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# FLUID-KINETIC PARALLEL CLOSURES*

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**THE PARALLEL CLOSURE PROBLEM CONCERNS THE EVALUATION OF THE CHEW-GOLDBERGER-LOW (GYROTROPIC) PART OF THE STRESS TENSOR:**

$$\mathsf{P}^{CGL} \;=\; p_\perp \mathsf{I} + (p_\parallel - p_\perp)\mathbf{bb} \;=\; p\mathsf{I} + (p_\parallel - p_\perp)(\mathbf{bb} - \mathsf{I}/3),$$

$$\textbf{with} \qquad p \;=\; (2p_\perp + p_\parallel)/3\ .$$

**(Species indices will be omitted when equations apply to either species or to the ions. The electron species index will be made explicit in the simplified form of the electron equations after taking the small electron mass limit.)**

**KEEPING $O(pv_{th\iota}/L)+O(\rho_\iota p v_{th\iota}/L^2)$ IN A FINITE-LARMOR-RADIUS, FAST DYNAMICS ORDERING [i.e. $\partial/\partial t = O(\rho_\iota \Omega_{c\iota}/L) + O(\rho_\iota^2 \Omega_{c\iota}/L^2)$ and $u = O(v_{th\iota}) + O(\rho_\iota v_{th\iota}/L)$], THE EVOLUTION EQUATIONS FOR THE COMPONENTS OF $\mathsf{P}^{CGL}$ ARE:**

$$\frac{3}{2}\Big[\frac{\partial p}{\partial t} + \nabla\cdot\big(p\mathbf{u}\big)\Big] \; + \; p\nabla\cdot\mathbf{u} \; + \; (p_\parallel - p_\perp)\Big\{\mathbf{b}\cdot\big[(\mathbf{b}\cdot\nabla)\mathbf{u}\big] - \nabla\cdot\mathbf{u}/3\Big\} \; + \; \nabla\cdot\big(q_\parallel \mathbf{b}\big) \; +$$

$$+ \; \mathsf{P}^{gyr} : (\nabla\mathbf{u}) \; + \; \nabla\cdot\mathbf{q}_\perp \; - g^{coll} \; = \; 0$$

**and**

$$\frac{\partial (p_\parallel - p_\perp)}{\partial t} \; + \; \nabla\cdot\big[(p_\parallel - p_\perp)\mathbf{u}\big] \; + \; (p_\parallel - p_\perp)\Big\{\mathbf{b}\cdot\big[(\mathbf{b}\cdot\nabla)\mathbf{u}\big] + \nabla\cdot\mathbf{u}/3\Big\} \; +$$

$$+ \; p\Big\{\mathbf{b}\cdot\big[3(\mathbf{b}\cdot\nabla)\mathbf{u}\big] - \nabla\cdot\mathbf{u}\Big\} \; + \; \nabla\cdot\big[(3q_{B\parallel} - q_\parallel)\mathbf{b}\big] \; + \; 3(q_\parallel - q_{B\parallel})\mathbf{b}\cdot\nabla(\ln\ B) \; +$$

$$+3\mathbf{b}\cdot\mathsf{P}^{gyr}\cdot(\mathbf{b}\times\boldsymbol{\omega}) - \mathsf{P}^{gyr} : (\nabla\mathbf{u}) + \nabla\cdot(3\mathbf{q}_{B\perp} - \mathbf{q}_\perp) - 6\mathbf{q}_{B\perp}\cdot\boldsymbol{\kappa} + 3(q_\parallel - q_{B\parallel})\sigma + g^{coll} - 3g_B^{coll} \; = \; 0 \; .$$

**Here, the non-gyrotropic, FLR terms $\mathsf{P}^{gyr}$, $\mathbf{q}_\perp$, $\mathbf{q}_{B\perp}$ and $\sigma$ are known (except for a non-Maxwellian contribution to $\mathbf{q}_\perp$ and $\mathbf{q}_{B\perp}$). For the electrons, $\mathsf{P}^{gyr}$ and $\sigma$ can be neglected.**

