Numerical Studies of Gyroviscous Effects Using High-Order Finite Elements

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Abstract

We have developed a technique for incorporating a general expression of the gyroviscous force [5, Ramos, 2005] into an implicit solution algorithm for the two-fluid magnetohydrodynamic (MHD) equations. We present the results of numerical simulations of six-field extended-MHD equations in two dimensions, including Braginskii's gyroviscous stress tensor, using triangular finite elements with fifth-order accuracy and continuous first derivatives (C^1-continuity). Our model extends that used by [4, Jardin and Breslau, 2005] by including the evolution of pressure and flow compressibility, in addition to the inclusion of the gyroviscous force. The use of C^1-continuous finite elements allows up to four differentiations of any field variable, thus enabling the inclusion of the full gyroviscous stress tensor. The effect of this term on wave propagation and Harris-equilibrium reconnection is demonstrated.

The Gyroviscous Force

- The gyroviscous term represents the first order finite-Larmor radius (FLR) contributions to the Braginskii equations [1].

- It is not dissipative.

- It becomes important when β_i is (locally) large, such as in the dissipation region in null-helicity reconnection.

- We use a general expression for the gyroviscous force in the collisionless limit is given by [1, 5]:

$$\Pi_{ij} = \frac{p^{(i)}}{4B^2}\left[\epsilon_{ikl}\left(\delta_{jm} + 3\frac{B_j B_m}{B^2}\right) + \epsilon_{jkl}\left(\delta_{im} + 3\frac{B_i B_m}{B^2}\right)\right] B_k \left(\frac{\partial v_l}{\partial x_m} + \frac{\partial v_m}{\partial x_l}\right)$$

  where repeated indices are implicitly summed.

- Particles simulations have shown [2] that E_z is balanced by the ion pressure tensor at the X-point.

Finite-Element Representation

- Fields are represented as a linear combination of N basis functions ν_i:

$$\phi(\xi, \eta) = \sum_{i=1}^{N} v_i \phi_i(\xi, \eta)$$

where ξ and η are the local coordinates of the element.

- The **weak form** of equations are solved. For example,

$$\frac{\partial \nabla^2 \chi}{\partial t} = -\nabla^2 p \quad \Rightarrow \quad \int dA\, \nu_i \frac{\partial \nabla^2 \chi}{\partial t} = -\int dA\, \nu_i \nabla^2 p.$$

Thus every equation becomes a system of N equations.

- The equations can be integrated by parts to move derivatives off of field variables and onto the trial function. This relaxes the differentiability requirements. For example,

$$\int dA\, \nu_i \frac{\partial \nabla^2 \chi}{\partial t} = -\int dA\, \nu_i \nabla^2 p \quad \Rightarrow \quad -\int dA\, \nabla \nu_i \cdot \frac{\partial \nabla \chi}{\partial t} = \int dA\, \nabla \nu_i \cdot \nabla p$$

Surface terms vanish in the presence of periodic or homogeneous Dirichlet boundary conditions.

Weak form of Gyroviscous Force

The weak form of the equation $n\partial_t\mathbf{v} = -\nabla\cdot\Pi$, after integrating by parts, can be written in coordinate-independent form (omitting the integral symbols):

$$\nu_i n\frac{\partial\nabla^2\phi}{\partial t} = \frac{p^{(i)}}{2B^2}\left\{\begin{array}{l} I\left(1+\frac{3}{2}\frac{|\nabla\psi|^2}{B^2}\right)\left(\nabla^2[\nu_i,\phi]-[\nabla^2\nu_i,\phi]-[\nu_i,\nabla^2\phi]\right)+ \\ +\frac{1}{2}\left(1-3\frac{I^2}{B^2}\right)\left([\psi,(\nu_i],U)+[U,(\nu_i],\psi)\right)-\frac{3}{2B^2}[\psi,U]\left(\nabla^2\nu_i|\nabla\psi|^2+2[\psi,[\nu_i],\psi]\right)- \\ -I\Big[\left(1+\frac{3}{2}\frac{|\nabla\psi|^2}{B^2}\right)\left((\nabla^2\nu_i,\chi)+(\nu_i,\nabla^2\chi)-\nabla^2(\nu_i,\chi)\right)+\left(1+3\frac{|\nabla\psi|^2}{B^2}\right)\nabla^2\nu_i\nabla^2\chi+\frac{3}{B^2}\nabla^2\chi[\psi,[\nu_i],\psi]\Big] \end{array}\right\}$$

