Implementation and Verification of Braginskii's Gyroviscous Stress in M3D-C1

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The Gyroviscous Force

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- We currently use Braginskii's form of the gyroviscous stress for Π^{gv} [1, 2]:

$$\Pi^{gv}_{\mu\nu} = \frac{p^{(i)}}{4B^2}\varepsilon_{\rho\sigma(\mu}\left[\delta_{\nu)\tau} + 3\frac{B_{\nu)}B_{\tau}}{B^2}\right]B_{\rho}v_{(\sigma;\tau)}.$$

- This term represents the first order finite-Larmor radius (FLR) contribution to the fluid equations [1, 2], in the limit where $\Omega_i \gg \nu_i$.

- In toroidal geometry, it is the appropriate closure for the "high-collisionality" regime in which ν_i is greater than the bounce frequency.

- Though it is termed "viscosity," it is not dissipative.

- Particles simulations of reconnection have shown [3] that for the ions, the pressure tensor balances the reconnection electric field at the X-point.

- Gyroviscosity is stabilizing to "slow" instabilities, for which $\gamma/\Omega_i \lesssim k\rho_i$ [4, 5].

Gyroviscous Force in Galerkin Form

Operating on $\nabla\cdot\Pi$ with $-\hat{\mathbf{z}}\cdot\nabla\times$, $\hat{\mathbf{z}}\cdot$, and $\nabla\cdot$, multiplying by the trial function ν_i, and integrating by parts until no field is differentiated more than twice, yields:

$$\frac{p^{(i)}}{2B^2}\left\{\begin{array}{l} I\left(2+3\frac{\psi_{,\mu}\psi_{,\mu}}{B^2}\right)\nu_{i,\nu[x}U_{,y]\nu}+ \\ +\frac{1}{2}\left[\left(1-3\frac{I^2}{B^2}\right)\left(\psi_{,[x}\nu_{i,y]\mu}V_{,\mu}+V_{,[x}\nu_{i,y]\mu}\psi_{,\mu}\right)-\frac{3}{B^2}\psi_{,[x}V_{,y]}\left(\nu_{i,\mu\mu}\psi_{,\nu}\psi_{,\nu}+2\psi_{,[x}\nu_{i,y][x}\psi_{,y]}\right)\right]+ \\ +I\left[\left(2+3\frac{\psi_{,\mu}\psi_{,\mu}}{B^2}\right)\nu_{i,\nu\rho}\chi_{,\nu\rho}-\left(1+3\frac{\psi_{,\mu}\psi_{,\mu}}{B^2}\right)\nu_{i,\nu\nu}\chi_{,\rho\rho}-\frac{3}{B^2}\psi_{,[x}\nu_{i,y][x}\psi_{,y]}\chi_{,\mu\mu}\right] \end{array}\right\}$$

$$\frac{p^{(i)}}{2B^2}\left\{\begin{array}{l} \frac{1}{2}\left[\left(1-3\frac{I^2}{B^2}\right)\left(\psi_{,[x}U_{,y]\mu}\nu_{i,\mu}+\nu_{i,[x}U_{,y]\mu}\psi_{,\mu}\right)-\frac{3}{B^2}\psi_{,[x}\nu_{i,y]}\left(U_{,\mu\mu}\psi_{,\nu}\psi_{,\nu}+2\psi_{,[x}U_{,y][x}\psi_{,y]}\right)\right]- \\ -\frac{1}{2}I\left(1+3\frac{I^2-\psi_{,\mu}\psi_{,\mu}}{B^2}\right)\nu_{i,[x}V_{,y]}+ \\ +\nu_{i,\mu}\chi_{,\mu\nu}\psi_{,\nu}+\frac{3}{B^2}\left(\psi_{,[x}\nu_{i,y]}\psi_{,[x}\chi_{,y]\mu}\psi_{,\mu}-I^2\nu_{i,[x}\chi_{,y][x}\psi_{,y]}\right) \end{array}\right\}$$

