High-Accuracy, Implicit Solution of the Extended-MHD Equations using High-Continuity Finite Elements

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In collaboration with
the M3D group
and the
SciDAC Center for Extended MHD Modeling

Princeton University
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The Center for Extended Magnetohydrodynamic Modeling

(Global Stability of Magnetic Fusion Devices)

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a SciDAC activity…
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Considerations for a next-generation nonlinear MHD code for Magnetic Fusion Applications

i.e.: what have we learned?

- 2-fluid terms (Extended MHD) are essential to model real fusion experiments…but best form is uncertain
- Highly implicit treatment is needed to address long timescales
- There are advantages to using the potential/stream function form of the vector fields…avoids spec. pol.+ low order subsets
- High-order (4^{th} or more) finite elements are essential for describing highly anisotropic heat conduction.
- Direct sparse matrix inversions (vs iterative solvers) in the poloidal plane can be very efficient for the MHD system
- It is advantageous to have a fast linear option to scope runs
- Boundary conditions should be applied at infinity, but we need the capability to model a nearby resistive conducting structure

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Our center is comparing 5+ different Extended-MHD models and need to be able to change models without major code restructuring

Model	Momentum Equation	Ohm's law	Whistlers[1]	KAW[2]	GV[3]	Slow dynamics[4]
General	$mn\frac{d\mathbf{V}}{dt} = -\nabla(p_e + p_i) + \mathbf{J}\times\mathbf{B} - \nabla\cdot(\Pi_{\parallel e} + \Pi_{\parallel i}) - \nabla\cdot\Pi_i^{gv}$	$\mathbf{E} = -\mathbf{V}\times\mathbf{B} + \eta\mathbf{J} + \frac{1}{ne}\left(\mathbf{J}\times\mathbf{B} - \nabla p_e - \nabla\cdot\Pi_{\parallel e}\right)$	Yes	Yes	Yes	Either
Generalized Hall MHD[5]	$mn\frac{d\mathbf{V}}{dt} = -\nabla(p_e + p_i) + \mathbf{J}\times\mathbf{B} - \nabla\cdot(\Pi_{\parallel e} + \Pi_{\parallel i})$	$\mathbf{E} = -\mathbf{V}\times\mathbf{B} + \eta\mathbf{J} + \frac{1}{ne}\left(\mathbf{J}\times\mathbf{B} - \nabla p_e - \nabla\cdot\Pi_{\parallel e}\right)$	Yes	Yes	No	No
Neoclassical-MHD	$mn\frac{d\mathbf{V}}{dt} = -\nabla(p_e + p_i) + \mathbf{J}\times\mathbf{B} - \nabla\cdot(\Pi_{\parallel e} + \Pi_{\parallel i}) - \nabla\cdot\Pi_i^{gv}$	$\mathbf{E} = -\mathbf{V}\times\mathbf{B} + \eta\mathbf{J} - \frac{1}{ne}\nabla\cdot\Pi_{\parallel e}$	No	No	Yes	Yes
Generalized resistive MHD[5]	$mn\frac{d\mathbf{V}}{dt} = -\nabla p + \mathbf{J}\times\mathbf{B} - \nabla\cdot\Pi_{\parallel}$	$\mathbf{E} = -\mathbf{V}\times\mathbf{B} + \eta\mathbf{J}$	No	No	No	No
Generalized drift[6]	$mn\frac{d\mathbf{V}}{dt} = -mn\mathbf{V}_{di}\cdot\nabla\mathbf{V}_{\perp} + \upsilon_{gv} + nm\mu\nabla_{\perp}^2\mathbf{V} - \nabla\cdot(\Pi_{\parallel e} + \Pi_{\parallel i}) - \nabla(p_e + p_i) + \mathbf{J}\times\mathbf{B}$	$\mathbf{E} = -\mathbf{V}\times\mathbf{B} + \eta\mathbf{J}^* - \frac{1}{ne}\left[\nabla_{\parallel} p_e + \nabla\cdot\Pi_{\parallel e}\right]$	No	Yes	Yes	Yes

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All the MHD models beyond resistive MHD contain dispersive waves

Extended MHD

$$mn\frac{d\mathbf{V}}{dt} = -\nabla(p_e + p_i) + \mathbf{J}\times\mathbf{B} - \nabla\cdot(\Pi_{\parallel e} + \Pi_{\parallel i}) - \nabla\cdot\Pi_i^{gv}$$

Off-diagonal stress tensor terms lead to gyro-viscous waves

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla\times\mathbf{E}\ , \qquad \mathbf{E} = -\mathbf{V}\times\mathbf{B} + \eta\mathbf{J} \quad + \frac{1}{ne}\left(\mathbf{J}\times\mathbf{B} - \nabla p_e - \nabla\cdot\Pi_{\parallel e}\right)$$

Hall term leads to Whistler wave

Pressure gradient terms lead to Kinetic Alfven wave

All these new "Extended MHD" waves have similar structure

$$\frac{\partial^2 \mathbf{B}}{\partial t^2} = -\left(\frac{V_A^2}{\Omega}\right)^2 (\mathbf{b}\cdot\nabla)^2\,\nabla^2\mathbf{B}$$

Note 4th spatial derivatives

Limiting form gives wave-like equation where wave speed is inversely proportional to wavelength:

$$\text{i.e.}\quad \frac{\omega}{k} \sim k$$

Need viable implicit techniques for these 4th order (in space) equations to provide numerical stability for large timesteps.

Highly anisotropic heat conduction requires accurate spatial representation and implicit time differencing

$$\frac{\partial T}{\partial t} = \nabla \bullet \left[\kappa_{\parallel} \frac{\vec{B}\vec{B}}{B^2} \bullet \nabla T \right] + \nabla \bullet \kappa \nabla T + S$$

In a highly magnetized fusion plasma, $\kappa_{\parallel} >> \kappa$

- Low-order finite difference methods are not adequate
- AMR based on rectangles (or cubes) is probably not the most efficient approach
- Two approaches have been shown to be viable:
    - High order finite elements: C^0 vs C^1
    - Field aligned coordinates
- Similar considerations for anisotropy in mass diffusion and wave propagation

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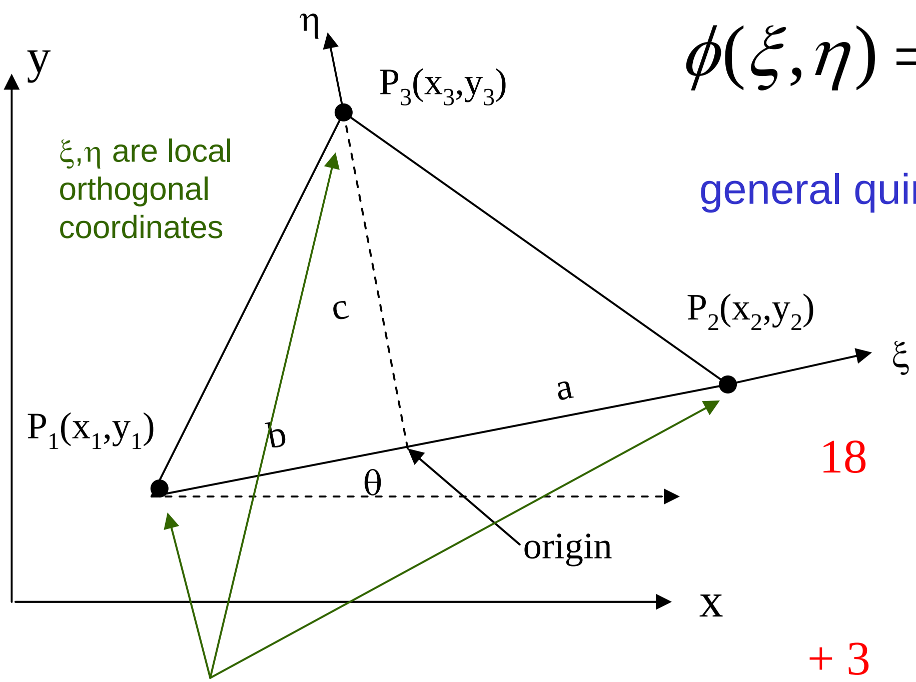
Approach

- Use high-order, *high-continuity* triangular finite elements in poloidal plane, spectral in the toroidal direction