$$\nu_i n\frac{\partial U}{\partial t} = \frac{p^{(i)}}{2B^2}\left\{\begin{array}{l} \frac{1}{2}\Big[\left(1-3\frac{I^2}{B^2}\right)\left([\psi,(\phi],\nu_i)+[\nu_i,(\phi],\psi)\right)-\frac{3}{B^2}[\psi,\nu_i]\left(\nabla^2\phi|\nabla\psi|^2+2[\psi,[\phi],\psi]\right)\Big]- \\ -\frac{1}{2}I\left(1+3\frac{I^2-|\nabla\psi|^2}{B^2}\right)[\nu_i,U]+ \\ +(\nu_i,(\chi),\psi)+\frac{3}{B^2}\left([\psi,\nu_i][\psi(\chi],\psi)-I^2[\nu_i,[\chi],\psi]\right) \end{array}\right\}$$

$$\nu_i n\frac{\partial\nabla^2\chi}{\partial t} = \frac{p^{(i)}}{2B^2}\left\{\begin{array}{l} I\Big[\left(1+\frac{3}{2}\frac{|\nabla\psi|^2}{B^2}\right)\left((\nabla^2\phi,\nu_i)+(\phi,\nabla^2\nu_i)-\nabla^2(\phi,\nu_i)\right)+\left(1+3\frac{|\nabla\psi|^2}{B^2}\right)\nabla^2\phi\nabla^2\nu_i+\frac{3}{B^2}\nabla^2\nu_i[\psi,[\phi],\psi]\Big]+ \\ +(U,(\nu_i),\psi)+\frac{3}{B^2}\left([\psi,U][\psi,(\nu_i],\psi)-I^2[U,[\nu_i],\psi]\right)- \\ -I\Big[\left(1+\frac{3}{2}\frac{|\nabla\psi|^2}{B^2}\right)\left([\nabla^2\nu_i,\chi]+[\nu_i,\nabla^2\chi]-\nabla^2[\nu_i,\chi]\right)+\frac{3}{B^2}\left(\nabla^2\nu_i[\psi,(\chi],\psi)-\nabla^2\chi[\psi,(\nu_i],\psi)\right)\Big] \end{array}\right\}$$

Where we have defined the following coordinate-independent functions:

$$\begin{aligned} [a,b] &= a_{,x}b_{,y}-a_{,y}b_{,x} \\ [a,\ [b],c] &= a_{,x}b_{,yx}c_{,y}-a_{,x}b_{,yy}c_{,x}+a_{,y}b_{,xy}c_{,x}-a_{,y}b_{,xx}c_{,y} \\ [a,\ (b],c) &= a_{,x}b_{,yx}c_{,x}+a_{,x}b_{,yy}c_{,y}-a_{,y}b_{,xx}c_{,x}-a_{,y}b_{,xy}c_{,y} \\ (a,(b),c) &= a_{,x}b_{,xx}c_{,x}+a_{,x}b_{,xy}c_{,y}+a_{,y}b_{,yx}c_{,x}+a_{,y}b_{,yy}c_{,y} \end{aligned}$$

where $a_{,x} = \partial a/\partial x$, etc..

