REDUCED TWO-FLUID EQUATIONS WITH ANISOTROPIC TEMPERATURE GRADIENTS FOR PLASMA DYNAMICS ON THE DIAMAGNETIC DRIFT SCALE

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I. FOREWORD

This note presents a reduced system of finite-Larmor-radius two-fluid equations based on the "paraxial" or "long-thin" geometrical approximation, namely a large-aspect-ratio and long-parallel-wavelength ordering for plasmas in a strong guide magnetic field of weak curvature. The main new features are the full account of diamagnetic effects associated with temperature gradients and the allowance for strong pressure anisotropies, i.e. $(p_\parallel - p_\perp) \sim p \equiv (p_\parallel + 2p_\perp)/3$, with dynamically evolved ion and electron parallel and perpendicular pressures. For the sake of conciseness, the analysis will be limited to the slow dynamics ordering where frequencies and flow velocities are respectively on their diamagnetic drift scales.

II. BASIC TWO-FLUID SYSTEM

The starting system is the set of quasineutral two-fluid equations (single ion species of unit charge), in the limit of negligible electron mass:

$$n_e = n_i \equiv n \ , \tag{1}$$

$$\mathbf{u}_e = \mathbf{u}_i - \frac{1}{en}\mathbf{j} \equiv \mathbf{u} - \frac{1}{en}\mathbf{j} \ , \tag{2}$$

$$\mathbf{j} = \nabla \times \mathbf{B} \ , \tag{3}$$

$$\frac{\partial \mathbf{B}}{\partial t} + \nabla \times \mathbf{E} = 0 \ , \tag{4}$$

$$\frac{\partial n}{\partial t} + \nabla \cdot (n\mathbf{u}) \; = \; 0 \; , \tag{5}$$

$$\mathbf{E} \; = \; -\mathbf{u} \times \mathbf{B} \; + \; \frac{1}{en}\Big(\mathbf{j} \times \mathbf{B} \; - \; \nabla \cdot \mathsf{P}_e \; + \; \mathbf{F}_e^{coll}\Big) \tag{6}$$

and

$$m_i \; n \; \left[\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u}\right] \; + \; \nabla \cdot (\mathsf{P}_e + \mathsf{P}_i) \; - \; \mathbf{j} \times \mathbf{B} \; = \; 0 \; . \tag{7}$$

The fluid-rest-frame stress tensors,

$$\mathsf{P}_\alpha(\mathbf{x}, t) \; \equiv \; m_\alpha \int d^3\mathbf{v}_\alpha \; (\mathbf{v}_\alpha - \mathbf{u}_\alpha)(\mathbf{v}_\alpha - \mathbf{u}_\alpha) \; f_\alpha(\mathbf{v}_\alpha, \mathbf{x}, t) \; , \tag{8}$$

are for the ions

$$\mathsf{P}_i \; = \; p_{i\perp}\mathsf{I} \; + \; (p_{i\|} - p_{i\perp})\mathbf{bb} \; + \; \Pi_i^{gyr} \; + \; \Pi_{i\perp}^{coll} \; , \tag{9}$$

and for the electrons

$$\mathsf{P}_e \; = \; p_{e\perp}\mathsf{I} \; + \; (p_{e\|} - p_{e\perp})\mathbf{bb} \; . \tag{10}$$

In addition, use will be made of the four (two per species) dynamic evolution equations for the CGL components of the stress tensors, $p_\alpha \equiv (p_{\alpha\|} + 2p_{\alpha\perp})/3 = \mathsf{P}_\alpha : \mathsf{I}/3$ and $p_{\alpha\|} = \mathsf{P}_\alpha : (\mathbf{bb})$,

$$\frac{3}{2}\left[\frac{\partial p_\alpha}{\partial t} + \nabla \cdot (p_\alpha \mathbf{u}_\alpha)\right] \; + \; \mathsf{P}_\alpha : (\nabla \mathbf{u}_\alpha) \; + \; \nabla \cdot \mathbf{q}_\alpha \; - \; h_\alpha^{coll} \; = \; 0 \tag{11}$$