$$\frac{p^{(i)}}{2B^2}\left\{\begin{array}{l} -I\left[\left(2+3\frac{\psi_{,\mu}\psi_{,\mu}}{B^2}\right)U_{,\nu\rho}\nu_{i,\nu\rho}-\left(1+3\frac{\psi_{,\mu}\psi_{,\mu}}{B^2}\right)U_{,\nu\nu}\nu_{i,\rho\rho}-\frac{3}{B^2}\psi_{,[x}U_{,y][x}\psi_{,y]}\nu_{i,\mu\mu}\right]+ \\ +V_{,\mu}\nu_{i,\mu\nu}\psi_{,\nu}+\frac{3}{B^2}(\psi_{,[x}V_{,y]}\psi_{,[x}\nu_{i,y]\mu}\psi_{,\mu}-I^2V_{,[x}\nu_{i,y][x}\psi_{,y]})+ \\ +I\left[\left(2+3\frac{\psi_{,\mu}\psi_{,\mu}}{B^2}\right)\nu_{i,\nu[x}\chi_{,y]\nu}-\frac{3}{B^2}\left(\nu_{i,\mu\mu}\psi_{,[x}\chi_{,y]\nu}\psi_{,\nu}-\chi_{,\mu\mu}\psi_{,[x}\nu_{i,y]\nu}\psi_{,\nu}\right)\right] \end{array}\right\}$$

Matrix Representation of $\nabla \cdot \Pi$

$n\partial_t \mathbf{v} = -\nabla \cdot \Pi$ can be written in the form:

$$\begin{pmatrix} L_{ij}^{11} & L_{ij}^{12} & L_{ij}^{13} \\ L_{ij}^{21} & L_{ij}^{22} & L_{ij}^{23} \\ L_{ij}^{31} & L_{ij}^{32} & L_{ij}^{33} \end{pmatrix} \begin{pmatrix} \dot{U}_j \\ \dot{V}_j \\ \dot{\chi}_j \end{pmatrix} = \begin{pmatrix} R_{ij}^{11} & R_{ij}^{12} & R_{ij}^{13} \\ R_{ij}^{21} & R_{ij}^{22} & R_{ij}^{23} \\ R_{ij}^{31} & R_{ij}^{32} & R_{ij}^{33} \end{pmatrix} \begin{pmatrix} U_j \\ V_j \\ \chi_j \end{pmatrix}$$

- In general, each element of $\overleftrightarrow{R}$ is sixth-order nonlinear. The evaluation of such a term using analytic integrations requires $\mathcal{O}(N^6)$ operations, where N is the number of trial functions.
- By introducing a few auxiliary fields, we have been able to rewrite each term as a fourth-order nonlinear term. The equations determining the auxiliary fields are also fourth-order nonlinear.
- This results in a reduction of operations by a factor of $\mathcal{O}(N^2)$.
- This does not introduce error of a higher order than the previous truncation error.

Matrix Representation of $\nabla \cdot \Pi$

We define the following auxiliary variables:

$$Y^{(1)} = p^{(i)}/B^2; \quad Y^{(2)} = 3Y^{(1)}/B^2; \quad Y^{(3)} = Y^{(2)} I^2;$$

$$Y^{(4)} = Y^{(2)} \psi_{,\nu} \psi_{,\nu}/2; \quad Y^{(5)} = Y^{(2)} (\psi_{,x}^2 - \psi_{,y}^2)/2; \quad Y^{(6)} = Y^{(2)} \psi_{,x} \psi_{,y}.$$

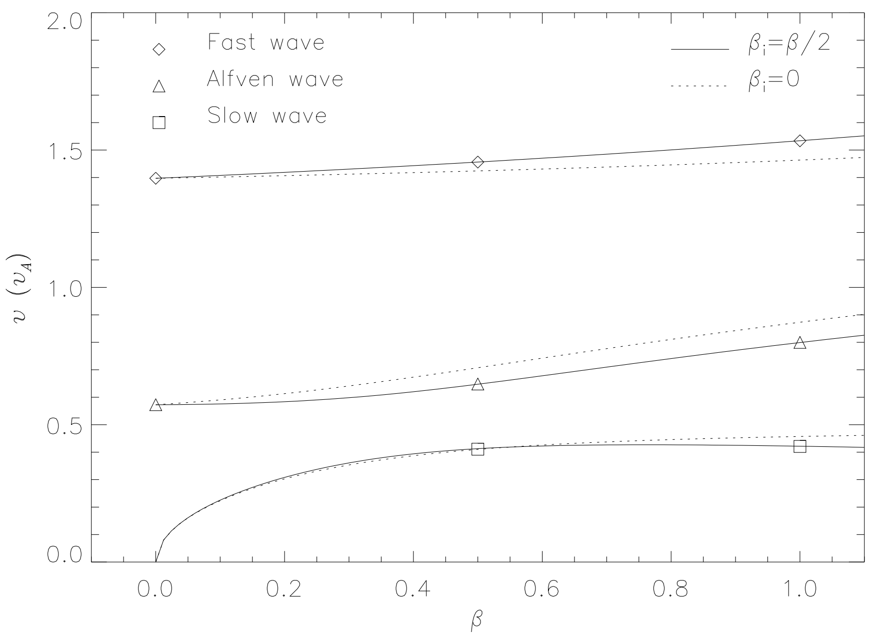
The elements R_{ij} can now be written:

