Low moment kinetic MHD closure

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Outline

- Model equations
- Including electron inertia "for free"
- Comparison with linear theory

Basic Idea:

- Starting with F=ma and Maxwell's equations, assume as little as possible, **but**
  - Use a generalized Ohm's law for determining E
  - Assume quasi-neutrality
  - Neglect the displacement current

MHD Part

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$$

$$\mathbf{E} = -\mathbf{u} \times \mathbf{B} + \frac{\eta}{\mu_0} \nabla \times \mathbf{B} + \frac{1}{\mu_0 e n} (\nabla \times \mathbf{B}) \times \mathbf{B}$$

$$-\frac{1}{en} \nabla \cdot \mathbf{\Pi}_e - \frac{m_e}{ne} \frac{\partial (n \mathbf{u}_e)}{\partial t}$$

$$\mathbf{\Pi}_e \equiv \int \mathbf{v}\mathbf{v}\delta f_e d^3 v$$

$$\mathbf{u} = \mathbf{u}_i + \frac{m_e}{m_i}\mathbf{u}_e$$

$$\mathbf{u}_\alpha = \frac{1}{n}\int \mathbf{v}\delta f_\alpha d^3 v$$

$$n = n_0 + \int \delta f d^3 v$$

Including Electron Inertia

$$\nabla \times (\nabla \times \mathbf{E}) = -\frac{\partial\,(\nabla \times \mathbf{B})}{\partial t} = \mu_0 e\left[\frac{\partial(n\mathbf{u}_e)}{\partial t} - \frac{\partial(n\mathbf{u}_i)}{\partial t}\right]$$

$$-\frac{m_e}{ne}\frac{\partial(n\mathbf{u}_e)}{\partial t} = -\frac{m_e}{\mu_0 n e^2}\nabla \times (\nabla \times \mathbf{E}) - \frac{m_e}{ne}\frac{\partial(n\mathbf{u}_i)}{\partial t}$$

$$\frac{\partial(n\mathbf{u}_i)}{\partial t} = \frac{en}{m_i}(\mathbf{E} + \mathbf{u}_i \times \mathbf{B}) - \frac{1}{m_i}\nabla \cdot \mathbf{\Pi}_i$$

$$-\frac{m_e}{ne}\frac{\partial(n\mathbf{u}_e)}{\partial t} = \frac{m_e}{\mu_0 n e^2}\nabla^2\mathbf{E} - \frac{m_e}{m_i}(\mathbf{E}+\mathbf{u}_i\times\mathbf{B}) + \frac{m_e}{m_i n e}\nabla\cdot\mathbf{\Pi}_i$$

$\frac{m_e}{m_i} \ll 1 \Rightarrow$

$$\left[1 - \frac{c^2}{\omega_{pe}^2}\nabla^2\right]\mathbf{E} = -\mathbf{u} \times \mathbf{B} + \frac{\eta}{\mu_0}\nabla \times \mathbf{B}$$

$$+\frac{1}{\mu_0 e n}(\nabla \times \mathbf{B}) \times \mathbf{B} - \frac{1}{en}\nabla \cdot \mathbf{\Pi}_e$$

Kinetics

$$\frac{\partial \delta f_\alpha}{\partial t} + \mathbf{v} \cdot \nabla \delta f_\alpha + \frac{q_\alpha}{m_\alpha}(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \frac{\partial \delta f_\alpha}{\partial \mathbf{v}} =$$

$$-\frac{q_\alpha}{m_\alpha}(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \frac{\partial f_{0,\alpha}}{\partial \mathbf{v}}$$

where $f = f_0 + \delta f$

$$f_0 = f_M = \frac{n}{\left(\sqrt{2\pi} v_t\right)^3} \exp\left(\frac{-v^2}{2v_t^2}\right)$$

$$\dot{\delta f}_\alpha = -\frac{q_\alpha}{m_\alpha v_{t,\alpha}^2} \mathbf{v} \cdot \mathbf{E} f_M$$

$$\delta f(\mathbf{x}, \mathbf{v}, t) = \sum_i \delta f_i \Delta V_i \delta^3(\mathbf{x} - \mathbf{x}_i) \delta^3(\mathbf{v} - \mathbf{v}_i)$$

Linear Theory

Faraday's and Ohm's law produces whistler

$$\omega = \pm \frac{k^2 v_A^2}{\Omega_i} \frac{1}{\left(1 + \frac{c^2 k^2}{\omega_{pe}^2}\right)}$$