- The compactness and high-continuity of this representation makes a full *implicit* solution practical: including whistler, gyroviscous, and kinetic Alfvén waves

Divide domain into triangular regions: represent solution as a quintic polynomial within each region



$$\phi(\xi,\eta) = \sum_{k=1}^{21} a_k \xi^{m_k} \eta^{n_k}$$

general quintic has 21 terms.

k	m_k	n_k
1	0	0
2	1	0
3	0	1
4	2	0
5	1	1
6	0	2
7	3	0
8	2	1
9	1	2
10	0	3
11	4	0
12	3	1
13	2	2
14	1	3
15	0	4
16	5	0
17	3	2
18	2	3
19	1	4
20	0	5
21	4	1

18 constraints to match the function and derivatives at nodes

+ 3 constraints on quintic coefficients to enforce C^1 continuity at edges

21 coefficients of the quintic polynomial

The function and it's first and second derivatives at the 3 nodes are the global unknowns (6 per node) $(\phi, \phi_x, \phi_y, \phi_{xx}, \phi_{xy}, \phi_{yy})$

Error ~ h^5 *(since complete Taylor series through h^4)*

C^1 continuity allows treatment of 4th spatial derivatives *(Galerkin Method)*

Most compact representation for this accuracy "reduced quintic"

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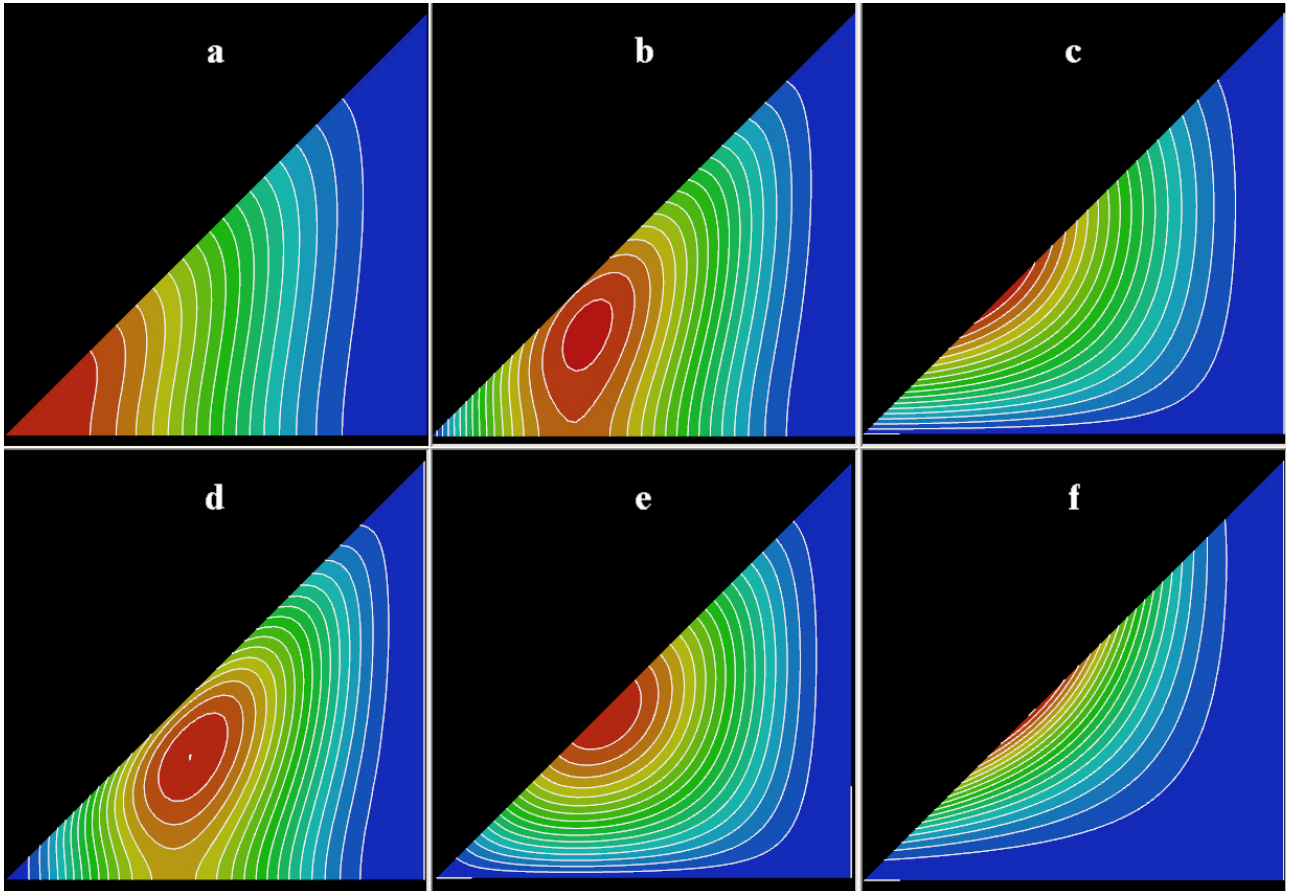
$a_i = g_{ij}\,\Phi_j$

The Trial Functions:

$$\phi = \sum_{i=1}^{21} a_i\,\xi^{m_i}\eta^{n_i} = \sum_{i=1}^{21}\sum_{j=1}^{18} g_{ij}\Phi_j\,\xi^{m_i}\eta^{n_i} = \sum_{j=1}^{18} v_j\Phi_j$$

$$v_j = \sum_{i=1}^{21} \xi^{m_i}\eta^{n_i} g_{ij}$$



These are the trial functions. There are 18 for each triangle.

The 6 shown here correspond to one node, and vanish at the other nodes, along with their derivatives

Each of the six has value 1 for the function or one of it's derivatives at the node, zero for the others.

Note that the function and it's derivatives (through 2nd) play the role of the amplitudes

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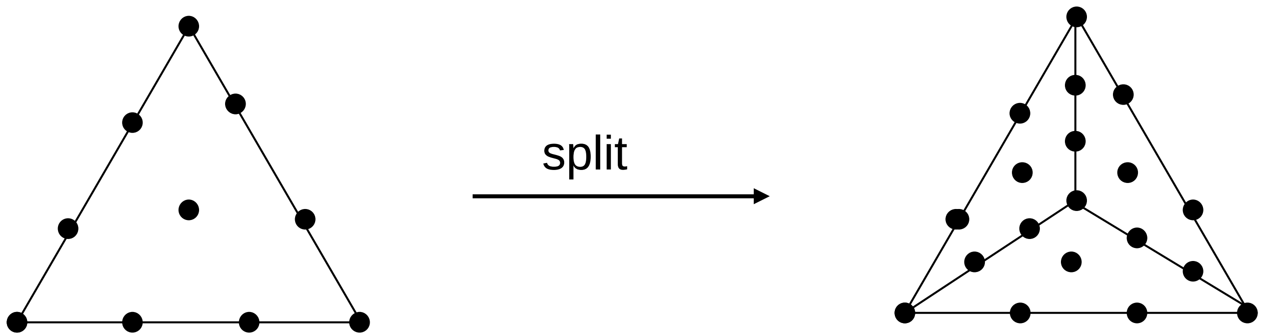
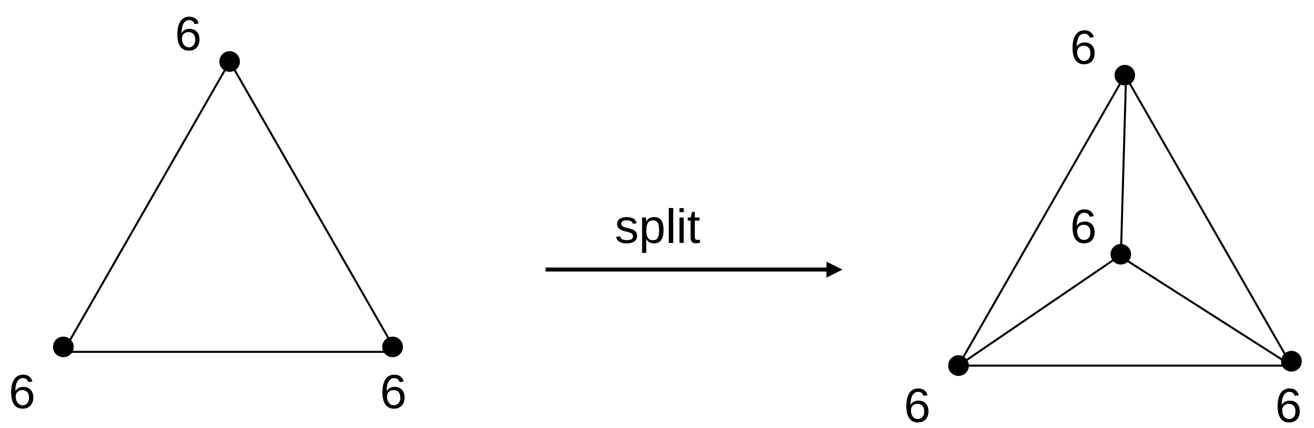
Comparison with a popular C^0 Element