Matrix Representation of $\partial_t \mathbf{v} = -\nabla \cdot \Pi$

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Writing $n\partial_t \mathbf{v} = -\nabla \cdot \Pi$ as:

$$\begin{pmatrix} L_{ij}^{11} & 0 & 0 \\ 0 & L_{ij}^{22} & 0 \\ 0 & 0 & L_{ij}^{33} \end{pmatrix} \frac{\partial}{\partial t} \begin{pmatrix} \phi_j \\ U_j \\ \chi_j \end{pmatrix} + \begin{pmatrix} R_{ij}^{11} & R_{ij}^{12} & R_{ij}^{13} \\ R_{ij}^{21} & R_{ij}^{22} & R_{ij}^{23} \\ R_{ij}^{31} & R_{ij}^{32} & R_{ij}^{33} \end{pmatrix} \begin{pmatrix} \phi_j \\ U_j \\ \chi_j \end{pmatrix} = 0$$

- In general, each element is sixth-order nonlinear. The evaluation of such a term requires $\mathcal{O}(N^6)$ operations, where N is the number of trial functions.
- By introducing a few auxiliary fields, we have been able to rewrite each term as a fourth-order nonlinear term. The equations determining the auxiliary fields are also fourth-order nonlinear.
- This results in a reduction of operations by a factor of $\mathcal{O}(N^2)$.
- This does not introduce error of a higher order than the previous truncation error.
- This method works for any C^1 finite element.

Matrix Representation of $\partial_t \mathbf{v} = -\nabla \cdot \Pi$

We define the following auxiliary variables:

$$\alpha = p^{(i)}/B^2; \quad \gamma = 3\alpha/B^2; \quad \mu = \gamma I^2$$

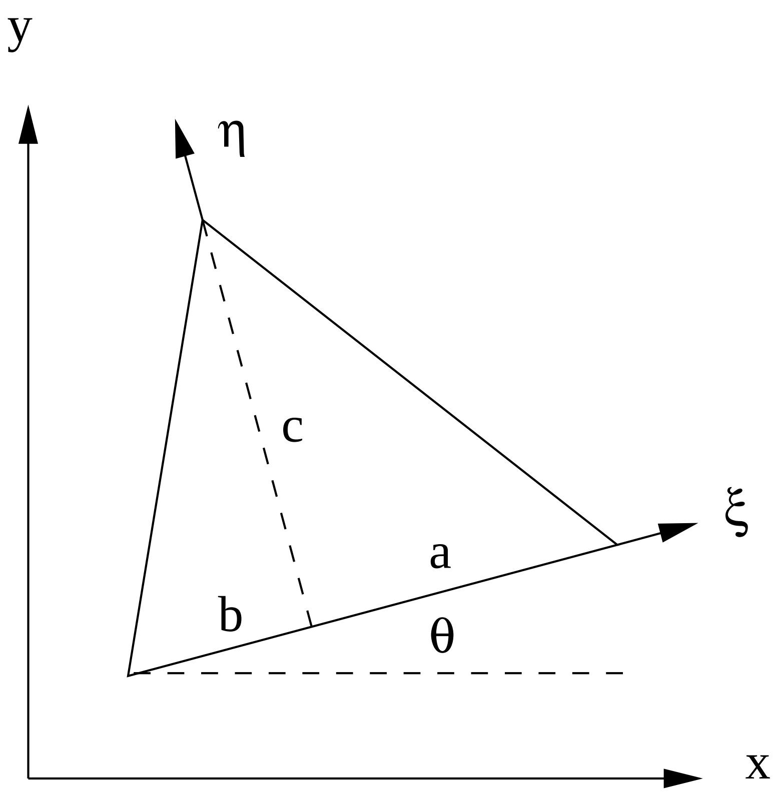
$$W^{(1)} = \tfrac{1}{2}\gamma|\nabla\psi|^2; \quad W^{(2)} = \tfrac{1}{2}\gamma(\psi_{,x}^2 - \psi_{,y}^2); \quad W^{(3)} = \gamma\psi_{,x}\psi_{,y}$$