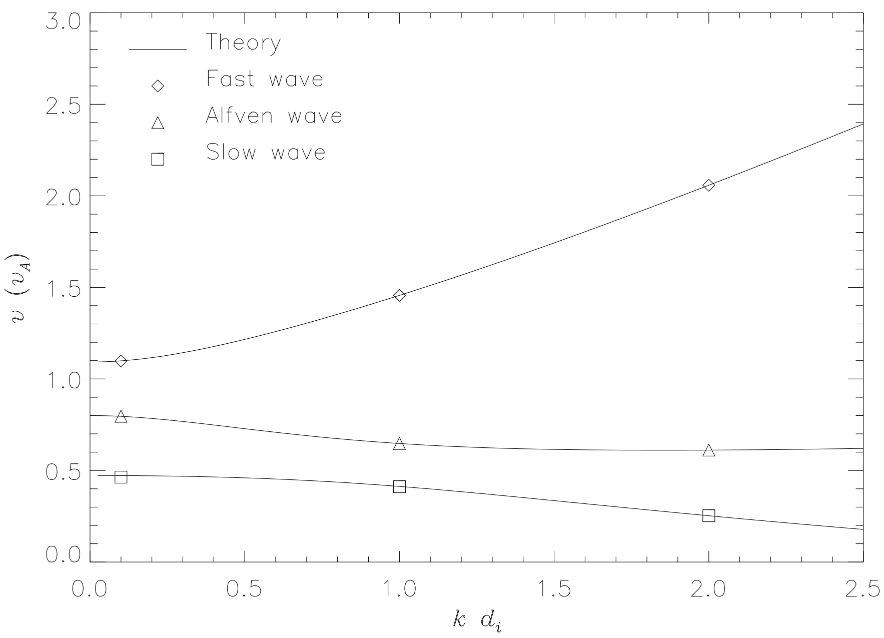
$$\begin{aligned}
R_{ij}^{11} &= G_{ijkl}^{(11a)} I_k (Y_l^{(1)} + Y_l^{(4)}) \\
R_{ij}^{12} &= G_{ijkl}^{(12a)} \psi_k (Y_l^{(1)} - Y_l^{(3)}) + G_{ijkl}^{(12b)} \psi_k Y_l^{(5)} + G_{ijkl}^{(12c)} \psi_k Y_l^{(6)} \\
R_{ij}^{13} &= G_{ijkl}^{(13a)} I_k (Y_l^{(1)} + Y_l^{(4)}) + G_{ijkl}^{(13b)} I_k Y_l^{(5)} + G_{ijkl}^{(13c)} I_k Y_l^{(6)} \\
R_{ij}^{21} &= G_{jikl}^{(12a)} \psi_k (Y_l^{(1)} - Y_l^{(3)}) + G_{jikl}^{(12b)} \psi_k Y_l^{(5)} + G_{jikl}^{(12c)} \psi_k Y_l^{(6)} \\
R_{ij}^{22} &= G_{ijkl}^{(22a)} I_k (Y_l^{(1)} + Y_l^{(3)} - 2Y_l^{(4)}) \\
R_{ij}^{23} &= G_{ijkl}^{(23a)} \psi_k Y_l^{(1)} + G_{ijkl}^{(23b)} \psi_k Y_l^{(3)} - G_{jikl}^{(12c)} \psi_k Y_l^{(5)} + G_{jikl}^{(12b)} \psi_k Y_l^{(6)} \\
R_{ij}^{31} &= -G_{ijkl}^{(13a)} I_k (Y_l^{(1)} + Y_l^{(4)}) - G_{jikl}^{(13b)} I_k Y_l^{(5)} - G_{jikl}^{(13c)} I_k Y_l^{(6)} \\
R_{ij}^{32} &= G_{jikl}^{(23a)} \psi_k Y_l^{(1)} + G_{jikl}^{(23b)} \psi_k Y_l^{(3)} - G_{ijkl}^{(12c)} \psi_k Y_l^{(5)} + G_{ijkl}^{(12b)} \psi_k Y_l^{(6)} \\
R_{ij}^{33} &= G_{ijkl}^{(11a)} I_k (Y_l^{(1)} + Y_l^{(4)}) + G_{ijkl}^{(33b)} I_k Y_k^{(5)} + G_{ijkl}^{(33c)} I_k Y_k^{(6)}
\end{aligned}$$