Lagrange Cubic: C^0, h^4

9 new unknowns: 2 new triangles

9/2 = $4^{1/2}$ unknowns/ triangle



Reduced Quintic: C^1, h^5

6 new unknowns: 2 new triangles

6/2 = 3 unknowns/ triangle

Comparison of reduced quintic to other popular triangular elements

	Vertex nodes	Line nodes	Interior nodes	accuracy order h^p	Unknowns per triangle	continuity
linear element	3	0	0	2	½	C^0
Lagrange quadratic	3	3	0	3	2	C^0
Lagrange cubic	3	6	1	4	4½	C^0
Lagrange quartic	3	9	3	5	8	C^0
reduced quintic	18	0	0	5	3	C^1

The "reduced quintic" is the most compact representation of an element of this order of accuracy (fewest unknowns/triangle)

- and -

It's C^1 continuity property allows it to represent spatial derivatives up to 4th order without introducing auxiliary variables

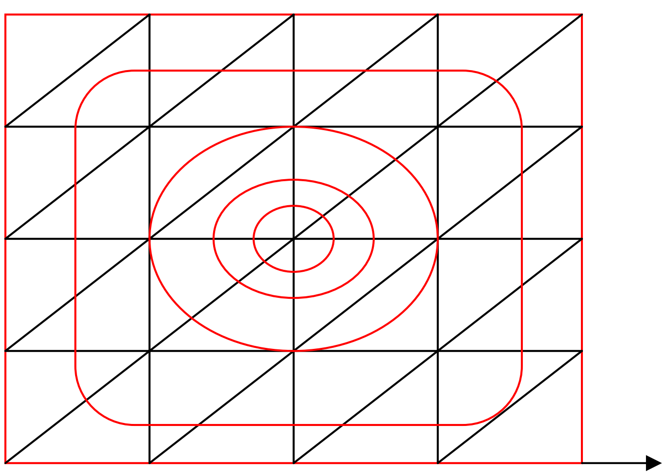
=> Smaller matrices to invert

Anisotropic Diffusion

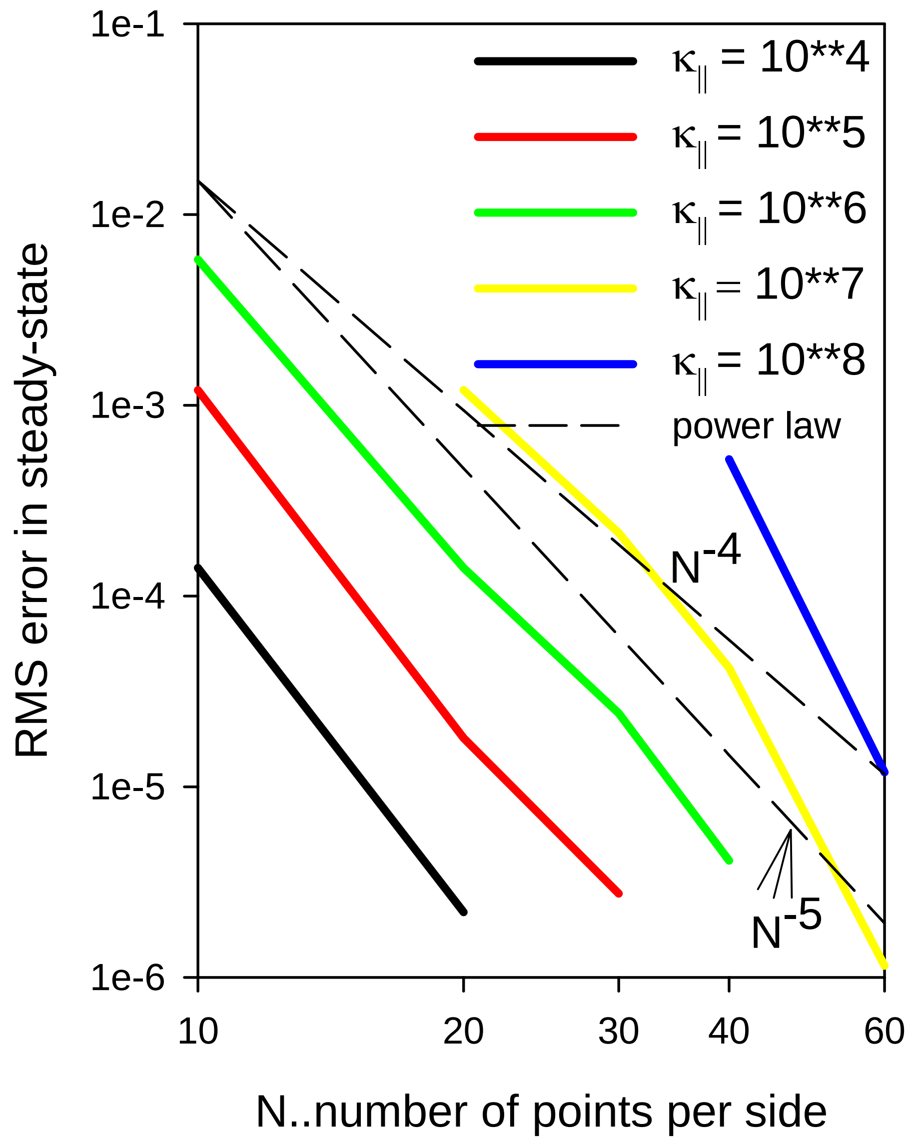
Shows greater than N^{-5} convergence

$$\frac{\partial \phi}{\partial t} = \nabla \bullet \left[\kappa_{\parallel} \frac{\vec{B}\vec{B}}{B^2} \bullet \nabla \phi \right] + \nabla \bullet \kappa \nabla \phi + S$$

Solve to steady state



$$S = \psi = \cos\frac{\pi x}{L}\cos\frac{\pi y}{L}$$



2D Incompressible MHD

$$\frac{\partial}{\partial t}\nabla^2\phi+\left[\nabla^2\phi,\phi\right]-\left[\nabla^2\psi,\psi\right]=\mu\nabla^4\phi$$

$$\frac{\partial\psi}{\partial t}+\left[\psi,\phi\right]=\eta\nabla^2\psi$$

note:

"reduced MHD"

ϕ is stream function

ψ is poloidal flux

$\left[a,b\right]\equiv\nabla a\times\nabla b\bullet\hat{z}$

θ-centering….Taylor expand in time (centered about n+1/2 for θ=0.5)

$$\nabla^2\dot{\phi}+\left[\nabla^2\phi^n+\theta\delta t\nabla^2\dot{\phi},\phi^n+\theta\delta t\dot{\phi}\right]-\left[\nabla^2\psi^n+\theta\delta t\nabla^2\dot{\psi},\psi+\theta\delta t\dot{\psi}\right]=\mu\left[\nabla^4\phi+\theta\delta t\nabla^4\dot{\phi}\right]$$

$$\dot{\psi}+\left[\psi^n+\theta\delta t\dot{\psi},\phi+\theta\delta t\dot{\phi}\right]=\eta\left[\nabla^2\psi^n+\theta\delta t\nabla^2\dot{\psi}\right]$$

Multiply out non-linear terms, neglecting terms ~ $(\delta t)^2$. Finite difference in time:

$$\dot{\phi}=\frac{\phi^{n+1}-\phi^n}{\delta t},\qquad\dot{\psi}=\frac{\psi^{n+1}-\psi^n}{\delta t}$$