The elements R_{ij} can now be written:

$$\begin{aligned}
R_{ij}^{11} &= G_{ijkl}^{(11a)} I_k(\alpha_l + W_l^{(1)}) \\
R_{ij}^{12} &= G_{ijkl}^{(12a)} \psi_k(\alpha_l - \mu_l) + G_{ijkl}^{(12b)} \psi_k W_l^{(2)} + G_{ijkl}^{(12c)} \psi_k W_l^{(3)} \\
R_{ij}^{13} &= G_{ijkl}^{(13a)} I_k(\alpha_l + W_l^{(1)}) + G_{ijkl}^{(13b)} I_k W_l^{(2)} + G_{ijkl}^{(13c)} I_k W_l^{(3)} \\
R_{ij}^{21} &= G_{jikl}^{(12a)} \psi_k(\alpha_l - \mu_l) + G_{jikl}^{(12b)} \psi_k W_l^{(2)} + G_{jikl}^{(12c)} \psi_k W_l^{(3)} \\
R_{ij}^{22} &= G_{ijkl}^{(22a)} I_k(\alpha_l + \mu_l - 2W_l^{(1)}) \\
R_{ij}^{23} &= G_{ijkl}^{(23a)} \psi_k \alpha_l + G_{ijkl}^{(23b)} \psi_k \mu_l - G_{jikl}^{(12c)} \psi_k W_l^{(2)} + G_{jikl}^{(12b)} \psi_k W_l^{(3)} \\
R_{ij}^{31} &= -G_{ijkl}^{(13a)} I_k(\alpha_l + W_l^{(1)}) - G_{jikl}^{(13b)} I_k W_l^{(2)} - G_{jikl}^{(13c)} I_k W_l^{(3)} \\
R_{ij}^{32} &= G_{jikl}^{(23a)} \psi_k \alpha_l + G_{jikl}^{(23b)} \psi_k \mu_l - G_{ijkl}^{(12c)} \psi_k W_l^{(2)} + G_{ijkl}^{(12b)} \psi_k W_l^{(3)} \\
R_{ij}^{33} &= G_{ijkl}^{(11a)} I_k(\alpha_l + W_l^{(1)}) + G_{ijkl}^{(33b)} I_k W_k^{(2)} + G_{ijkl}^{(33c)} I_k W_k^{(3)}
\end{aligned}$$

$$\begin{aligned}
G_{ijkl}^{(11a)} B_j C_k D_l &= \frac{1}{2}\Big([\nabla^2\nu_i, B] + [\nu_i, \nabla^2 B] - \nabla^2[\nu_i, B]\Big) CD \\
G_{ijkl}^{(12a)} B_j C_k D_l &= -\frac{1}{2}\left([C, (\nu_i], B) + [B, (\nu_i], C)\right) D \\
G_{ijkl}^{(12b)} B_j C_k D_l &= \frac{1}{2}[C, B](\nu_{i,xx} - \nu_{i,yy}) D \\
G_{ijkl}^{(12c)} B_j C_k D_l &= [C, B](\nu_{i,xy}) D \\
G_{ijkl}^{(13a)} B_j C_k D_l &= \frac{1}{2}\Big((\nabla^2\nu_i, B) + (\nu_i, \nabla^2 B) - \nabla^2(\nu_i, B) - \nabla^2\nu_i\nabla^2 B\Big) CD \\
G_{ijkl}^{(13b)} B_j C_k D_l &= \frac{1}{2}\nabla^2 B(\nu_{i,xx} - \nu_{i,yy}) CD \\
G_{ijkl}^{(13c)} B_j C_k D_l &= \nabla^2 B(\nu_{i,xy}) CD \\
G_{ijkl}^{(22a)} B_j C_k D_l &= \frac{1}{4}[\nu_i, B] CD \\
G_{ijkl}^{(23a)} B_j C_k D_l &= -\frac{1}{2}(\nu_i, (B), C) D \\
G_{ijkl}^{(23b)} B_j C_k D_l &= \frac{1}{2}[\nu_i, [B], C] D \\
G_{ijkl}^{(33b)} B_j C_k D_l &= \Big(\nabla^2\nu_i B_{,xy} - \nabla^2 B \nu_{i,xy}\Big) CD \\
G_{ijkl}^{(33c)} B_j C_k D_l &= (\nu_{i,xx} B_{,yy} - \nu_{i,yy} B_{,xx}) CD
\end{aligned}$$