**Specifically,**

$$\mathsf{P}^{gyr} : (\nabla \mathbf{u}) \;=\; \mathbf{b}\cdot\mathsf{P}^{gyr}\cdot\left[2(\mathbf{b}\cdot\nabla)\mathbf{u}+\mathbf{b}\times\boldsymbol{\omega}\right] \;+\; (q_\parallel - q_{B\parallel})\sigma \;,$$

$$\mathbf{b}\cdot\mathsf{P}^{gyr} \;=\; \frac{m}{eB}\;\mathbf{b}\times\left[2p_\parallel(\mathbf{b}\cdot\nabla)\mathbf{u}+p_\perp\mathbf{b}\times\boldsymbol{\omega}+\nabla(q_\parallel-q_{B\parallel})+2(2q_{B\parallel}-q_\parallel)\boldsymbol{\kappa}\right] \;,$$

$$\sigma \;=\; \frac{m}{4eB}\epsilon_{jkl}b_j\Big(\frac{\partial b_k}{\partial x_m}+\frac{\partial b_m}{\partial x_k}\Big)(\delta_{mn}-b_m b_n)\Big(\frac{\partial u_l}{\partial x_n}+\frac{\partial u_n}{\partial x_l}\Big) \;,$$

$$\mathbf{q}_\perp \;=\; \frac{1}{eB}\mathbf{b}\times\Big[p_\perp\nabla\Big(\frac{p_\parallel+4p_\perp}{2n}\Big)+\frac{p_\parallel(p_\parallel-p_\perp)}{n}\boldsymbol{\kappa}+2m(2q_\parallel-q_{B\parallel})(\mathbf{b}\cdot\nabla)\mathbf{u}+m(q_\parallel-q_{B\parallel})\mathbf{b}\times\boldsymbol{\omega}\Big] \;+\; \tilde{\mathbf{q}}_\perp \;,$$

$$\mathbf{q}_{B\perp} \;=\; \frac{1}{eB}\mathbf{b}\times\Big[p_\perp\nabla\Big(\frac{p_\parallel}{2n}\Big)+\frac{p_\parallel(p_\parallel-p_\perp)}{n}\boldsymbol{\kappa}+2mq_{B\parallel}(\mathbf{b}\cdot\nabla)\mathbf{u}+m(q_\parallel-q_{B\parallel})\mathbf{b}\times\boldsymbol{\omega}\Big] \;+\; \tilde{\mathbf{q}}_{B\perp} \;,$$

**where the $m\to 0$ limit can be taken for the electrons.**

**The non-Maxwellian contributions $\tilde{\mathbf{q}}_\perp$ and $\tilde{\mathbf{q}}_{B\perp}$ to the perpendicular heat fluxes are:**

$$\tilde{\mathbf{q}}_\perp = \frac{1}{eB}\mathbf{b} \times \left[\nabla \tilde{r}_\perp^{(0)} + (\tilde{r}_\parallel^{(0)} - \tilde{r}_\perp^{(0)})\boldsymbol{\kappa}\right]$$

**and**

$$\tilde{\mathbf{q}}_{B\perp} = \frac{1}{eB}\mathbf{b} \times \left[\nabla (\tilde{r}_\perp^{(0)} + \tilde{r}_\Delta^{(0)})/5 + (\tilde{r}_\parallel^{(0)} - \tilde{r}_\perp^{(0)} - \tilde{r}_\Delta^{(0)})\boldsymbol{\kappa}\right] ,$$

**where**

$$\tilde{r}_\perp^{(0)} \;=\; (m^2/4)\int d^3\mathbf{v}\; |\mathbf{v}-\mathbf{u}|^2 \left\{|\mathbf{v}-\mathbf{u}|^2 - [\mathbf{b}\cdot(\mathbf{v}-\mathbf{u})]^2\right\} \left[f^{(0)}(\mathbf{v}) - f_{2M}(\mathbf{v})\right] ,$$

$$\tilde{r}_\parallel^{(0)} \;=\; (m^2/2)\int d^3\mathbf{v}\; |\mathbf{v}-\mathbf{u}|^2 \, [\mathbf{b}\cdot(\mathbf{v}-\mathbf{u})]^2 \left[f^{(0)}(\mathbf{v}) - f_{2M}(\mathbf{v})\right] ,$$

$$\tilde{r}_\Delta^{(0)} \;=\; (m^2/4)\int d^3\mathbf{v} \left\{|\mathbf{v}-\mathbf{u}|^2 - [\mathbf{b}\cdot(\mathbf{v}-\mathbf{u})]^2\right\} \left\{5[\mathbf{b}\cdot(\mathbf{v}-\mathbf{u})]^2 - |\mathbf{v}-\mathbf{u}|^2\right\} \left[f^{(0)}(\mathbf{v}) - f_{2M}(\mathbf{v})\right] .$$

**Besides $\tilde{r}_{\perp}^{(0)}$, $\tilde{r}_{\|}^{(0)}$ and $\tilde{r}_{\Delta}^{(0)}$, the parallel closure terms that must be provided by kinetic theory are the two independent parallel heat fluxes:**

$$q_{\|} \;=\; (m/2)\int d^3\mathbf{v}\;[(\mathbf{v}-\mathbf{u})\cdot\mathbf{b}]\;|\mathbf{v}-\mathbf{u}|^2\;f\;,$$

$$q_{B\|} \;=\; (m/2)\int d^3\mathbf{v}\;[(\mathbf{v}-\mathbf{u})\cdot\mathbf{b}]^3\;f\;.$$

