Gravitational Instability with 2-Fluid and FLR Effects

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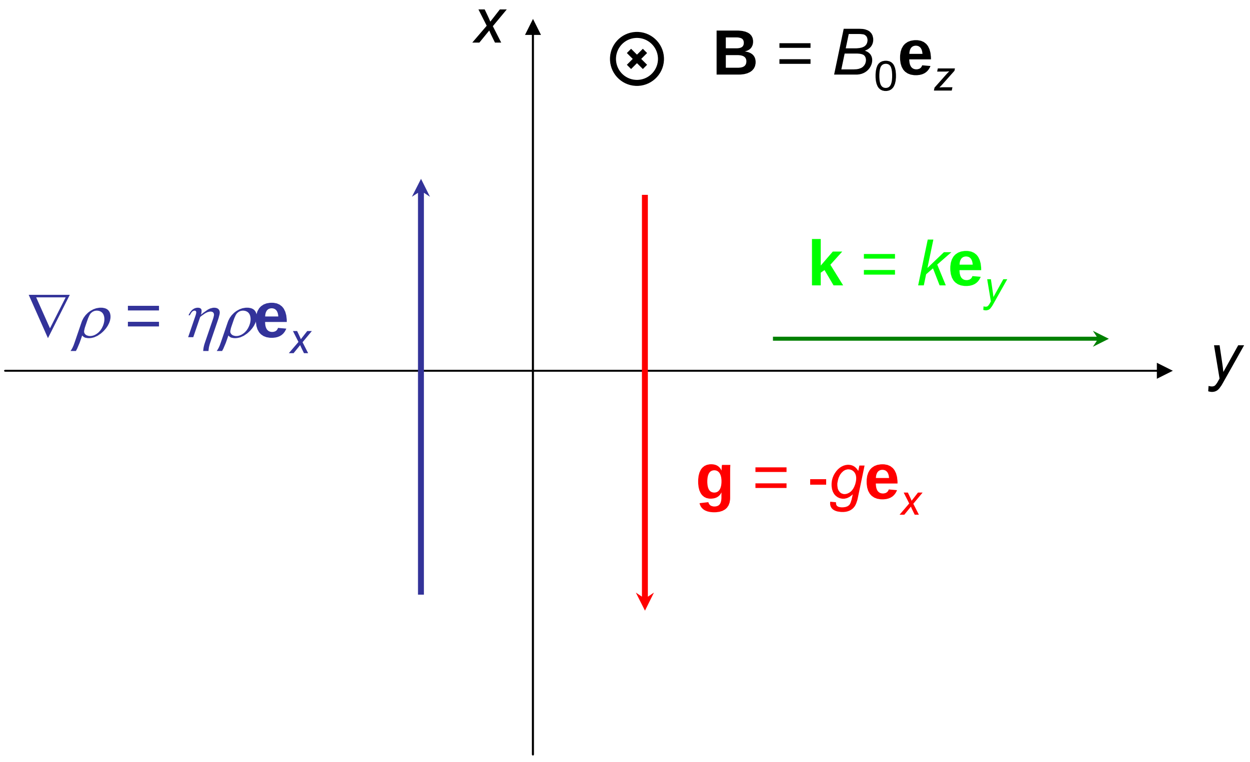
Opening Remarks….

- *There is nothing fundamentally new here!*
- The results are “well known”
- It’s all been done before (1962)!
- Looking for “simple” test case for 2-fluid/FLR
- Self-education

Motivation

- Need “simple” problem with known solution for testing/benchmarking new NIMROD capabilities
- Revisit Roberts & Taylor [PRL **8**, 197 (1962)]
  - A classic!
  - Terse! 2 pages in PRL, including references
- Gravitational instability in 2-D slab with 2-fluid Ohm’s law and Gyro-viscosity
  - Analytic solution
  - 2-fluid and FLR stabilization
- ** Investigate effect of ion heat stress

Geometry



x
$\mathbf{B} = B_0\mathbf{e}_z$
$\mathbf{k} = k\mathbf{e}_y$
$\nabla\rho = \eta\rho\mathbf{e}_x$
y
$\mathbf{g} = -g\mathbf{e}_x$