Reduced Quintic Finite Element [3]



- The basis 18 functions are 5^{th} degree polynomials

$$\nu_j(\xi,\eta) = \sum_{i=1}^{20} g_{ij}\xi^{m_i}\eta^{n_i}$$

where m_i and n_i are integers between 0 and 5.

- $g_{ij}(a,b,c,\theta)$ are functions only of the orientation and shape of the finite-elements, and can be pre-computed for a static mesh.

- Fields represented on these elements automatically have continuous first derivatives (C^1 continuity) across element boundaries. Therefore fields may be differentiated twice without introducing auxiliary fields.

Numerical Model

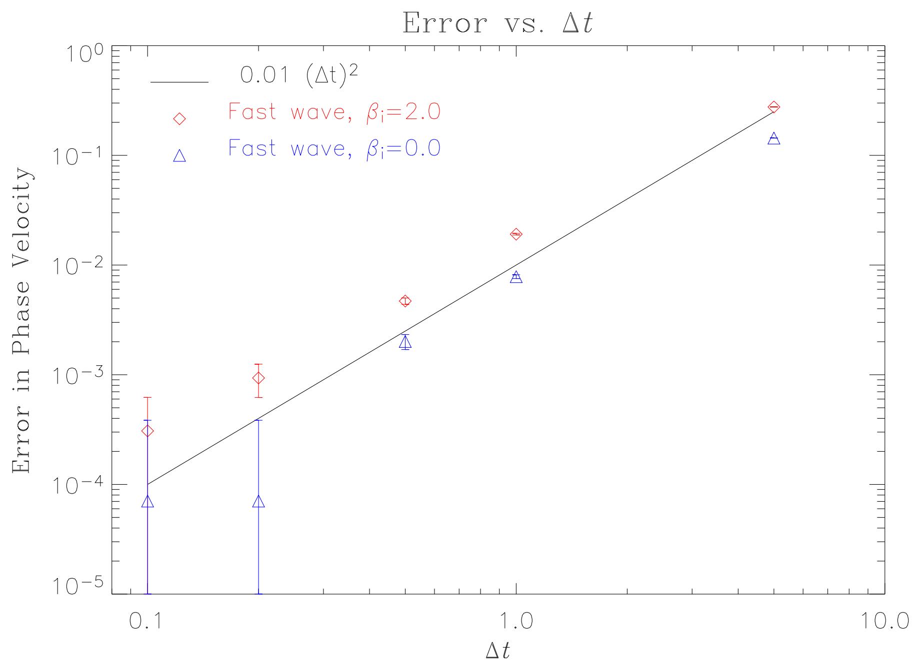
$$\frac{\partial n}{\partial t} = -\nabla \cdot n\mathbf{v}$$
$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E}$$
$$\frac{\partial \mathbf{v}}{\partial t} = -\mathbf{v} \cdot \nabla \mathbf{v} + \mathbf{J} \times \mathbf{B} - \nabla p - \nabla \cdot \Pi_i$$
$$\frac{\partial p_e}{\partial t} = -\nabla \cdot (p_e \mathbf{v}) - \frac{2}{3}(p_i \nabla \cdot \mathbf{v} + \Pi_i : \nabla \mathbf{v} + \nabla \cdot \mathbf{q}_i - Q_\Delta)$$
$$\frac{\partial p}{\partial t} = -\nabla \cdot (p\mathbf{v}_i) - \frac{2}{3}(p\nabla \cdot \mathbf{v}_i + \Pi_i : \nabla \mathbf{v}_i + \nabla \cdot (\mathbf{q}_i + \mathbf{q_e})) + \\ + \frac{\mathbf{J}}{n} \cdot \left(\nabla p_e - \frac{5}{3}\frac{p_e}{n}\nabla n + \frac{2}{3}\mathbf{R}\right)$$