Move all terms at time level (n+1) to left of equal sign. Expand in trial functions. Multiply equations by each trial function and integrate over space. Integrate by parts as needed. *(Galerkin Method)*

$$\phi^n=\sum_{j=1}^{18}v_j\Phi_j^n\qquad\psi^n=\sum_{j=1}^{18}v_j\Psi_j^n$$

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Leads to the Matrix Implicit System

$$\begin{bmatrix} S_j^{11} & S_j^{12} \\ S_j^{21} & S_j^{22} \end{bmatrix} \begin{bmatrix} \Phi_j^{n+1} \\ \Psi_j^{n+1} \end{bmatrix} = \begin{bmatrix} D_j^{11} & D_j^{12} \\ D_j^{21} & D_j^{22} \end{bmatrix} \begin{bmatrix} \Phi_j^{n} \\ \Psi_j^{n} \end{bmatrix}$$

unknown
at time n+1

unknown
at time n

Block-Sparse
coefficient matrices

$$\begin{bmatrix} S_j^{11} & S_j^{12} \\ S_j^{21} & S_j^{22} \end{bmatrix} = \begin{bmatrix} A_{i,j} + \theta\delta t[\bar{G}_{i,j,k}\Phi_k^{*} - \mu B_{i,j}] & -\theta\delta t \bar{G}_{i,j,k}\Psi_k^{*} \\ \theta\delta t K_{i,j,k}\Psi_k^{*} & M_{i,j} + \theta\delta t[K_{i,k,j}\Phi_k^{*} - \eta A_{i,j}] \end{bmatrix}$$

$$\begin{bmatrix} D_j^{11} & D_j^{12} \\ D_j^{21} & D_j^{22} \end{bmatrix} = \begin{bmatrix} \left\{ \begin{matrix} A_{i,j} - \delta t[G_{i,j,k}\Phi_k^{n} - \theta\bar{G}_{i,j,k}\Phi_k^{*} \\ +(1-\theta)\mu B_{i,j}] \end{matrix} \right\} & \delta t(G_{i,j,k}\Psi_k^{n} - \theta\bar{G}_{i,j,k}\Psi_k^{*}) \\ \delta t K_{i,j,k}(-\tfrac{1}{2}\Psi_k^{n} + \theta\Psi_k^{*}) & \left\{ \begin{matrix} M_{i,j} - \delta t[K_{i,k,j}(\tfrac{1}{2}\Phi_k^{n} - \theta\Phi_k^{*}) \\ -(1-\theta)\eta A_{i,j}] \end{matrix} \right\} \end{bmatrix}$$

Solve each time step using SuperLU direct solver

For linear problem, only need to form LU decomposition once and do a back-substitution each time step.

Note that stream function and vorticity are solved together

Each spatial operator becomes a submatrix

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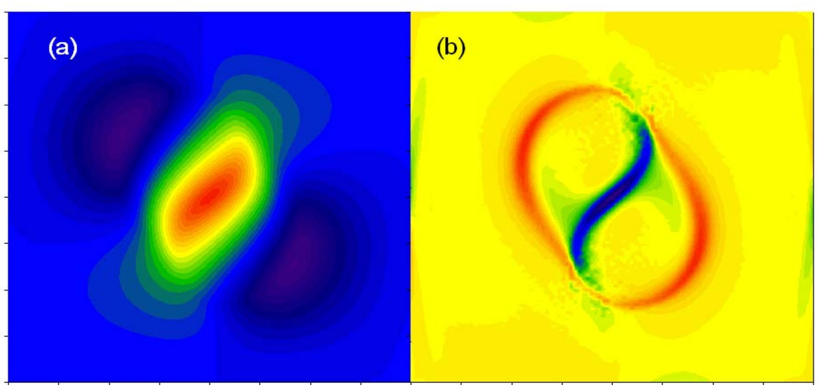
Tilting of a Plasma Column

Initial Condition:

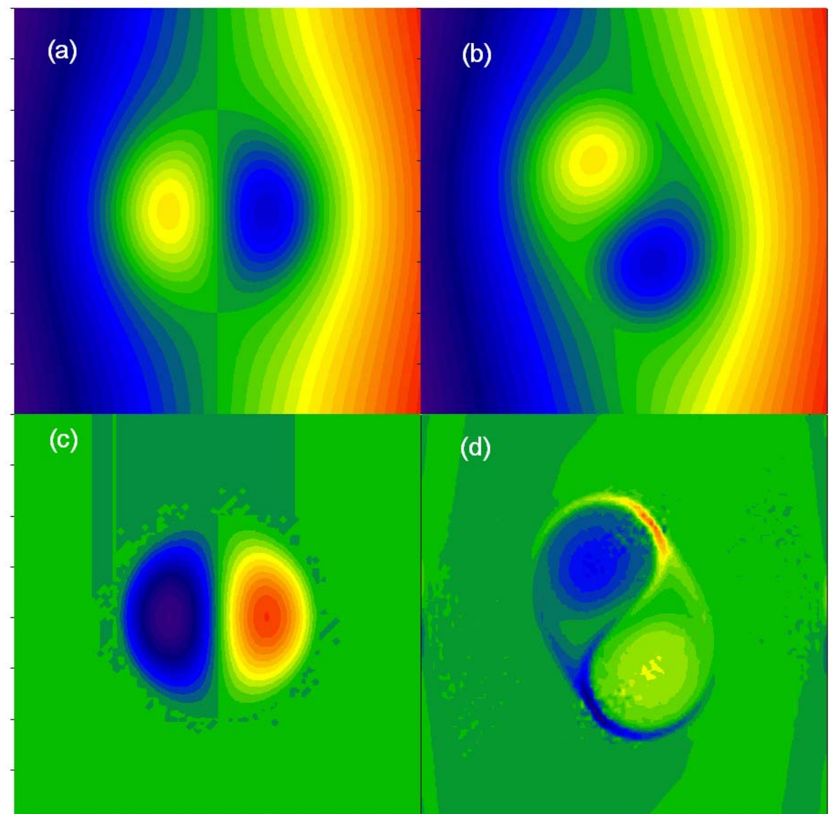
$$\psi = \begin{cases} [2/kJ_0(k)]J_1(kr)\cos\theta, & r<1 \\ (r-1/r)\cos\theta, & r>1 \end{cases}$$