Equations

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \mathbf{V} = 0$$

$$\rho \frac{d\mathbf{V}}{dt} = -\nabla\left(p + \frac{B^2}{2\mu_0}\right) + \rho \mathbf{g} - \nabla \cdot \Pi$$

$$\mathbf{E} = -\mathbf{V} \times \mathbf{B} + \frac{M}{\rho e}\left[\rho \frac{d\mathbf{V}}{dt} + \nabla p_i - \rho \mathbf{g} + \nabla \cdot \Pi\right]$$

$$\Pi_{xx} = -\Pi_{yy} = -\rho\nu\left(\frac{\partial V_y}{\partial x} + \frac{\partial V_x}{\partial y}\right)$$

$$\Pi_{xy} = \Pi_{yx} = \rho\nu\left(\frac{\partial V_x}{\partial x} - \frac{\partial V_y}{\partial y}\right)$$

$$\nu = a^2\Omega / 2 \qquad a^2 = V_{th}^2 / \Omega^2$$

Equilibrium

$$\left.\begin{aligned} &\frac{d}{dx}\left(p_0+\frac{B_0^2}{2\mu_0}\right)=-\rho_0 g \\ &\frac{d\rho_0}{dx}=\eta\rho_0 \quad , \qquad \eta\equiv 1/L_n \end{aligned}\right\} \textit{Only } p_T=p_0+\frac{B_0^2}{2\mu_0} \quad \textit{matters}$$

$p=p(\rho)$ *(barotropic) :*

$$\frac{\partial p_0}{\partial x}=\left(\frac{\partial p}{\partial \rho}\right)_0\frac{\partial \rho_0}{\partial x}=C_s^2\eta\rho_0>0 \qquad p_0 \;\; \textit{must increase with } x$$

$$\frac{d}{dx}\left(\frac{B_0^2}{2\mu_0}\right)=-\left(g+\eta C_s^2\right)\rho_0(x) \qquad B_0 \;\; \textit{cannot be constant}$$

Simplifying Assumptions

Only variations in $p_T = p + B^2 / 2\mu_0$ *affect dynamics*

$\Rightarrow$ *Ignore perturbations to B*

$\Rightarrow$ $\nabla \times \mathbf{E} = 0$ *(low* β*, electrostatic)*

Assume ions are barotropic, $p_i = p_i(\rho)$

$\Rightarrow$ *Simplifies Ohm's law*

Variation in x much weaker than variation in y

$\Rightarrow$ $\eta^2 << k^2$

$\Rightarrow$ *Can ignore explicit x-dependence of equilibrium*

Assume $\exp(i\omega t + iky)$ *dependence*

$\Rightarrow$ *Linearized equations are algebraic*

Final Equations

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \mathbf{V} = 0$$

$$\rho \frac{d\mathbf{V}}{dt} = -\nabla p_T + \rho \mathbf{g} - \nabla \cdot \Pi$$

$$\nabla \cdot \mathbf{V} + \frac{1}{\Omega} \nabla \times \left[\frac{d\mathbf{V}}{dt} - \frac{1}{\rho^2} \nabla \rho \times \nabla \cdot \Pi \right] = 0$$

Plus definition of Π

4 equations in 4 unknowns : ρ, $\mathbf{V}$, p_T

Last equation serves as "equation of state", or closure

Linearized Equations

$$(\nabla\cdot\Pi)_x = -\underbrace{\nu_0\,\eta\rho_0 ikV_x}_{\text{Equilibrium variation}} + \rho_0\,\nu_0 k^2 V_y \quad , \quad (\nabla\cdot\Pi)_y = -\underbrace{\nu_0\,\eta\rho_0 ikV_y}_{\text{Equilibrium variation}} - \rho_0\,\nu_0 k^2 V_x$$

Ignores equilibrium variaton of Ω_0

$$i\omega\rho \quad + \quad \eta\rho_0 V_x \quad + \quad ik\rho_0 V_y \quad = 0$$

$$\frac{g}{\rho_0}\rho \quad + \quad (i\omega - \zeta\eta\nu_0 ik)V_x \quad + \quad \zeta\nu_0 k^2 V_y \quad = 0$$

$$-\zeta\nu_0 k^2 V_x \quad + \quad (i\omega - \zeta\eta\nu_0 ik)V_y \quad + \quad \frac{ik}{\rho_0}p_T = 0$$

$$\left(\xi\frac{\omega k}{\Omega_0} + \xi\zeta\frac{\nu_0\,\eta}{\Omega_0}k^2\right)V_x \quad + \quad \left(1 + \xi\zeta\frac{\eta^2\,\nu_0}{\Omega_0}\right)ikV_y \quad = 0$$