$$\mathbf{J} = \nabla \times \mathbf{B}; \quad \mathbf{R} = \eta n \mathbf{J}; \quad p = p_i + p_e; \quad Q_\Delta = 3\tfrac{m_e}{m_i}(p - 2p_e)\nu_e$$

$$\mathbf{E} + \mathbf{v} \times \mathbf{B} = \tfrac{1}{n}(\mathbf{R} + \mathbf{J} \times \mathbf{B} - \nabla p_e)$$

The equations in blue, together with $\nabla \cdot \mathbf{v} = 0$ represent the **four-field equations**.

Four Field Results: Convergence



- With $\theta = 0.5$, the error scales as $(\Delta t)^2$
- For $\theta > 0.5$, or for larger k, the error scales as Δt.

Four Field Linear Dispersion Analysis

For the case where $k_z = 0$, the two right-traveling normal modes are:

Slow Wave

$$v_\phi = \frac{b_k}{2B}\left(\sqrt{4B^4 + K_-^2} - K_+\right)$$

$$\begin{pmatrix} \psi_1 \\ \phi_1 \\ I_1 \\ U_1 \end{pmatrix} = \begin{pmatrix} \sqrt{4B^4 + K_-^2} - K_- \\ -2B^2 \\ -k(\sqrt{4B^4 + K_-^2} - K_-) \\ 2kB^2 \end{pmatrix}$$

Fast Wave

$$v_\phi = \frac{b_k}{2B}\left(\sqrt{4B^4 + K_-^2} + K_+\right)$$

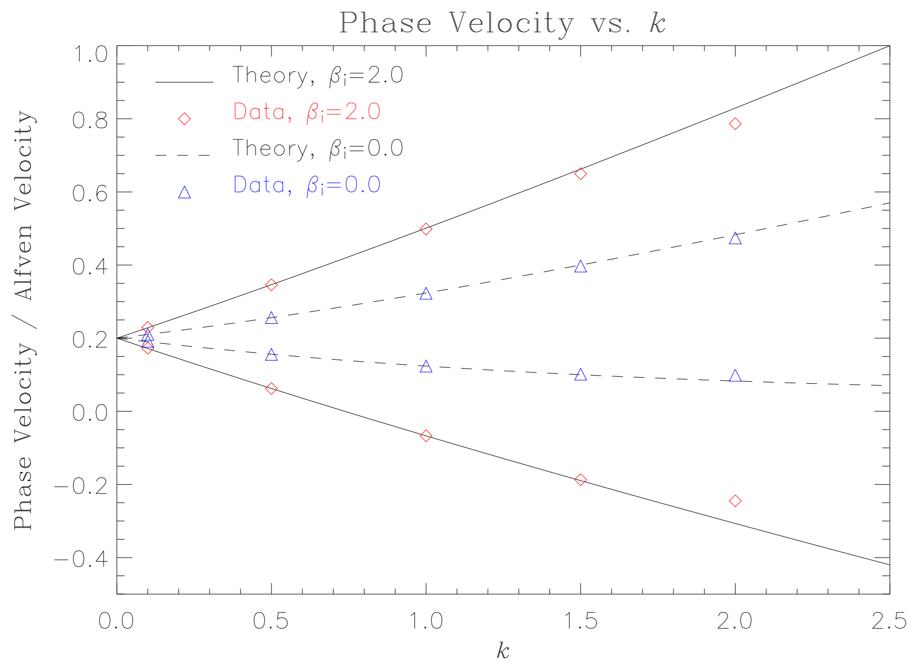
$$\begin{pmatrix} \psi_1 \\ \phi_1 \\ I_1 \\ U_1 \end{pmatrix} = \begin{pmatrix} \sqrt{4B^4 + K_-^2} + K_- \\ -2B^2 \\ k(\sqrt{4B^4 + K_-^2} + K_-) \\ -2kB^2 \end{pmatrix}$$

$$b_k = (\mathbf{k} \cdot \mathbf{B})/(kB); \quad \beta_i = p_{i0}\beta; \quad K_\pm = k\left(B^2 \pm \frac{1}{4}\beta_i(3b_k^2 - 1)\right)$$

