary conditions are Neumann **dƒ/dn=0 (Reflecting) or Dirichlet***
- Multigrid convergence is sensitive to block size
- Flux corrections at coarse-fine boundaries to maintain conservation
  - *A consequence of this step: **r¢** B=0 is violated on coarse meshes in cells adjacent to fine meshes.*
- Code is parallel
- Second order accurate in space and time

Adaptive Mesh Refinement with Chombo

- ***Chombo*** is a collection of C++ libraries for implementing block-structured adaptive mesh refinement (AMR) finite difference calculations (http://www.seesar.lbl.gov/ANAG/chombo)
  - *(Chombo is an AMR developer's toolkit)*
- Mixed language model
  - *C++ for higher-level data structures*
  - *FORTRAN for regular single grid calculations*
  - *C++ abstractions map to high-level mathematical description of AMR algorithm components*
- Reusable components. Component design based on mathematical abstractions to classes
- Based on public-domain standards
  - *MPI, HDF5*
- Chombovis: visualization package based on VTK, HDF5
- Layered hierarchical properly nested meshes
- Adaptivity in both space and time

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Chombo Layered Design

- ***Chombo*** layers correspond to different levels of functionality in the AMR algorithm space
- Layer 1: basic data layout
  - *Multidimensional arrays and set calculus*
  - *data on unions of rectangles mapped onto distributed memory*
- Layer 2: operators that couple different levels
  - *conservative prolongation and restriction*
  - *averaging between AMR levels*
  - *interpolation of boundary conditions at coarse-fine interfaces*
  - *refluxing to maintain conservation at coarse-fine interfaces*
- Layer 3: implementation of multilevel control structures
  - *Berger-Oliger time stepping*
  - *multigrid iteration*
- Layer 4: complete PDE solvers
  - *Godunov methods for gas dynamics*
  - *Ideal and single-fluid resistive MHD*
  - *elliptic AMR solvers*

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