$$\begin{aligned}
G_{ijkl}^{(11a)} B_j C_k D_l &= -\nu_{i,\mu[x} B_{,y]\mu} C D \\
G_{ijkl}^{(12a)} B_j C_k D_l &= -\frac{1}{2} \left(C_{,[x} \nu_{i,y]\mu} B_{,\mu} + B_{,[x} \nu_{i,y]\mu} C_{,\mu} \right) D \\
G_{ijkl}^{(12b)} B_j C_k D_l &= \frac{1}{2} C_{,[x} B_{,y]} (\nu_{i,xx} - \nu_{i,yy}) D \\
G_{ijkl}^{(12c)} B_j C_k D_l &= C_{,[x} B_{,y]} (\nu_{i,xy}) D \\
G_{ijkl}^{(13a)} B_j C_k D_l &= -\left(\nu_{i,\mu\nu} B_{,\mu\nu} + \frac{1}{2} \nu_{i,\mu\mu} B_{,\nu\nu} \right) C D \\
G_{ijkl}^{(13b)} B_j C_k D_l &= \frac{1}{2} B_{,\mu\mu} (\nu_{i,xx} - \nu_{i,yy}) C D \\
G_{ijkl}^{(13c)} B_j C_k D_l &= B_{,\mu\mu} (\nu_{i,xy}) C D \\
G_{ijkl}^{(22a)} B_j C_k D_l &= \frac{1}{4} \nu_{i,[x} B_{,y]} C D \\
G_{ijkl}^{(23a)} B_j C_k D_l &= -\frac{1}{2} \nu_{i,\mu} B_{,\mu\nu} C_{,\nu} D \\
G_{ijkl}^{(23b)} B_j C_k D_l &= \frac{1}{2} \nu_{i,[x} B_{,y][x} C_{,y]} D \\
G_{ijkl}^{(33b)} B_j C_k D_l &= (\nu_{i,\mu\mu} B_{,xy} - B_{,\mu\mu} \nu_{i,xy}) C D \\
G_{ijkl}^{(33c)} B_j C_k D_l &= (\nu_{i,xx} B_{,yy} - \nu_{i,yy} B_{,xx}) C D
\end{aligned}$$

Code Validation: Linear Waves

- We have analytically calculated the eigenmodes of a homogeneous, stationary equilibrium for our full extended-MHD model.
- Our simulations are in excellent agreement with these calculations.





Gravitational Instability

- An equilibrium where $\mathbf{g} \cdot \nabla \rho < 0$ may be unstable.
- This gravitational instability (GI) is known to be stabilized by FLR effects [4], and its linear growth rate has been previously calculated taking gyroviscosity into account [5]. Schnack suggested using the GI to validate implementations of gyroviscosity [6].
- We have derived a more general linear dispersion relation, taking into account compressibility, finite β, and $\mathbf{k} \cdot \mathbf{g} \neq 0$. Starting from an equilibrium where

$$n(y) = n_0 e^{y/L_n}; \quad p(y) = Tn(y); \quad I(y) = \sqrt{I_0^2 - 2(gL_n + \Gamma T)[n(y) - n_0]}$$