$\xi = 0,\ \zeta = 0$ *Ideal MHD*

$\xi = 1,\ \zeta = 0$ *2 - fluid*

$\xi = 0,\ \zeta = 1$ *Gyro - viscosity*

$\xi = 1,\ \zeta = 1$ *Extended MHD*

Ideal MHD

$$\xi = 0, \quad \zeta = 0:$$

$$\omega^2 + g\eta = 0$$ *Always unstable!*

$$\gamma = \sqrt{g\eta}$$

2-Fluid Only

$\xi = 1, \;\; \zeta = 0:$

$$\omega^2 - \frac{gk}{\Omega_0}\omega + g\eta = 0$$

$$2\omega = \frac{gk}{\Omega_0} \pm \sqrt{\left(\frac{gk}{\Omega_0}\right)^2 - 4g\eta}$$

Stable for :

$$k^2 > k_{2F}^2 = \frac{4\eta\Omega_0^2}{g}$$

Gyro-viscosity Only

$\xi = 0, \ \ \zeta = 1:$

$$\omega^2 - \nu_0 \eta k \omega + g\eta = 0$$

$$2\omega = \nu_0 \eta k \pm \sqrt{(\nu_0 \eta k)^2 - 4g\eta}$$

Stable for :

$$k^2 > k_{GV}^2 = \frac{4g}{\nu_0^2 \eta}$$

Extended MHD

$\xi = 1, \;\; \zeta = 1:$

$$\left[1+\frac{\nu_0}{\Omega_0}\left(\eta^2+k^2\right)\right]\omega^2-\left\{\frac{gk}{\Omega_0}+\nu_0\eta k\left[1+\frac{\nu_0}{\Omega_0}\left(\eta^2-k^2\right)\right]\right\}\omega$$
$$+g\eta\left[1+\frac{\nu_0}{\Omega_0}\left(\eta^2-k^2\right)\right]=0$$

$\nu_0 k^2/\Omega_0=(ka)^2/2<<1, \quad \eta^2<<k^2:$

$$\omega^2-\left(\frac{gk}{\Omega_0}+\nu_0\eta k\right)\omega+g\eta=0$$

$$2\omega=\frac{gk}{\Omega_0}+\nu_0\eta k\pm\sqrt{\left(\frac{gk}{\Omega_0}+\nu_0\eta k\right)^2-4g\eta} \quad .$$

Stable for :

$$k^2>k_{EMHD}^2=\frac{4g\eta}{\left(\dfrac{g}{\Omega_0}+\nu_0\eta\right)^2}$$

Other Possible Benchmarks

- Huba, J.D., *Finite Larmor radius magnetohydrodynamics of the Rayleigh-Taylor instability*, Phys. Plasmas **3**, 2523, 1996.
- Huba, J.D., *The Kelvin-Helmholtz instablity: Finite Larmor radius magnetohydrodynamics*, Geophys. Res. Lett. **23**, 2907, 1996.
- Huba, J.D. and D. Winske, *Rayleigh-Taylor instability: Comparisonof hybrid and nonideal magnetohydrodynamic simulations*, Phys. Plasmas **5**, 2305, 1998.

What about the Ion Heat Stress?

- GV stress should contain additional terms (Mikhailovskii & Tsypin, Hazeltine & Meiss, Simakov & Catto, Ramos, …..)

$$\Pi\wedge = \Pi\wedge_{gv} + \Pi\wedge_{q}$$

$$\Pi\wedge_{q} = \frac{2}{5\Omega}\left[\mathbf{b}\times\mathbf{W}_{q}\cdot(\mathbf{I}+3\mathbf{bb})+\mathit{transpose}\right]$$

$$\mathbf{W}_{q} = \nabla\mathbf{q}_{i} + \nabla\mathbf{q}_{i}^{T} - \frac{2}{3}\mathbf{I}\nabla\cdot\mathbf{q}_{i}$$

- What is the effect of this term on
  - Wave propagation?
  - Stability?