For parallel propagation, expansion in k reveals an $\mathcal{O}(\beta_i)$ correction to the whistler wave:

$$v_{\phi\|} = 1 \pm \frac{1}{2}(1 + \beta_i/2)k + \frac{1}{8}(1 - \beta_i/2)^2 k^2 + \mathcal{O}(k^3)$$

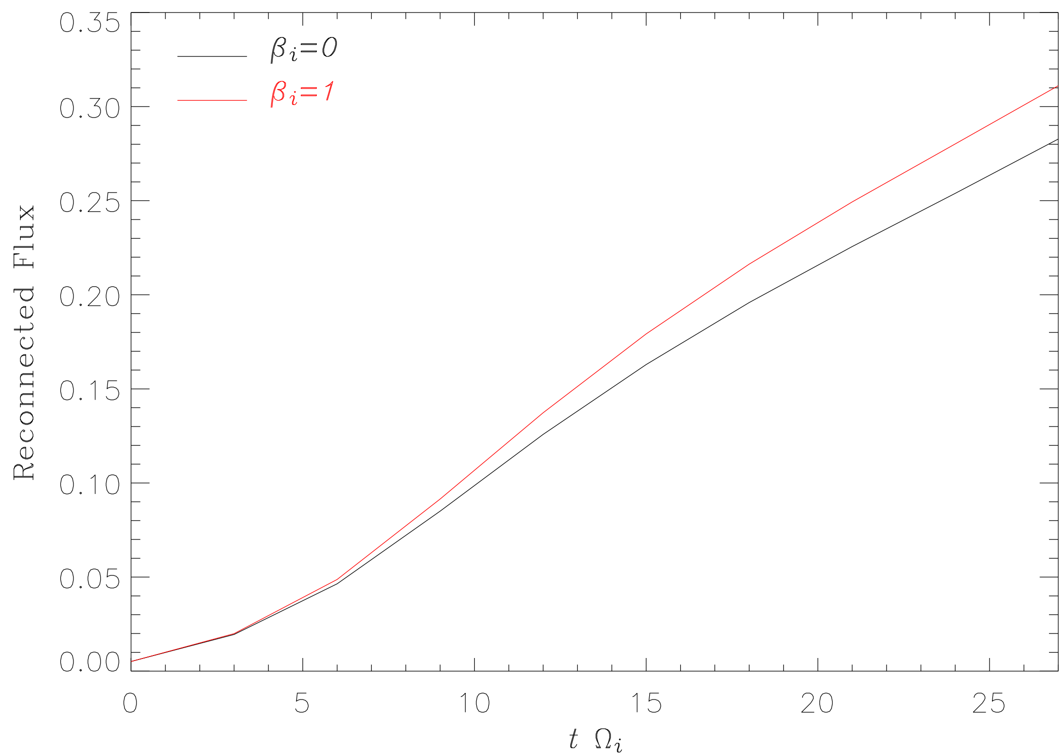
Four Field Results: Dispersion



The code is very accurate at low k. At high k, the discrepancy between the theoretical result and the numerical result is due to a finite viscosity included in the numerical simulation, but neglected in the theory. This discrepancy can be made arbitrarily small by choosing a sufficiently small time step so that viscosity can be reduced.