and

$$\frac{1}{2}\left[\frac{\partial p_{\alpha\|}}{\partial t} + \nabla \cdot (p_{\alpha\|}\mathbf{u}_\alpha)\right] \; - \; \mathbf{b} \cdot \mathsf{P}_\alpha \cdot \left[\frac{\partial \mathbf{b}}{\partial t} + (\mathbf{u}_\alpha \cdot \nabla)\mathbf{b} - (\mathbf{b} \cdot \nabla)\mathbf{u}_\alpha - \mathbf{b} \times (\nabla \times \mathbf{u}_\alpha)\right] -$$

$$- \; \mathbf{b} \cdot \mathsf{Q}_\alpha : (\nabla \mathbf{b}) \; + \; \nabla \cdot \mathbf{q}_{\alpha B} \; - \; h_{\alpha B}^{coll} \; = \; 0 \; . \tag{12}$$

These involve the fluid-rest-frame stress flux tensors

$$\mathsf{Q}_\alpha(\mathbf{x}, t) \; \equiv \; m_\alpha \int d^3\mathbf{v}_\alpha \; (\mathbf{v}_\alpha - \mathbf{u}_\alpha)(\mathbf{v}_\alpha - \mathbf{u}_\alpha)(\mathbf{v}_\alpha - \mathbf{u}_\alpha) \; f_\alpha(\mathbf{v}_\alpha, \mathbf{x}, t) \; , \tag{13}$$

and, in particular, the heat flux vectors $\mathbf{q}_\alpha = \mathsf{Q}_\alpha : \mathsf{I}/2$ and $\mathbf{q}_{\alpha B} = \mathsf{Q}_\alpha : (\mathbf{bb})/2$.

The collisional friction forces and heat generation rates are

$$\mathbf{F}_\alpha^{coll}(\mathbf{x}, t) \;\equiv\; m_\alpha \int d^3\mathbf{v}_\alpha \; (\mathbf{v}_\alpha - \mathbf{u}_\alpha) \; C_\alpha(\mathbf{v}_\alpha, \mathbf{x}, t) \;, \tag{14}$$

$$h_\alpha^{coll}(\mathbf{x}, t) \;\equiv\; \frac{m_\alpha}{2} \int d^3\mathbf{v}_\alpha \; |\mathbf{v}_\alpha - \mathbf{u}_\alpha|^2 \; C_\alpha(\mathbf{v}_\alpha, \mathbf{x}, t) \tag{15}$$

and

$$h_{\alpha B}^{coll}(\mathbf{x}, t) \;\equiv\; \frac{m_\alpha}{2} \int d^3\mathbf{v}_\alpha \; [(\mathbf{v}_\alpha - \mathbf{u}_\alpha) \cdot \mathbf{b}]^2 \; C_\alpha(\mathbf{v}_\alpha, \mathbf{x}, t) \;, \tag{16}$$

where C_α are the collision operators. Conservation of momentum and energy imply $\mathbf{F}_i^{coll} + \mathbf{F}_e^{coll} = 0$ and $en(h_i^{coll} + h_e^{coll}) = \mathbf{j} \cdot \mathbf{F}_e^{coll}$.

III. ASYMPTOTIC ORDERINGS

This section lists the asymptotic orderings assumed in order to derive the reduced two-fluid system: large-aspect-ratio and small-parallel-gradient paraxial geometry, slow (diamagnetic drift scale) dynamics, low plasma compressibility and low collisionality.