New Coupling with Energy Equation

- Coupling between stress tensor and ion energy equation

$$\rho_0 \frac{\partial \mathbf{V}}{\partial t} = -\nabla p - \nabla \cdot \Pi_{\wedge q}$$

$$\frac{\partial p}{\partial t} = -\gamma p_0 \nabla \cdot \mathbf{V}$$

$$\mathbf{q} = -\kappa_{||} \nabla_{||} T - \kappa_{\perp} \nabla_{\perp} T - \kappa_{\wedge} \mathbf{b} \times \nabla_{\perp} T$$

Effect on Sound Waves

$$i\omega\rho_0 V_x = \frac{ik^3}{5\Omega n_0} F_1(\theta) p$$

$$i\omega\rho_0 V_y = \left(-ik\sin\theta - \frac{ik^3}{5\Omega n_0} F_2(\theta)\right) p$$

$$i\omega\rho_0 V_z = \left(-ik\cos\theta + \frac{ik^3}{5\Omega n_0} F_3(\theta)\right) p$$

$$i\omega p = -i\gamma p_0 k\left(V_y \sin\theta + V_z \cos\theta\right)$$

$$F_1(\theta) = \left(2\sin^2\theta + \cos^2\theta\right)\left(\kappa_\perp \sin\theta - \kappa_\wedge \cos\theta\right) - \kappa_{||}\cos^2\theta\sin\theta$$

$$F_2(\theta) = \kappa_\wedge \sin\theta\left(2\sin^2\theta + \cos^2\theta\right)$$

$$F_3(\theta) = \kappa_\wedge \sin^2\theta\cos\theta$$

Sound Wave Dispersion Relation

$$\omega^2 = C_s^2 k^2 \left[1 + \frac{\kappa_\wedge}{5 n_0 \Omega} f(\theta) k^2 \right]$$

$$f(\theta) = \frac{1}{2} \sin\theta \left(2\sin^2\theta + \cos^2\theta - \sin\theta\cos\theta \right) \qquad f(0) = 0 \qquad f(\pi/2) = 1$$

$$\kappa_\wedge = 5 n_0 T / 2 m_i \Omega$$

$$\omega^2 = C_s^2 k^2 \left[1 + f(\theta) (\rho_i k)^2 \right]$$

- Parallel sound wave unaffected!

Effect on g-mode

$$i\omega\rho + \eta\rho_0 V_x + ik\rho_0 V_y = 0$$

$$\frac{g}{\rho_0}\rho + (i\omega - \eta\nu_0 ik)V_x + \nu_0 k^2 V_y = 0$$

$$-\nu_0 k^2 V_x + (i\omega - \eta\nu_0 ik)V_y + \frac{ik}{\rho_0}\left(1+\boxed{\frac{2\kappa_{\wedge} k^2}{5n_0\Omega}}\right)p_T = 0$$

$$\left(\frac{\omega k}{\Omega_0} + \frac{\nu_0\eta}{\Omega_0}k^2\right)V_x + \left(1+\frac{\eta^2\nu_0}{\Omega_0}\right)ikV_y - \boxed{\frac{ik}{\rho_0}\frac{2\kappa_{\wedge} k^2}{5n_0\Omega}}\,p_T = 0$$

g-mode Stability

$$2\omega = \frac{gk}{\Omega}\left(1+\frac{\eta^2}{k^2}\right)+\nu_0\eta k$$

$$\pm\sqrt{\left[\frac{gk}{\Omega}\left(1+\frac{\eta^2}{k^2}\right)+\nu_0\eta k\right]^2-4g\eta}$$

- Same as Roberts-Taylor result to within

$$O(\eta^2/k^2) << 1$$

- Negligible effect within the assumptions of R-T treatment

Effect of Ion Heat Stress

- Parallel sound propagation unaffected (uniform medium, etc.)
  - Dispersive FLR correction for oblique propagation
- g-mode stability unaffected [to $O(1/(kL)^2) \ll 1$] with R-T assumptions
- Action: *Ignore this term for now!*

Summary

- Gravitational instability in extended MHD has analytic solution (known in 1962!)
- Relatively simple, 2-D slab problem
- Makes definite predictions about growth rate, real frequency, and stabilization
- Can be used to make quantitative benchmark of extended MHD computations
- Ion heat stress will be ignored in initial implementations