$$J_1(k)=0$$

Give small perturbation and evolve in time

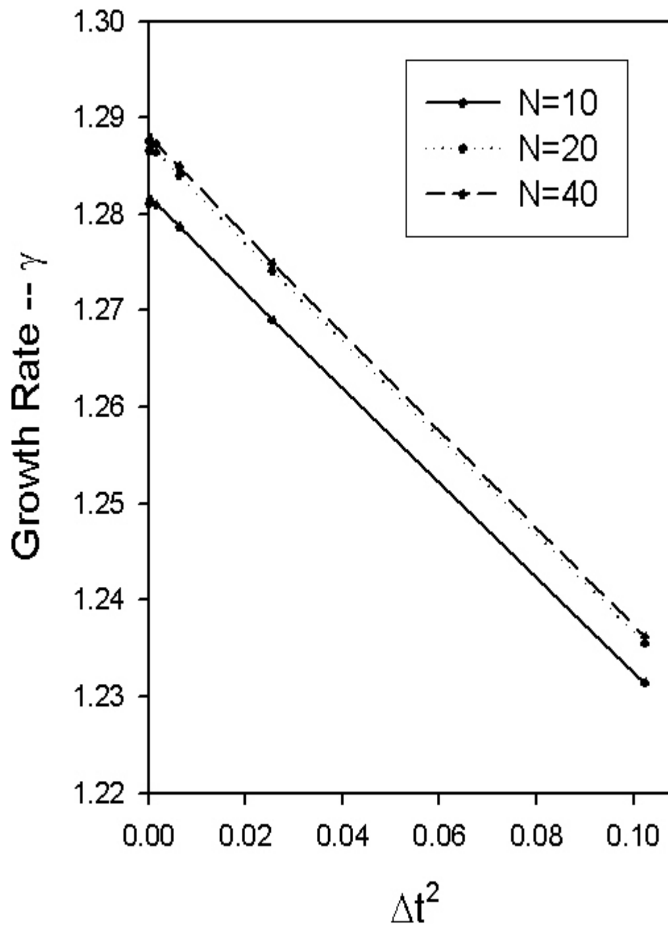


Stream function and vorticity at final time

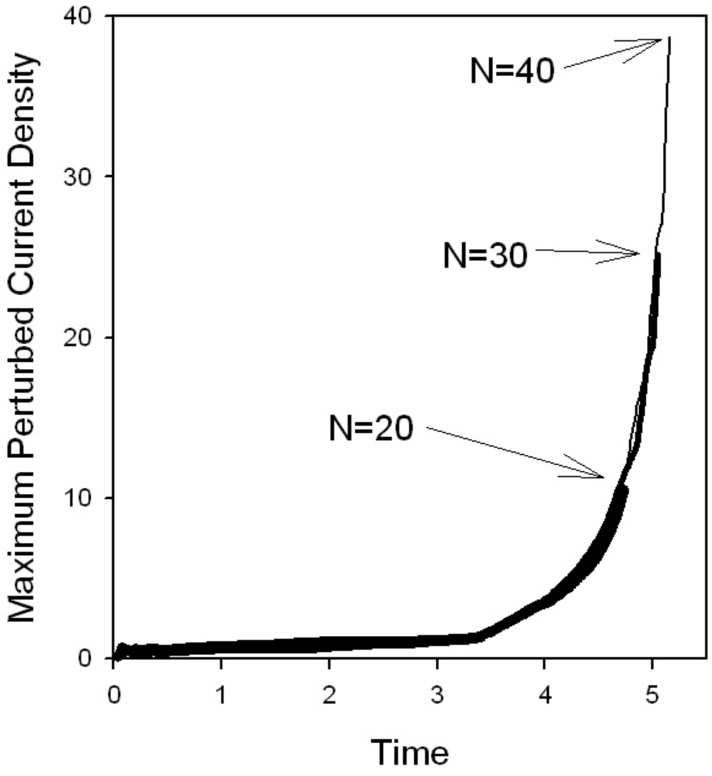


Flux (top) and current (bottom) at initial and final times

Tilting of a Plasma Column-cont



Converged (in time) growth rate the same for N=30,40 out to 6 decimal places



Calculation stopped each time when energy error reached 1%.

Higher order formulation

By further manipulation, it is possible to get a 4th order (in space) PDE for Φ^{n+1} that is independent of Ψ^{n+1}

$$\begin{bmatrix} S_j'^{11} & \boxed{0} \\ S_j^{21} & S_j^{22} \end{bmatrix} \begin{bmatrix} \Phi_j^{n+1} \\ \Psi_j^{n+1} \end{bmatrix} = \begin{bmatrix} D_j'^{11} & D_j'^{12} \\ D_j^{21} & D_j^{22} \end{bmatrix} \begin{bmatrix} \Phi_j^{n} \\ \Psi_j^{n} \end{bmatrix}$$

Note: $S_j'^{11}$ now is a 4th order operator: contains all the linear Ideal MHD (Alfven wave) response

Instead of inverting full S matrix, invert two sub-matrices sequentially. Gives same results in 1/8th – 1/4th the time

$$S_j'^{11}\Phi_j^{n+1} = D_j'^{11}\Phi_j^{n} + D_j'^{12}\Psi_j^{n}$$

$$S_j^{22}\Psi_j^{n+1} = -S_j^{21}\Phi_j^{n+1} + D_j^{21}\Phi_j^{n} + D_j^{22}\Psi_j^{n}$$

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M3D-*C1* code has full Extended MHD equations expressed in a form that allows non-trivial subsets of lower rank equations:

$$\begin{bmatrix} S_{11}^v & S_{12}^v & S_{13}^v \\ S_{21}^v & S_{22}^v & S_{23}^v \\ S_{31}^v & S_{32}^v & S_{33}^v \end{bmatrix} \bullet \begin{bmatrix} \phi \\ V_z \\ \chi \end{bmatrix}^{n+1} = \begin{bmatrix} D_{11}^v & D_{12}^v & D_{13}^v \\ D_{21}^v & D_{22}^v & D_{23}^v \\ D_{31}^v & D_{32}^v & D_{33}^v \end{bmatrix} \bullet \begin{bmatrix} \phi \\ V_z \\ \chi \end{bmatrix}^{n} + \begin{bmatrix} R_{11}^v & R_{12}^v & R_{13}^v \\ R_{21}^v & R_{22}^v & R_{23}^v \\ R_{31}^v & R_{32}^v & R_{33}^v \end{bmatrix} \bullet \begin{bmatrix} \psi \\ I \\ T_e \end{bmatrix}^{n}$$

$$\begin{bmatrix} S_{11}^p & S_{12}^p & S_{13}^p \\ S_{21}^p & S_{22}^p & S_{23}^p \\ S_{31}^p & S_{32}^p & S_{33}^p \end{bmatrix} \bullet \begin{bmatrix} \psi \\ I \\ T_e \end{bmatrix}^{n+1} = \begin{bmatrix} D_{11}^p & D_{12}^p & D_{13}^p \\ D_{21}^p & D_{22}^p & D_{23}^p \\ D_{31}^p & D_{32}^p & D_{33}^p \end{bmatrix} \bullet \begin{bmatrix} \psi \\ I \\ T_e \end{bmatrix}^{n} + \begin{bmatrix} R_{11}^p & R_{12}^p & R_{13}^p \\ R_{21}^p & R_{22}^p & R_{23}^p \\ R_{31}^p & R_{32}^p & R_{33}^p \end{bmatrix} \bullet \begin{bmatrix} \phi \\ V_z \\ \chi \end{bmatrix}^{n+1} + \begin{bmatrix} Q_{11}^p & Q_{12}^p & Q_{13}^p \\ Q_{21}^p & Q_{22}^p & Q_{23}^p \\ Q_{31}^p & Q_{32}^p & Q_{33}^p \end{bmatrix} \bullet \begin{bmatrix} \phi \\ V_z \\ \chi \end{bmatrix}^{n}$$

Phase-I: Resistive MHD:

$$\frac{\partial}{\partial t}\nabla^2\phi + \left[\nabla^2\phi, \phi\right] - \left[\nabla^2\psi, \psi\right] = \mu\nabla^4\phi$$

$$\frac{\partial\psi}{\partial t} + \left[\psi, \phi\right] = \eta\nabla^2\psi$$

Phase-II: Fitzpatrick-Porcelli 4-field model:

$$\frac{\partial}{\partial t}\nabla^2\phi = \left[\phi, \nabla^2\phi\right] + \left[\nabla^2\psi, \psi\right] + \mu\nabla^4\phi$$

$$\frac{\partial V_z}{\partial t} = \left[\phi, V_z\right] + c_\beta\left[I, \psi\right] + \mu\nabla^2 V_z$$

$$\frac{\partial\psi}{\partial t} = \left[\phi, \psi\right] + d_\beta\left[\psi, I\right] + \eta\nabla^2\psi$$

$$\frac{\partial I}{\partial t} = \left[\phi, I\right] + d_\beta\left[\nabla^2\psi, \psi\right] + c_\beta\left[V_z, \psi\right] + c_\beta^2\eta\nabla^2 I$$