A. Paraxial geometry approximation

This is the main asymptotic ordering. It uses as expansion parameter the ratio $\epsilon \sim a/L_\parallel \ll 1$ between characteristic lengths perpendicular and parallel to the magnetic field. The magnetic field is assumed to be made of a time independent, weakly inhomogeneous background plus a dynamic part of order ϵ relative to the static background:

$$\mathbf{B} \equiv B\mathbf{b} = \mathbf{B}_0 + \mathbf{B}_1 \equiv B_0\mathbf{b}_0 + \mathbf{B}_1 \;, \tag{17}$$

where

$$|\mathbf{b}| = |\mathbf{b}_0| = 1, \quad B_0 = constant \;, \quad \partial\mathbf{b}_0/\partial t = 0 \;, \quad \|\nabla\mathbf{b}_0\| \sim 1/L_\parallel \sim \epsilon/a \quad and \quad |\mathbf{B}_1|/B_0 \sim \epsilon \;. \tag{18}$$

All parallel gradients are assumed to be first order in ϵ relative to the perpendicular gradients:

$$k_\parallel \sim \mathbf{b} \cdot \nabla \sim \mathbf{b}_0 \cdot \nabla \sim \epsilon k_\perp \sim \epsilon/a \;. \tag{19}$$

The ion and electron betas are assumed to be of order ϵ,

$$p_\alpha/B_0^2 \sim v_{thi}^2/v_A^2 \sim \epsilon \ , \tag{20}$$

and the ion and electron pressure anisotropies are assumeed to be of order unity,

$$(p_{\alpha\|} - p_{\alpha\perp}) \sim p_\alpha \ . \tag{21}$$

In the case of an axisymmetric background geometry such as in the application to tokamaks, the background magnetic unit vector $\mathbf{b}_0$ is the azimuthal unit vector $\mathbf{e}_\zeta$ of the cylindrical coordinate system (R, ζ, Z) and, in the plasma domain, $R = R_0 + O(a)$ with $R_0 = O(a/\epsilon) = O(L_\|)$.

B. Slow dynamics ordering

The second expansion parameter is the ratio $\delta \sim \rho_i/a \sim \rho_i k_\perp \ll 1$ between the ion Larmor radius and the perpendicular lengths, and it is required to satisfy $\delta^2 \lesssim \epsilon$. A slow dynamics ordering is assumed, whereby flow velocities and time derivatives are respectively comparable to the diamagnetic drift velocities and frequencies:

$$u_\alpha \sim \frac{j}{en} \sim \delta v_{thi} \qquad and \qquad \frac{\partial}{\partial t} \sim \frac{u_\alpha}{a} \sim \delta^2 \Omega_{ci} \ . \tag{22}$$

In addition, the heat fluxes are also assumed to satisfy their slow dynamics ordering:

$$q_\alpha \sim u_\alpha p_\alpha \sim \delta m_i n v_{thi}^3 \ . \tag{23}$$

Whereas the small ion Larmor radius assumption is necessary in the present fluid description, the slow dynamics ordering is not essential and is adopted here just for the sake of conciseness. A similar analysis can be carried out for the fast (MHD-like) dynamics ordering, $u_\alpha \sim v_{thi}$ and $\partial/\partial t \sim \delta\Omega_{ci}$, or keeping enough terms to cover both the fast and slow orderings. In all cases, the condition $q_{\alpha\|} \sim u_\alpha p_\alpha$ on the parallel heat fluxes is necessary for the asymptotic closure of the reduced two-fluid system.

C. Low compressibility

It is assumed that the particle density is constant in lowest order:

$$n \ = n_0 + n_1 \ , \qquad with \qquad n_0 = constant \qquad and \qquad n_1/n_0 \sim \epsilon \ . \tag{24}$$

Then, consistent with the continuity equation and the parallel component of Faraday's law, the divergence of the flow velocity is first order in ϵ:

$$\nabla \cdot \mathbf{u} \sim \epsilon u/a \ . \tag{25}$$

This assumption is made for simplicity. It allows to express the lowest-order flow velocity in terms of only two scalars and to write the lowest-order convective derivative in the form of a Poisson bracket. The analysis could be extended to the general case of strong density variation and flow compressibility, where the three scalar components of the velocity vector must be considered and the Poisson bracket form of the convective derivative no longer holds.