Four Field Results: Reconnection

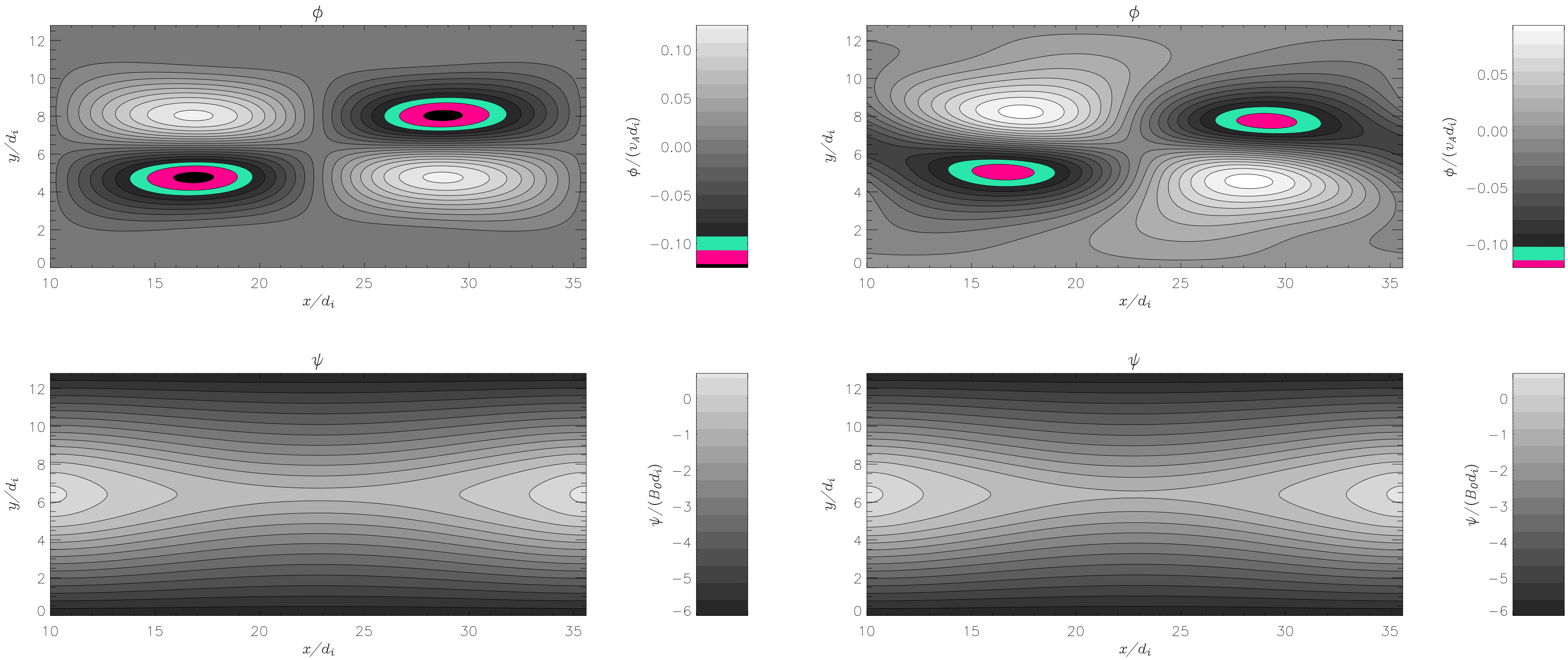
Gyroviscosity appears to increase the rate of reconnection slightly.



This simulation had a guide field $I = 0.5$. The magnitude of the gyroviscous force scales as I^{-2} at the X-point, so the gyroviscous force would be expected to have the greatest effect in null-helicity reconnection.

Four Field Results: Reconnection

ϕ and ψ fields at $t\Omega_i = 27$.



$\beta_i = 0.0$ $\beta_i = 1.0$

- The simulation shows some shear in the velocity field in the case where the gyroviscous force has been included.

Conclusions

- The gyroviscous force causes as much dispersion as the Hall term when $\beta_i \sim 50\%$
- We are successfully able to model the full gyroviscous force in the collisionless regime.
- We are currently testing the eight field linear wave propagation.

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