we find the linear dispersion relation to be

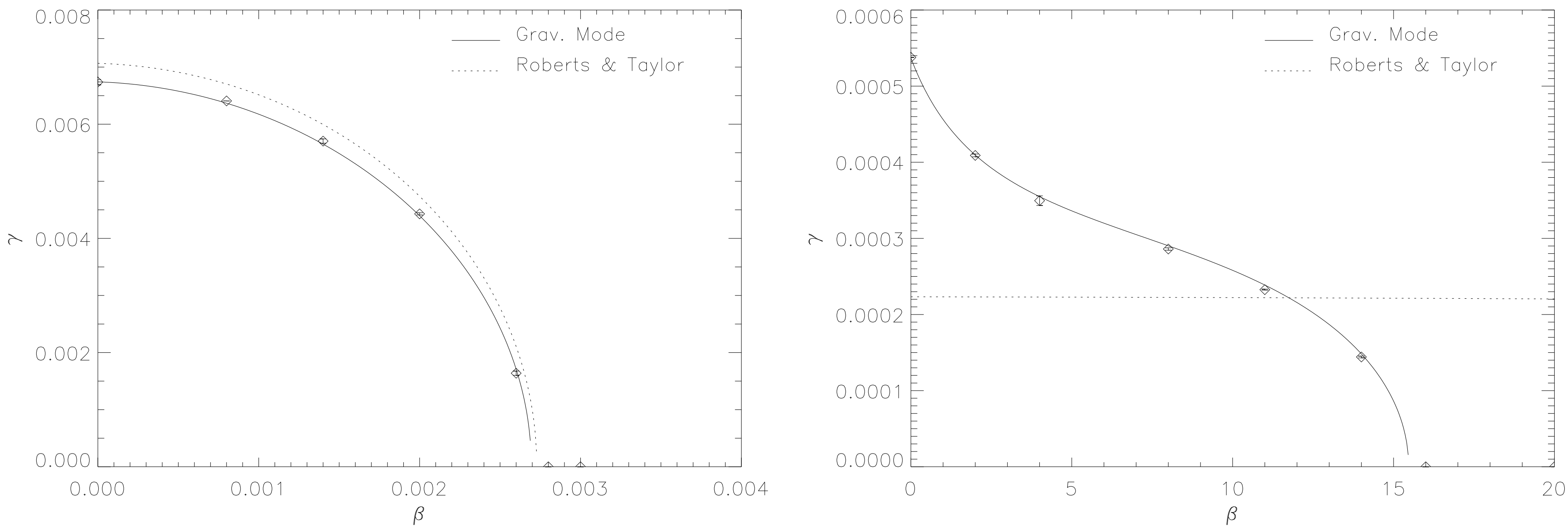
$$0 = 1 + \bar{g} + \bar{\beta} - \left[2\bar{\nu}\left(1 + \bar{g} + \bar{\beta}\right)\left(1 + \bar{\beta}\right) + 2\bar{\nu}\bar{g} + \frac{1}{\bar{\Omega}}\right]\bar{\omega} + \left(1 + \bar{\beta}\right)\bar{\omega}^2.$$

in the limit where $kL_n \gg 1 \gg k\rho_i$ and $\bar{g} \sim 1$, where

$$\bar{\omega} = \frac{k\omega}{k_x\sqrt{g/L_n}}; \quad \bar{\Omega} = \frac{\Omega_i}{k\sqrt{gL_n}}; \quad \bar{\nu} = \frac{\rho_i^2 \Omega_i k}{2\sqrt{gL_n}}; \quad \bar{\beta} = \Gamma \frac{p_0}{B^2}; \quad \bar{g} = \frac{gL_n}{v_A^2}.$$

Gravitational Instability: Linear Results

Our simulations agree closely with our analytic dispersion relation in both low-β and high-β regimes.



The dispersion relation given by Roberts and Taylor is not valid in the high-β regime.

$\mathbf{v}_*$ Approximation

- Ramos has shown that the Braginskii's form of the gyroviscous force can be written [2]:

$$\nabla \cdot \Pi = -mn\mathbf{v}_* \cdot \nabla \mathbf{v} - \nabla \chi - \nabla \times \left\{ \frac{mp_\perp}{eB} \left[\hat{\mathbf{b}} \cdot \nabla \mathbf{v} + \frac{1}{2} \left(\nabla \cdot \mathbf{v} - 3\hat{\mathbf{b}} \cdot [\hat{\mathbf{b}} \cdot \nabla \mathbf{v}] \right) \hat{\mathbf{b}} \right] \right\} + \\ + \mathbf{B} \cdot \nabla \left[\frac{mp_\perp}{eB} \hat{\mathbf{b}} \times (3\hat{\mathbf{b}} \cdot \nabla \mathbf{v} + \hat{\mathbf{b}} \times \nabla \times \mathbf{v}) + \frac{\chi}{B} \hat{\mathbf{b}} \right]$$

  where

$$\mathbf{v}_* = -\frac{1}{ne} \nabla \times \left(\frac{p_\perp}{B} \hat{\mathbf{b}} \right); \quad \chi = \frac{mp_\perp}{2eB} \hat{\mathbf{b}} \cdot \nabla \times \mathbf{v}.$$