Cold ions produces

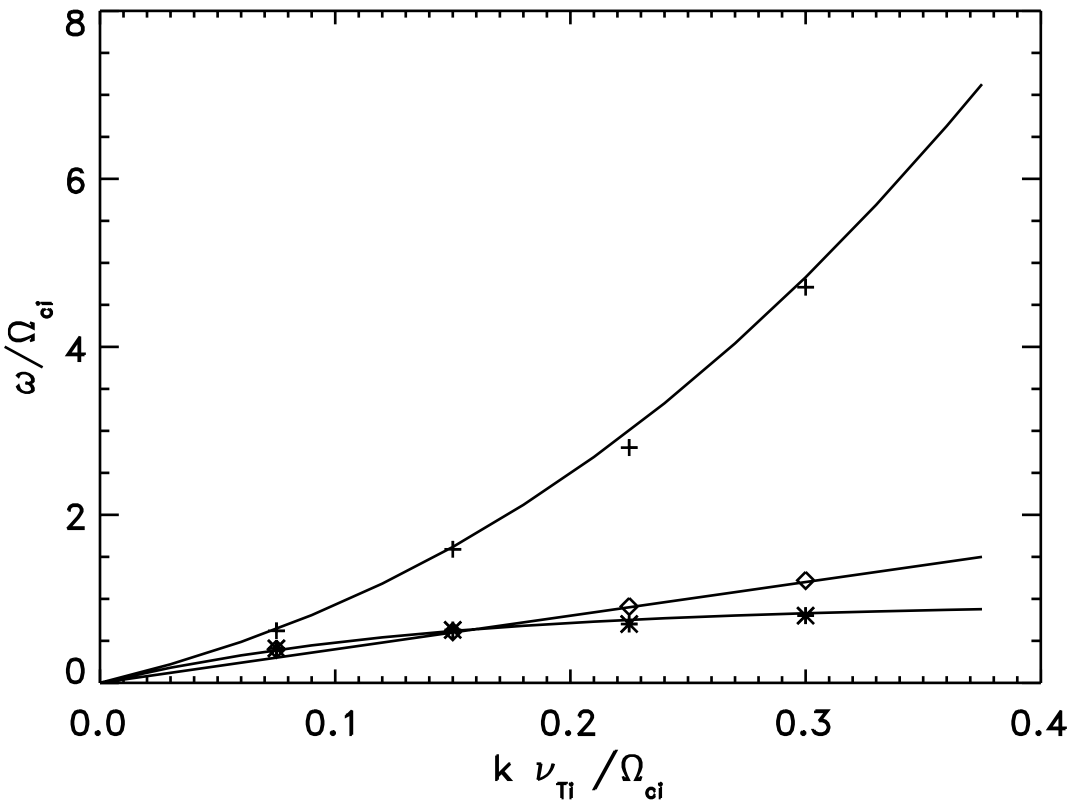
$$1 + \frac{c^2 k^2}{\omega_{pe}^2} + \frac{\Omega_i^2}{(\omega^2 - \Omega_i^2)} + i \frac{k^2 \eta}{\mu_0 \omega} = \pm \left[\frac{k^2 v_A^2}{\omega \Omega_i} + \frac{\omega \Omega_i}{(\omega^2 - \Omega_i^2)} \right]$$

Assuming $\eta = 0$, $\frac{c^2 k^2}{\omega_{pe}^2} \ll \frac{k^2 v_A^2}{\Omega_i^2}$,

$\frac{k^2 v_A^2}{\Omega_i^2} \gg 1 \Rightarrow$ whistler wave

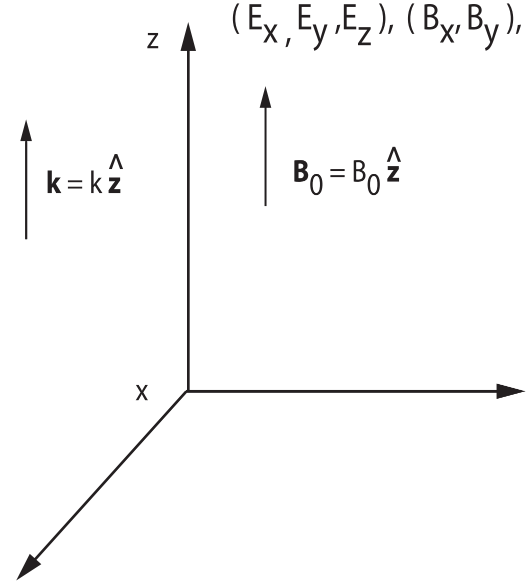
$\frac{c^2 k^2}{\omega_{pe}^2} \ll \frac{k^2 v_A^2}{\Omega_i^2} \Rightarrow$ Alfvén wave: $\omega^2 = \dfrac{k^2 v_A^2}{\left(1 + \frac{c^2 k^2}{\omega_{pe}^2}\right)}$

3 waves obtained simultaneously, agree with theory
(shear Alfven, whistler, ion-acoustic wave)