**(And the pressure anisotropy, also called "parallel viscosity":**

$$p_{\|}-p_{\perp} \;=\; (m/2)\int d^3\mathbf{v}\;\left\{3[(\mathbf{v}-\mathbf{u})\cdot\mathbf{b}]^2-|\mathbf{v}-\mathbf{u}|^2\right\}\;f\;,$$

**if one chooses not to use its fluid theory evolution equation.)**

# DRIFT-KINETIC EVALUATION OF THE FLUID CLOSURES

**1.) All the yet unknown terms needed to close the two-fluid system can be derived from moments of the gyrophase-averaged distribution functions, $\bar{f}$.**

**2.) The velocity moments of $\bar{f}$ needed for the fluid closure are evaluated most conveniently in the moving frame of the full macroscopic flow velocity, $\mathbf{u}(\mathbf{x}, t)$.**

**3.) For the fast dynamics under consideration, with large perpendicular electric fields $E_\perp \sim v_{th\iota} B$, the drift-kinetic equation must be derived in a moving frame close to the electric drift velocity $\mathbf{u}_E(\mathbf{x}, t) = \mathbf{E} \times \mathbf{B}/B^2$, such as $\mathbf{u}_E(\mathbf{x}, t)$ itself or $\mathbf{u}(\mathbf{x}, t)$.**

**4.) To determine the collisional moments in a low-collisionality regime ($\nu_\iota \lesssim \omega_*$) and the perpendicular heat flux closure terms $\tilde{\mathbf{q}}_\perp$ and $\tilde{\mathbf{q}}_{B\perp}$ within the required accuracy, only the lowest-order or zero-Larmor-radius distribution functions $\bar{f} = \bar{f}^{(0)}$ are needed.**

**5.) To determine the parallel heat fluxes and the pressure anisotropy (or the coefficient functions in their evolution equations) within the required accuracy, first-order FLR solutions of the drift-kinetic equation, $\bar{f}_\alpha = \bar{f}_\alpha^{(0)} + \bar{f}_\alpha^{(1)}$, are necessary.**

**A FINITE-LARMOR-RADIUS FORM OF THE DRIFT-KINETIC EQUATION HAS BEEN DERIVED, THAT MEETS THE DESIRED CONDITIONS FOR EVALUATION OF THE FLUID CLOSURES:**

**Accurate to the first FLR order in the fast dynamics ordering and valid for sonic macroscopic flows.**

**Use of the full macroscopic flow velocity, $\mathbf{u}(\mathbf{x}, t)$, to define the moving frame. Exact algebraic elimination of the electric field and no reference to $\mathbf{u}_E$ or any other drifts.**

**Formulation in terms of the standard MHD variables (macroscopic flow velocity and magnetic field) only. This facilitates the coupling to the extended-MHD formalism.**

**Velocity moments reproduce all the previously derived fluid results, including the higher-moment FLR results.**

**Our FLR drift-kinetic equation is [with $\mathbf{v}' = \mathbf{v} - \mathbf{u}(\mathbf{x}, t)$]:**

$$\frac{\partial \bar{f}(v'_\parallel, v'_\perp, \mathbf{x}, t)}{\partial t} \; + \; \dot{\mathbf{x}} \cdot \frac{\partial \bar{f}}{\partial \mathbf{x}} \; + \; \dot{v}'_\parallel \, \frac{\partial \bar{f}}{\partial v'_\parallel} \; + \; \dot{v}'_\perp \, \frac{\partial \bar{f}}{\partial v'_\perp} \; = \; \frac{D^{coll} \bar{f}}{Dt} \; ,$$

**where the coefficient functions are:**

$$\dot{\mathbf{x}} \; = \; \mathbf{u} + v'_\parallel \mathbf{b} \; + \; \frac{v'^2_\perp}{2} \nabla \times \frac{\mathbf{b}}{\Omega_c} \; - \; \frac{\mathbf{b}}{\Omega_c} \times \Big[ \frac{\nabla \cdot \mathsf{P}}{mn} - 2 v'_\parallel (\mathbf{b} \cdot \nabla) \mathbf{u} - \Big( v'^2_\parallel - \frac{v'^2_\perp}{2} \Big) \boldsymbol{\kappa} \Big] \; ,$$