D. Low collisionality

Finally, it is assumed that the collision frequencies are much smaller than the ion cyclotron frequency, $\nu_{\alpha\beta}^{coll} \ll \Omega_{ci}$. Again, this is not an essential assumption. It is made for the sake of simplicity and of applicability to regimes of interest in magnetic fusion plasmas. The main consequence is that the ion collisional perpendicular stress $\Pi_{i\perp}^{coll}$ is neglected compared to the gyroviscous stress Π_i^{gyr} and the collisional parts of the perpendicular heat fluxes are neglected compared to the corresponding diamagnetic parts.

IV. LOWEST-ORDER FIELDS IN TOKAMAK GEOMETRY

In the tokamak-relevant case of axisymmetric background geometry, the lowest-significant-order magnetic field and current density are

$$\mathbf{B}(R,\zeta,Z,t) \ = \ [B_0 + B_{1\zeta}(R,\zeta,Z,t)]\ \mathbf{e}_\zeta \ + \ \nabla\psi(R,\zeta,Z,t) \times \mathbf{e}_\zeta \ + \ O(\epsilon^2 B_0) \tag{26}$$

and

$$\mathbf{j}(R,\zeta,Z,t) \ = \ \frac{B_0}{R_0}\,\mathbf{e}_Z \ + \ \nabla B_{1\zeta}(R,\zeta,Z,t) \times \mathbf{e}_\zeta \ - \ \Delta_2\psi(R,\zeta,Z,t)\ \mathbf{e}_\zeta \ + \ O(\epsilon^2 B_0/a) \ , \tag{27}$$

with

$$B_{1\zeta}(R,\zeta,Z,t) \ = \ -B_0\left(\frac{R-R_0}{R_0}\right) \ - \ \frac{1}{B_0}\left[p_{i\perp}(R,\zeta,Z,t) + p_{e\perp}(R,\zeta,Z,t)\right] \ + \ O(\epsilon^2 B_0) \tag{28}$$

and the two-dimensional Laplacian operator Δ_2 being defined as

$$\Delta_2 \ \equiv \ \frac{\partial^2}{\partial R^2} + \frac{\partial^2}{\partial Z^2} \ . \tag{29}$$

The ion flow velocity is

$$\mathbf{u}(R,\zeta,Z,t) \ = \ u_\zeta(R,\zeta,Z,t)\ \mathbf{e}_\zeta \ - \ \nabla\phi(R,\zeta,Z,t)\times\mathbf{e}_\zeta \ + \ O(\epsilon u) \tag{30}$$

and the electric potential is

$$\Phi(R,\zeta,Z,t) \ = \ B_0\ \phi(R,\zeta,Z,t) \ - \ \frac{1}{en_0}\ p_{i\perp}(R,\zeta,Z,t) \ + \ O(\epsilon u a B_0) \ + \ O\left(\frac{aF_e^{coll}}{en}\right) . \tag{31}$$

V. REDUCED TWO-FLUID SYSTEM IN TOKAMAK GEOMETRY

After carrying out the reduction of the two-fluid system (1-7, 11-12) under the above discussed orderings, considering in particular the tokamak geometry, the following six-field coupled system is obtained for the lowest-order representations of the ion and electron parallel and perpendicular pressures and the magnetic and flow potentials ψ and ϕ:

$$\frac{1}{2}\frac{d'p_{i\|}}{dt} \ - \ h_{iB}^{coll} \ = \ 0 \ , \tag{32}$$

$$\frac{d'p_{i\perp}}{dt} \ - \ h_i^{coll} \ + \ h_{iB}^{coll} \ = \ 0 \ , \tag{33}$$

$$\frac{1}{2}\frac{d'p_{e\|}}{dt} \ - \ h_{eB}^{coll} \ = \ 0 \ , \tag{34}$$

$$\frac{d'p_{e\perp}}{dt} \ - \ h_i^{coll} \ + \ h_{eB}^{coll} \ = \ 0 \ , \tag{35}$$