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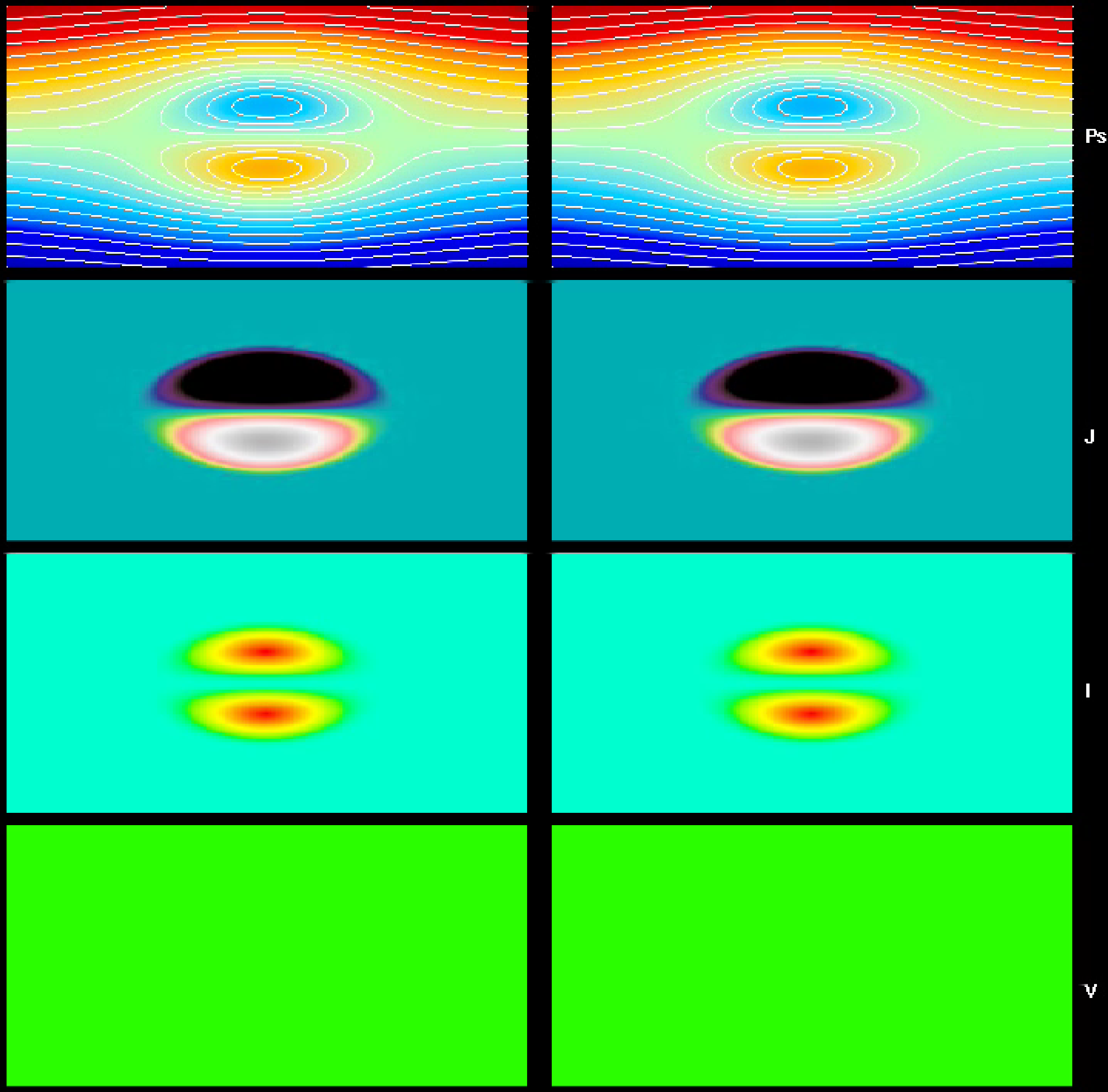


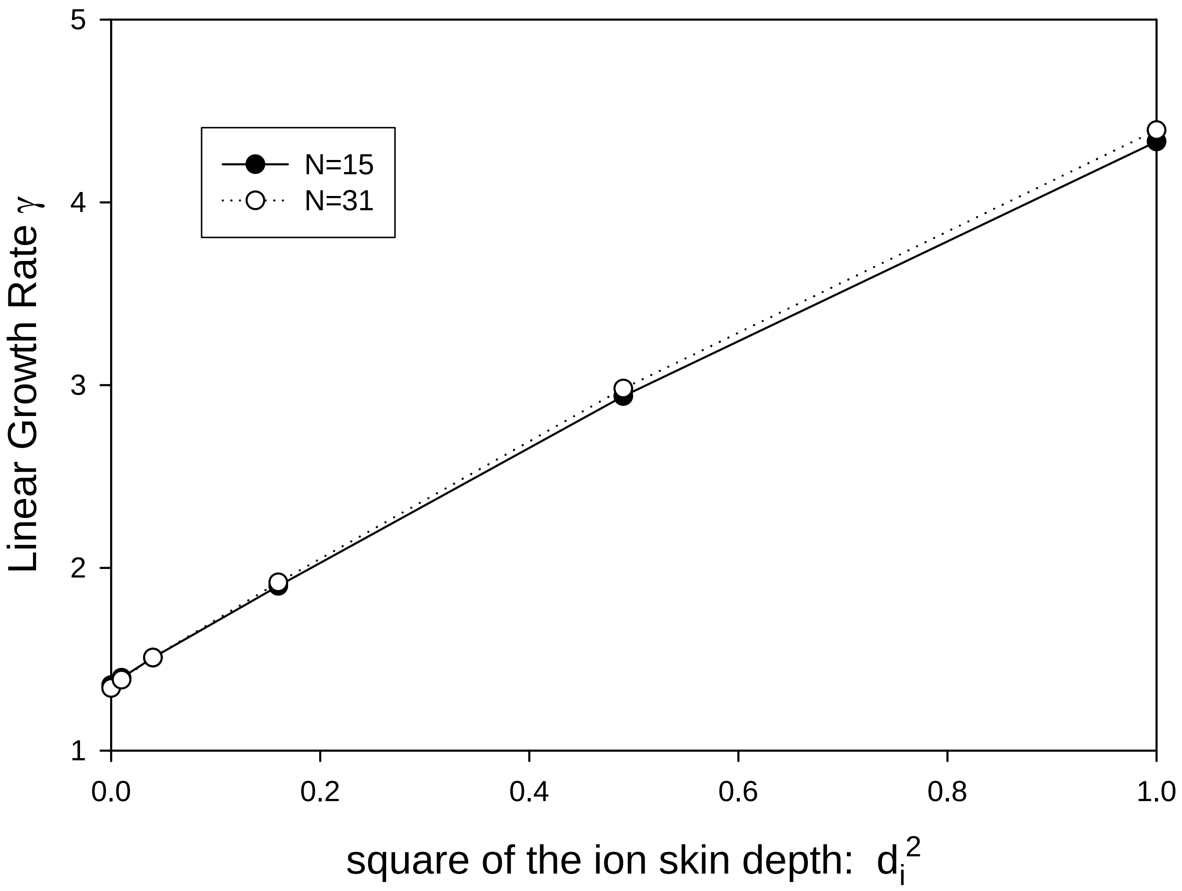
Tilting spheromak in 2-field (left) and 4-field (right) models.