$$\dot{v}'_\parallel \; = \; \frac{\mathbf{b} \cdot (\nabla \cdot \mathsf{P})}{mn} \; - \; v'_\parallel \mathbf{b} \cdot [(\mathbf{b} \cdot \nabla) \mathbf{u}] \; - \; \frac{v'^2_\perp}{2} \mathbf{b} \cdot \nabla \ln B \; + \; \frac{v'^2_\perp}{2} \nabla \cdot \Big[ \frac{\mathbf{b}}{\Omega_c} \times (\boldsymbol{\omega} \times \mathbf{b} + v'_\parallel \boldsymbol{\kappa}) \Big] \, +$$

$$+ \Big[ \frac{\mathbf{b}}{\Omega_c} \times (\boldsymbol{\omega} \times \mathbf{b} + v'_\parallel \boldsymbol{\kappa}) \Big] \cdot \Big[ \frac{\nabla \cdot \mathsf{P}}{mn} - 2 v'_\parallel (\mathbf{b} \cdot \nabla) \mathbf{u} - v'^2_\parallel \boldsymbol{\kappa} \Big] \; - \; \frac{v'^2_\perp}{2} \sigma \; ,$$

$$\dot{v}'_\perp \; = \; \frac{v'_\perp}{2} \Big\{ \mathbf{b} \cdot [(\mathbf{b} \cdot \nabla) \mathbf{u}] \; - \nabla \cdot \mathbf{u} \; + \; v'_\parallel \mathbf{b} \cdot \nabla \ln B \; + \; \nabla \cdot \Big[ \frac{\mathbf{b}}{\Omega_c} \times \Big( \frac{\nabla \cdot \mathsf{P}}{mn} - 2 v'_\parallel (\mathbf{b} \cdot \nabla) \mathbf{u} - v'^2_\parallel \boldsymbol{\kappa} \Big) \Big] \, +$$

$$+ \, 2 \Big[ \frac{\mathbf{b}}{\Omega_c} \times (\boldsymbol{\omega} \times \mathbf{b}) \Big] \cdot \Big[ (\mathbf{b} \cdot \nabla) \mathbf{u} + v'_\parallel \boldsymbol{\kappa} \Big] \; - \; \Big( \frac{\mathbf{b}}{\Omega_c} \times \boldsymbol{\kappa} \Big) \cdot \Big[ \frac{\nabla \cdot \mathsf{P}}{mn} - 4 v'_\parallel (\mathbf{b} \cdot \nabla) \mathbf{u} \Big] \Big\} \; .$$

THE FLUID AND DRIFT-KINETIC EQUATIONS SHOWN MAKE THE FOLLOWING DIAGRAM COMMUTATIVE, INCLUDING THE FIRST-ORDER FLR TERMS FOR SONIC-SCALE TIME EVOLUTION AND MEAN FLOWS:

$$
\begin{tikzcd}
\boxed{\text{VLASOV-BOLTZMAN KINETIC EQUATION}} \arrow[r, "\frac{1}{\Omega_c}\ \text{EXPANSION}"] \arrow[d, "\text{MOMENTS}"] & \boxed{\text{DRIFT-KINETIC EQUATION}} \arrow[d, "\text{MOMENTS}"] \\
\boxed{\text{FLUID MOMENT EQUATIONS}} \arrow[r, "\frac{1}{\Omega_c}\ \text{EXPANSION}"] & \boxed{\text{MAGNETIZED FLUID EQUATIONS}}
\end{tikzcd}
$$

**FOR SMALL-MASS ELECTRONS AND USING AS PHASE-SPACE VARIABLES THE MOVING FRAME KINETIC ENERGY $\varepsilon' = m_e(v_\|'^2 + v_\perp'^2)/2$ AND MAGNETIC MOMENT $\mu' = m_e v_\perp'^2/(2B)$ :**

$$\frac{\partial \bar{f}_e(\varepsilon', \mu', \mathbf{x}, t)}{\partial t} \; + \; \dot{\mathbf{x}} \cdot \frac{\partial \bar{f}_e}{\partial \mathbf{x}} \; + \; \dot{\varepsilon}' \, \frac{\partial \bar{f}_e}{\partial \varepsilon'} \; + \; \dot{\mu}' \, \frac{\partial \bar{f}_e}{\partial \mu'} \; = \; \frac{D^{coll} \bar{f}_e}{Dt} \; ,$$