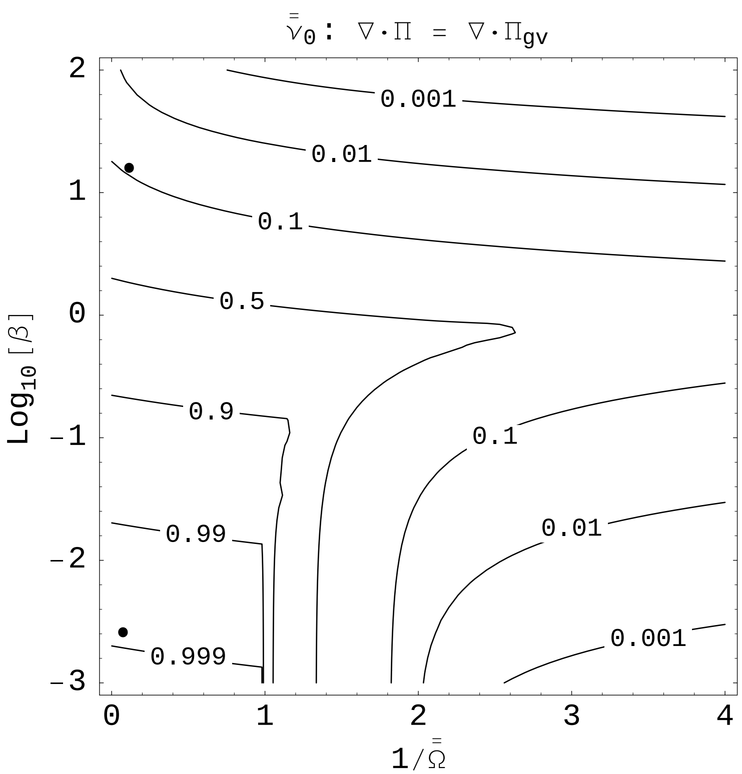
- $\mathbf{v}_*$ is the "magnetization velocity," which is the same as the diamagnetic drift velocity $\mathbf{v}_d$ only when $\nabla \times (\hat{\mathbf{b}}/B) = 0$.

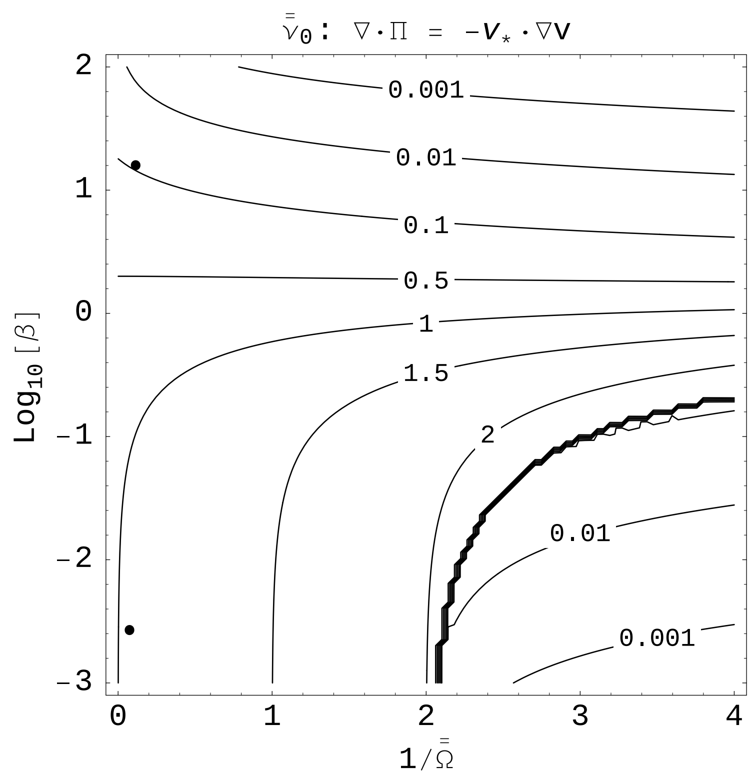
- Frequently, the approximation $\nabla \cdot \Pi = -mn\mathbf{v}_d \cdot \nabla \mathbf{v}$ is used.

- We considered some consequences of using the approximation $\nabla \cdot \Pi = -mn\mathbf{v}_* \cdot \nabla \mathbf{v}$.

$\mathbf{v}_*$ Approximation: Gravitational Instability

- The $\mathbf{v}_*$ approximation is generally sufficient for understanding the linear stabilization of the gravitational instability.
- Re-deriving the dispersion relation of the gravitational instability using the $\mathbf{v}_*$ approximation yields almost the same dispersion relation.