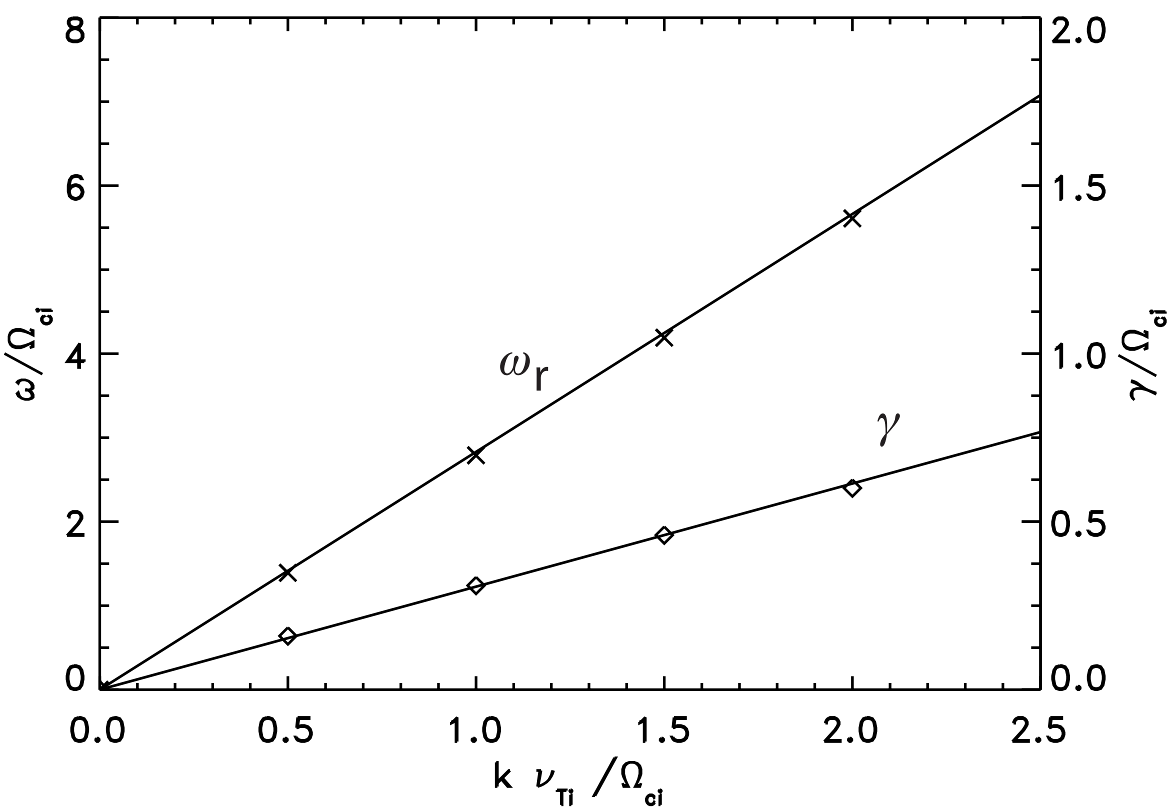
Particles:
(z, v_x, v_y, v_z)

Fields:
(E_x, E_y, E_z), (B_x, B_y), (u_x, u_y)



Simulation shows proper ion landau damping of ion-acoustic wave

Ion acoustic wave is obtained by assuming adiabatic electrons: $\delta p_e = T_e \delta n_e$ in the generalized Ohm's law



Summary

The model works very well!

- Low-moment model that assumes very little works quite well
    - assumes quasi-neutrality
    - neglects the displacement current
    - a simple electron closure

- Whistler and shear Alfven waves agree with linear theory
- Ion Landau damping of ion acoustic waves agrees with linear theory

Next step is to try model on a nonlinear problem.

Using Gyrokinetics

$\mathbf{u}_\perp$ must be calculated carefully

Gyroaveraging $\mathbf{v}_\perp \sim v_t$ produces zero to lowest order

$$\delta f = \delta f_{gc} + q\left[(\phi - \langle\phi\rangle) - v_\|(A_\| - \langle A_\|\rangle)\right]\frac{\partial f_M}{\partial \epsilon}$$

$$\mathbf{u}_\perp \approx \frac{1}{n}\int \mathbf{v}_\perp \delta f[\mathbf{x} - \rho(\gamma, v_\perp), v_\perp, v_\|] v_\perp dv_\perp dv_\| d\gamma$$

$$\mathbf{u}_\perp = \frac{1}{qnB}\widehat{b} \times \nabla_\perp \Pi_\perp - \frac{\widehat{b} \times \mathbf{E}_\perp}{B}$$

$$\Pi_\perp = \int v_\perp^2 \delta f_{gc} v_\perp dv_\perp dv_\|$$

One obtains $\mathbf{E}_\perp = -\mathbf{u}_\perp \times \mathbf{B} + \cdots = \mathbf{E}_\perp + \cdots$

$x = x + 0.1$ not solvable!

Better to use one-fluid momentum equation

$$\rho\frac{d\mathbf{u}}{dt} = \mathbf{J}\times\mathbf{B} - \sum_{\alpha}\nabla\mathbf{P}_{\alpha}$$

$$\mathbf{P}_{\alpha} = P_{\perp}\mathbf{I} + (P_{\|} - P_{\perp})\widehat{b}\widehat{b}$$

$$P_{\|} = \Pi_{\|} - u_{\|}^{2}$$

$$P_{\perp} = \Pi_{\perp} - u_{\perp}^{2}$$

No such problem calculating $\mathbf{u}_{\|}$, could be obtained directly from δf.