**where**

$$\dot{\mathbf{x}} \; = \; \Big[2(\varepsilon' - \mu' B)/m_e\Big]^{1/2} \mathbf{b} \; + \; \mu' B \; \nabla \times \frac{\mathbf{b}}{m_e \Omega_{ce}} \; - \; \frac{\mathbf{b}}{m_e \Omega_{ce}} \times \Big[\frac{\nabla \cdot \mathsf{P}_e^{CGL}}{n} - (2\varepsilon' - 3\mu' B)\boldsymbol{\kappa}\Big] \; + \; \mathbf{u}_e \; ,$$

$$\dot{\varepsilon}' \; = \; \Big[2(\varepsilon' - \mu' B)/m_e\Big]^{1/2} \, \frac{\mathbf{b} \cdot (\nabla \cdot \mathsf{P}_e^{CGL})}{n} \; + \; \mu' B \; \nabla \cdot \Big(\frac{\mathbf{b}}{m_e \Omega_{ce}} \times \frac{\nabla \cdot \mathsf{P}_e^{CGL}}{n}\Big) \; -$$

$$- \; (2\varepsilon' - 3\mu' B)\Big(\frac{\mathbf{b}}{m_e \Omega_{ce}} \times \frac{\nabla \cdot \mathsf{P}_e^{CGL}}{n}\Big) \cdot \boldsymbol{\kappa} \; - \mu' B \; \nabla \cdot \mathbf{u}_e \; - \; (2\varepsilon' - 3\mu' B)\mathbf{b} \cdot [(\mathbf{b} \cdot \nabla)\mathbf{u}_e] \; ,$$

$$\dot{\mu}' \; = \; \frac{\mu'}{m_e \Omega_{ce}} \Big\{[\mathbf{b} \cdot (\nabla \times \mathbf{b})] \; \Big[\mathbf{b} \cdot \Big(\frac{\nabla \cdot \mathsf{P}_e^{CGL}}{n} - \mu' \nabla B\Big)\Big] \; + \; 2(\varepsilon' - \mu' B)\mathbf{b} \cdot (\nabla \times \boldsymbol{\kappa})\Big\} \; .$$

**IF THE PERPENDICULAR ELECTRIC FIELD, HENCE THE PERPENDICULAR FLOW VELOCITY, WERE SMALL (i.e. $E_\perp \sim \delta v_{th\iota} B$ and $u_\perp \sim \delta v_{th\iota}$), THE DRIFT-KINETIC ANALYSIS COULD BE CARRIED OUT IN THE LABORATORY FRAME. THEN, USING AS PHASE-SPACE VARIABLES THE LABORATORY FRAME KINETIC ENERGY $\varepsilon = m(v_\parallel^2 + v_\perp^2)/2$ AND MAGNETIC MOMENT $\mu = m v_\perp^2/(2B)$, THE ELECTRON DRIF-KINETIC EQUATION BECOMES:**

$$\frac{\partial \bar{f}_e(\varepsilon, \mu, \mathbf{x}, t)}{\partial t} \; + \; \dot{\mathbf{x}} \cdot \frac{\partial \bar{f}_e}{\partial \mathbf{x}} \; + \; \dot{\varepsilon} \; \frac{\partial \bar{f}_e}{\partial \varepsilon} \; + \; \dot{\mu} \; \frac{\partial \bar{f}_e}{\partial \mu} \; = \; \frac{D^{coll} \bar{f}_e}{Dt} \; ,$$

**where**

$$\dot{\mathbf{x}} \; = \; \Big[2(\varepsilon - \mu B)/m_e\Big]^{1/2} \mathbf{b} \; + \; \mu B \; \nabla \times \frac{\mathbf{b}}{m_e \Omega_{ce}} \; + \; \frac{\mathbf{b}}{m_e \Omega_{ce}} \times \Big[e\mathbf{E} + (2\varepsilon - 3\mu B)\boldsymbol{\kappa}\Big] \; ,$$

$$\dot{\varepsilon} \; = \; - \; \Big[2(\varepsilon - \mu B)/m_e\Big]^{1/2} e\mathbf{b} \cdot \mathbf{E} \; + \; \mu B \; \nabla \cdot \Big(\frac{\mathbf{b}}{B} \times \mathbf{E}\Big) \; - \; (2\varepsilon - 3\mu B)\Big(\frac{\mathbf{b}}{B} \times \mathbf{E}\Big) \cdot \boldsymbol{\kappa} \; ,$$

$$\dot{\mu} \; = \; \frac{\mu}{m_e \Omega_{ce}} \Big\{ - [\mathbf{b} \cdot (\nabla \times \mathbf{b})] \Big[\mathbf{b} \cdot \Big(e\mathbf{E} + \mu \nabla B\Big)\Big] \; + \; 2(\varepsilon - \mu B)\mathbf{b} \cdot (\nabla \times \boldsymbol{\kappa})\Big\} \; ,$$

**in agreement with the conventional analyses.**