Poloidal Magnetic Flux

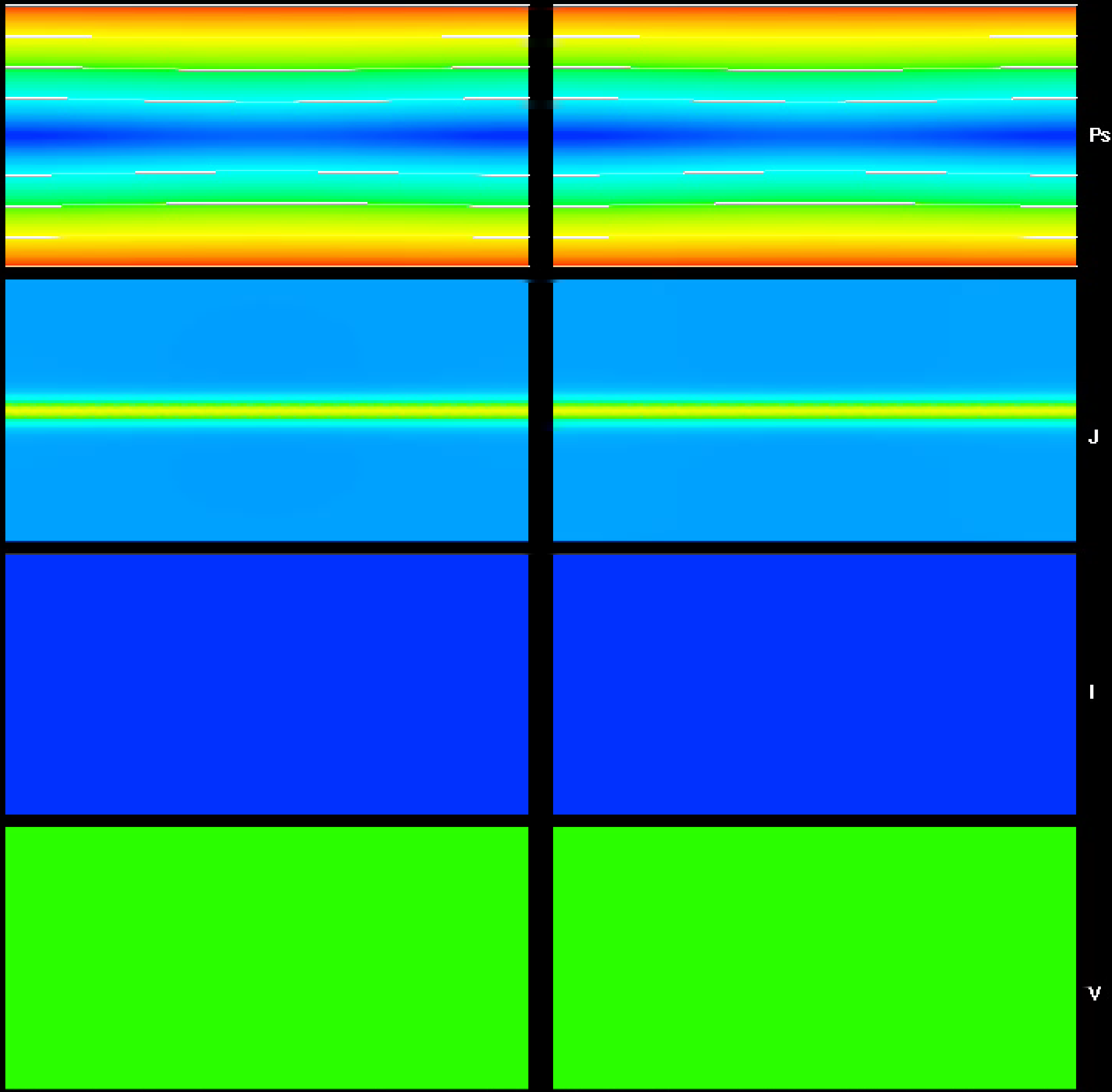
Toroidal Current Density

Toroidal Magnetic Field

Toroidal Velocity



4-field (2-fluid) model predicts that growth rate of tilt mode increases linearly with the square of the ion skin depth d_i