$$\frac{\partial\psi}{\partial t} \ + \ (\mathbf{B}\cdot\nabla)_0\left(\phi - \frac{p_{i\perp}+p_{e\|}}{en_0B_0}\right) \ + \ \mathbf{e}_\zeta\cdot\left(\frac{\mathbf{F}_e^{coll}}{en}\right) \ = \ 0 \tag{36}$$

and

$$\frac{d'}{dt}(\Delta_2\phi) \ + \ (\mathbf{B}\cdot\nabla)_0\left(\frac{\Delta_2\psi}{m_in_0}\right) \ - \ \frac{\Lambda(p_{i\perp},\phi)}{en_0B_0} \ - \ \mathbf{e}_Z\cdot\nabla\left(\frac{p_{i\|}+p_{i\perp}+p_{e\|}+p_{e\perp}}{m_in_0R_0}\right) = 0 \ . \tag{37}$$

Here, use has been made of the shorthand notations:

$$\frac{d'}{dt} \;\equiv\; \frac{\partial}{\partial t} \;+\; \left[\mathbf{e}_\zeta \times \nabla\left(\phi - \frac{p_{i\perp}}{en_0B_0}\right)\right] \cdot \nabla \;, \tag{38}$$

$$(\mathbf{B}\cdot\nabla)_0 \;\equiv\; (B_0\mathbf{e}_\zeta + \nabla\psi \times \mathbf{e}_\zeta)\cdot\nabla \tag{39}$$

and

$$\Lambda(p_{i\perp},\phi) \;\equiv\; \frac{\partial^2 p_{i\perp}}{\partial R\partial Z}\left(\frac{\partial^2\phi}{\partial R^2} - \frac{\partial^2\phi}{\partial Z^2}\right) \;-\; \frac{\partial^2\phi}{\partial R\partial Z}\left(\frac{\partial^2 p_{i\perp}}{\partial R^2} - \frac{\partial^2 p_{i\perp}}{\partial Z^2}\right) . \tag{40}$$

The azimuthal component of the velocity u_ζ and the first-order density n_1 are not involved in the reduced system (32-37) that couples the six primary fields $p_{\alpha\|}$, $p_{\alpha\perp}$, ψ and ϕ. Their evolution equations, which can be integrated after the solution for the primary fields has been obtained, are:

$$\frac{d'u_\zeta}{dt} \;+\; (\mathbf{B}\cdot\nabla)_0\left(\frac{p_{i\|}+p_{e\|}}{m_in_0B_0}\right) \;=\; 0\;. \tag{41}$$

and

$$\frac{d'n_1}{dt} + (\mathbf{B}\cdot\nabla)_0\left(\frac{n_0u_\zeta}{B_0}+\frac{\Delta_2\psi}{eB_0}\right) + \mathbf{e}_\zeta\cdot\left(\frac{\nabla p_{i\perp}\times\nabla p_{e\perp}}{eB_0^3}\right) + \mathbf{e}_Z\cdot\nabla\left(\frac{2n_0\phi}{R_0} - \frac{2p_{i\perp}+p_{e\|}+p_{e\perp}}{eB_0R_0}\right) \;=\; 0\,. \tag{42}$$

It is worth stressing that, despite its formal simplicity, this reduced system takes into account all the two-fluid effects associated with the Hall physics in the generalized Ohm's law, the ion gyroviscosity, the ion and electron pressure anisotropies (sometimes called parallel viscosities) and the diamagnetic perpendicular heat fluxes, within the assumed orderings. The parallel heat fluxes do not contribute to these lowest-order reduced equations by virtue of the orderings specified in Eqs.(19) and (23), with the result that the system is consistently closed except for the collisional terms. The toroidal effects that break the $R \leftrightarrow Z$ symmetry are represented by the last terms in Eqs.(37) and (42); these two terms would be missing in the case of a straight ($R_0 = \infty$) background geometry.