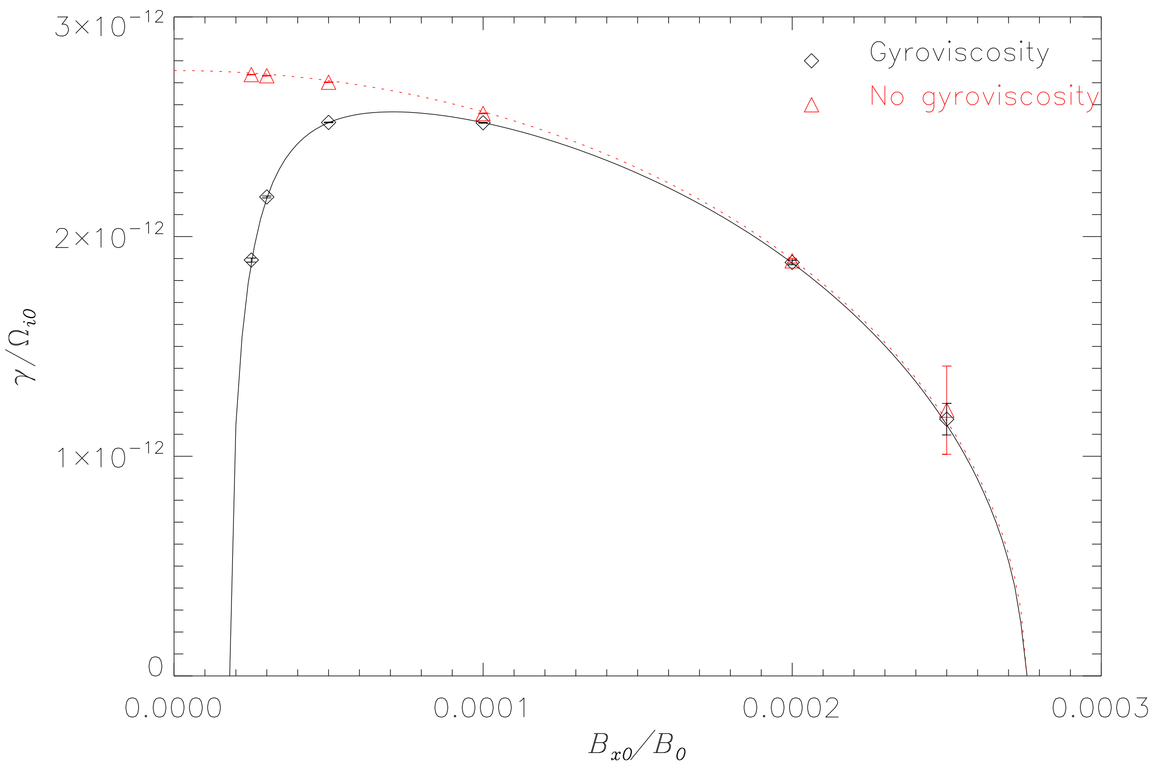




Shown are contours of the marginally stable values of ν.