Harris sheet reconnection in 2-field (left) and 4-field (right) models

Poloidal
Magnetic Flux

Toroidal
Current Density

Toroidal
Magnetic Field

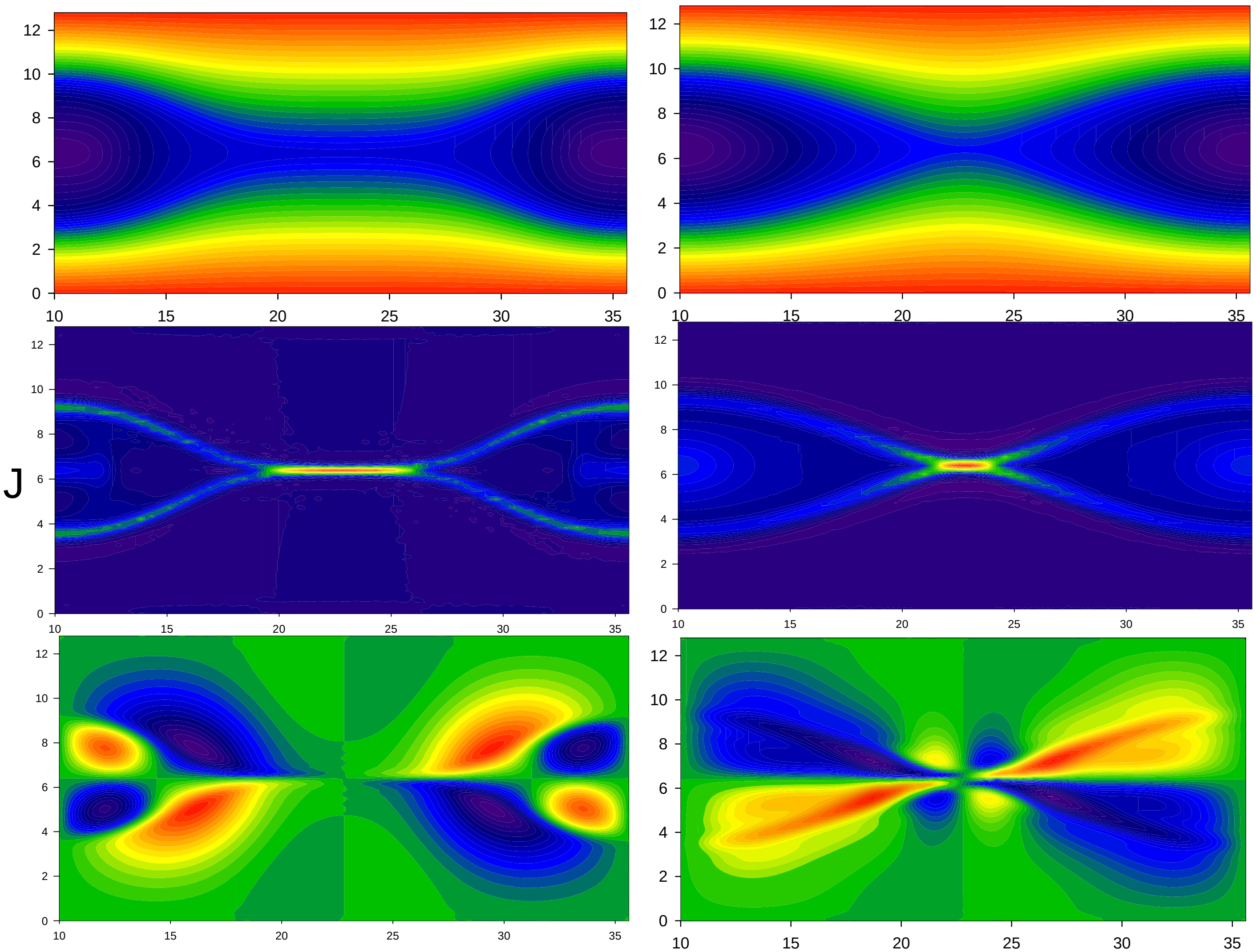
Toroidal
Velocity

Comparison of GEM reconnection with 2-field and 4-field models

Poloidal Flux Ψ

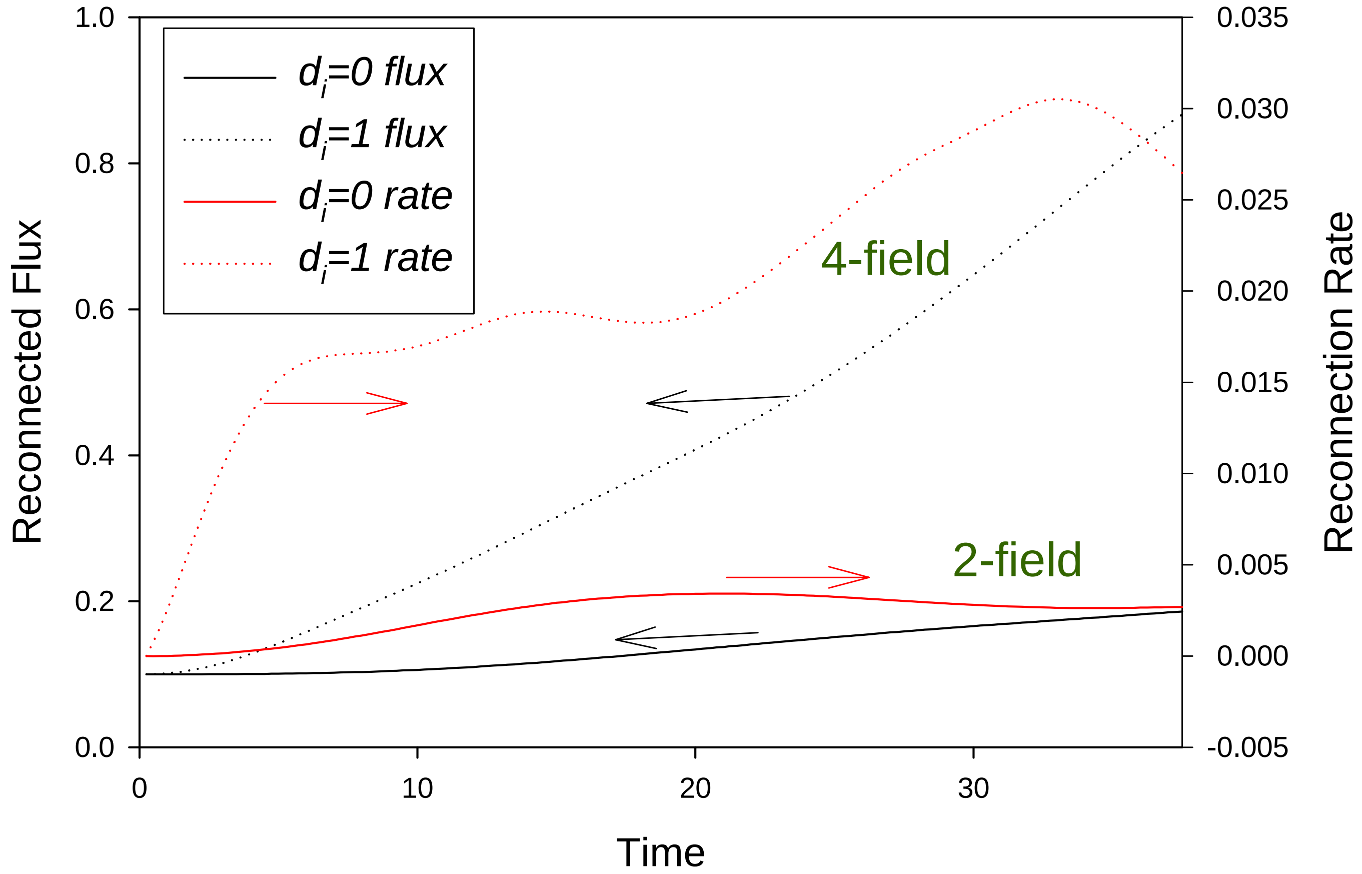
Current Density J

Velocity stream function ϕ



2-field reduced resistive MHD

4-field 2-fluid model



4-field (2-fluid) equations with di=1 show much greater reconnection rate that 2-field (reduced MHD) description

The 2D cylindrical two-fluid MHD equations and definition of the variables.

$$\vec{V} = \nabla \phi \times \hat{z} + \nabla_{\perp} \chi + v_{\varphi}$$

$$\vec{B} = \nabla \psi \times \hat{z} + I\hat{z}$$

$$\frac{\partial \vec{B}}{\partial t} = -\nabla \times \vec{E}$$

$$\vec{E} + \vec{V} \times \vec{B} = \eta \vec{J} + \frac{1}{ne}\left(\vec{J} \times \vec{B} - \nabla p_e\right)$$

$$\mu_0 \vec{J} = \nabla \times \vec{B}$$

$$nM_i\left(\frac{\partial \vec{V}}{\partial t} + \vec{V} \bullet \nabla \vec{V}\right) + \nabla p = \vec{J} \times \vec{B} - \nabla \bullet \overleftrightarrow{\Pi}_i^{gv} + \mu n \nabla \bullet \left[\nabla \vec{V} + \nabla \vec{V}^{\dagger}\right]$$

$$\frac{\partial n}{\partial t} + \nabla \bullet (n\vec{V}) = 0$$

$$\frac{3}{2}\frac{\partial p_e}{\partial t} + \nabla \bullet \left(\frac{3}{2} p_e \vec{V}_i\right) = -p_e \nabla \bullet \vec{V}_i + \frac{\vec{J}}{ne} \bullet \left[\frac{3}{2}\nabla p_e - \frac{5}{2}\frac{p_e}{n}\nabla n + \vec{R}\right] - \nabla \bullet \vec{q}_e - Q_{\Delta}$$

$$\frac{3}{2}\frac{\partial p_i}{\partial t} + \nabla \bullet \left(\frac{3}{2} p_i \vec{V}_i\right) = -p_i \nabla \bullet \vec{V}_i - \Pi_i : \nabla V_i + \nabla(\mu n \vec{V}) : \left[\nabla \vec{V} + \nabla \vec{V}^{\dagger}\right] - \nabla \bullet \vec{q}_i + Q_{\Delta}$$

$$\frac{3}{2}\frac{\partial p}{\partial t} + \nabla \bullet \left(\frac{3}{2} p \vec{V}\right) = -p \nabla \bullet V - \Pi_i : \nabla V_i + \nabla(\mu n \vec{V}) : \left[\nabla \vec{V} + \nabla \vec{V}^{\dagger}\right] - \nabla \bullet (\vec{q}_i + \vec{q}_e)$$

$$+ \frac{\vec{J}}{ne} \bullet \left[\frac{3}{2}\nabla p_e - \frac{5}{2}\frac{p_e}{n}\nabla n + \vec{R}\right]$$

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Numerical stability analysis for 2-fluid equations shows sequential inverstion method leads to stability for arbitrary timestep

$$\frac{\partial \vec{V}}{\partial t} = \vec{J} \times \vec{B}_0$$

$$\frac{\partial \vec{J}}{\partial t} = \nabla \times \frac{\partial \vec{B}}{\partial t} = \nabla \times \nabla \times \left[(\vec{V} - d_i \vec{J}) \times \vec{B}_0 \right] = \vec{B}_0 \times \nabla^2 \vec{V} - d_i \vec{B}_0 \times \nabla^2 \vec{J}$$

Alfven Waves

Whistler waves

$$\dot{V} = -\hat{z} \times \left\{ J + \theta \delta t \left[\hat{z} \times \nabla^2 (\vec{V} + \theta \delta t \dot{V}) - d_i \hat{z} \times \nabla^2 \vec{J} \right] \right\}$$

$$\dot{J} = \hat{z} \times \nabla^2 \vec{V} - d_i \hat{z} \times \nabla^2 (\vec{J} + \theta \delta t \dot{J})$$

$$\left[1 - (\theta \delta t)^2 \nabla^2 \right] (V^{n+1} - V^n) = \delta t \left\{ \theta \delta t \left[\nabla^2 V^n - d_i \nabla^2 J^n \right] \right\} - \delta t \hat{z} \times J^n$$

$$\left[1 + d_i \theta \delta t \hat{z} \times \nabla^2 \right] (J^{n+1} - J^n) = \delta t \hat{z} \times \nabla^2 [\theta V^{n+1} + (1-\theta) V^n] - d_i \delta t \hat{z} \times \nabla^2 J^n$$

Note: these can be solved sequentially!

Note!

$$\begin{bmatrix} 1-(\theta\delta t)^2\nabla^2 & 0 & 0 & 0 \\ 0 & 1-(\theta\delta t)^2\nabla^2 & 0 & 0 \\ 0 & \theta\delta t\nabla^2 & 1 & -\theta\delta t d_i\nabla^2 \\ -\theta\delta t\nabla^2 & 0 & \theta\delta t d_i\nabla^2 & 1 \end{bmatrix} \begin{bmatrix} V_x \\ V_y \\ J_x \\ J_y \end{bmatrix}^{n+1} = \begin{bmatrix} 1-\theta(\theta-1)(\delta t)^2\nabla^2 & 0 & -\theta(\delta t)^2 d_i\nabla^2 & \delta t \\ 0 & 1-\theta(\theta-1)(\delta t)^2\nabla^2 & -\delta t & -\theta(\delta t)^2 d_i\nabla^2 \\ 0 & (\theta-1)\delta t\nabla^2 & 1 & -\delta t d_i\nabla^2(\theta-1) \\ -(\theta-1)\delta t\nabla^2 & 0 & \delta t d_i\nabla^2(\theta-1) & 1 \end{bmatrix} \begin{bmatrix} V_x \\ V_y \\ J_x \\ J_y \end{bmatrix}^{n}$$

Summary

- Major upgrade to the M3D code is being explored--based on quintic C^1 finite elements
- Primary motivation is to allow efficient, high order, implicit solution of extended MHD equations with whister and KAW
- Staged implementation using reduced sets of equations with 2, 4, and then 6 variables
- Initial results look promising!

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