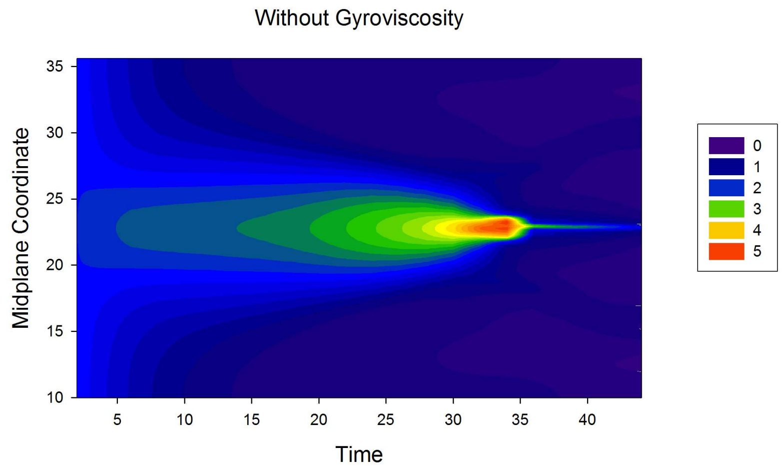
Magnetothermal Instability

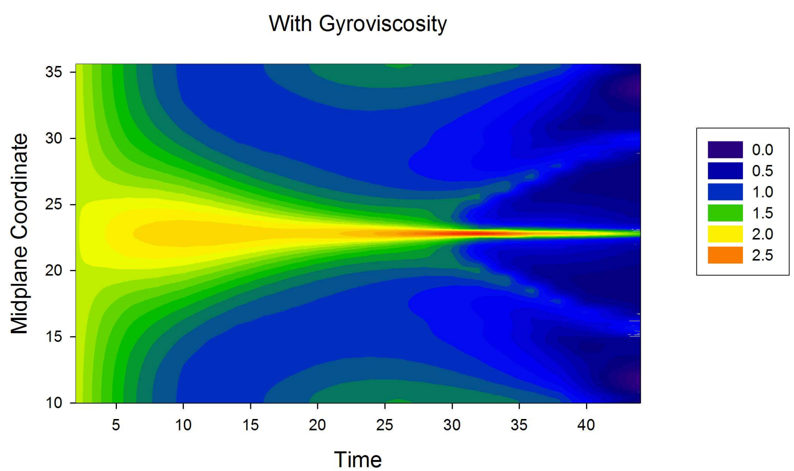
- We have implemented a field-aligned heat conduction term, $\mathbf{q} = \kappa_{\parallel}\hat{\mathbf{b}}\hat{\mathbf{b}} \cdot \nabla T$.
- We have successfully simulated the magnetothermal instability[7] (MTI) in the linear regime, including gyroviscosity.

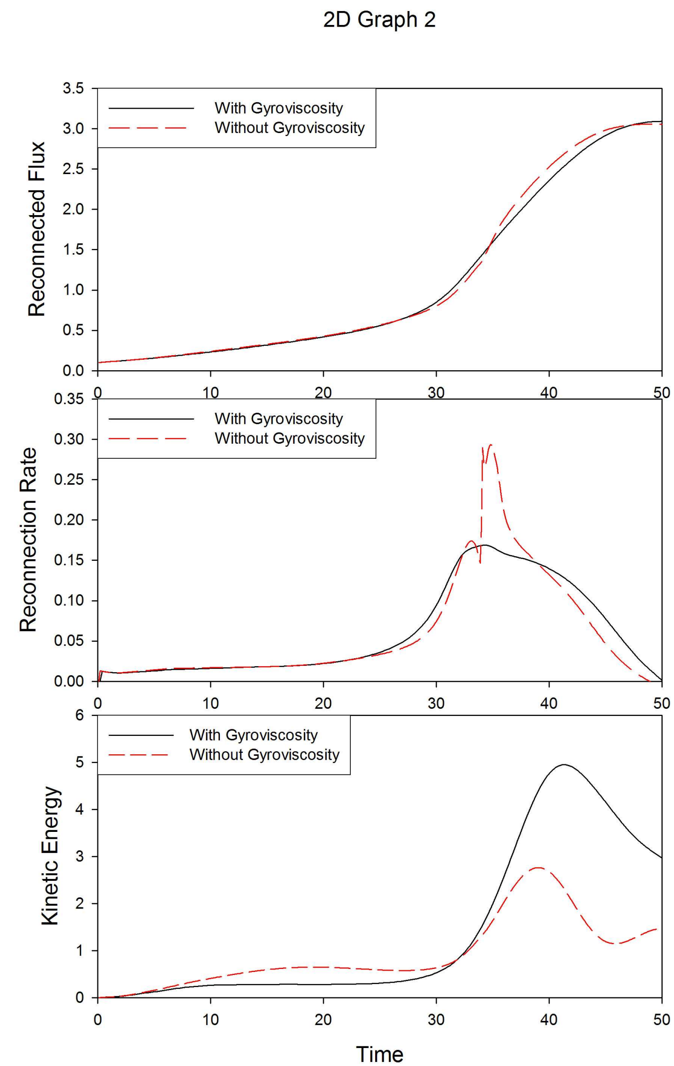


(For this case, $\kappa_{\parallel}/\kappa_{\perp} = 10^{12}$.)

GEM Reconnection







Current and Future Work

We have derived the flux form of the extended-MHD equations in **cylindrical geometry**, and have implemented them in a new code which:

- Uses **numerical integration** instead of analytic integration. This has resulted in a significant ($\sim 500\%$) speedup.
- Uses an **unstructured mesh**, in preparation for use of adaptive mesh refinement.

In the future, we plan to

- Extend this to **three dimensions** using multiple poloidal planes.
- Incorporate **neoclassical effects**: bootstrap current, particle trapping, etc..
- Use the code to find self-consistant equilibria with flows, and to simulate linear and nonlinear deviations from these equilibria.

References

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