Lorentz ion - gyro/drift-kinetic electron closure

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CEMM Meeting APS-DPP 2008

Outline

* Model equations with drift kinetic electrons
* Including gyrokinetic electrons
* Implicit algorithm
* Linear tests of the model
* GEM TAE results showing energetic particle destabilization
  and damping due to kinetic core ions

Motivation

- Current Gyrokinetic-Maxwell Equations in use are not fully electromagnetic
  - The $A_\parallel - \phi$ field model does not have $\delta B_\parallel$
- Use of the quasi-neutrality condition for determining ϕ might requires terms nonlinear in ϕ, because the polarization density $\sim k_\perp^2 \phi$ is very small form long wavelength modes.
- More accurate GK equations needed but might not be solvable in the edge or ITB with strong "equilibrium" variations in
  - $\mathbf{E} \times \mathbf{B}$ flow of thermal speed
  - density or temperature over $\sim 10\rho_i$
- For ETG simulations with non-adiabatic ion, N-point averaging with $N \gg 4$ is needed. It could be easier to follow the gyro-motion.

The Vlasov ion/Drift kinetic electron model

Vlasov ions:

$$\frac{d\mathbf{v}_i}{dt} = \frac{q}{m_i}(\mathbf{E} + \mathbf{v}_i \times \mathbf{B}), \qquad \frac{d\mathbf{x}_i}{dt} = \mathbf{v}_i$$

Drift kinetic electrons: $\varepsilon = \frac{1}{2} m_e v^2$

$$\frac{d\mathbf{x}}{dt} = \mathbf{v}_G \equiv v_\parallel \left(\mathbf{b} + \frac{\delta \mathbf{B}_\perp}{B_0} \right) + \mathbf{v}_D + \mathbf{v}_E$$

$$\frac{d\varepsilon}{dt} = -e\mathbf{v}_G \cdot \mathbf{E} + \mu \frac{\partial B}{\partial t}, \qquad \frac{d\mu}{dt} = 0$$

Ampere's equation

$$\nabla \times \mathbf{B} = \mu_0 (\mathbf{J}_i - e n_e (\mathbf{V}_{\mathrm{e}\perp} + u_{\parallel \mathrm{e}} \mathbf{b}))$$

$$\mathbf{V}_{\mathrm{e}\perp} = \frac{1}{B} \mathbf{E} \times \mathbf{b} - \frac{1}{enB} \mathbf{b} \times \nabla P_{\perp \mathrm{e}}$$

$$\mathbf{J}_i = \int f_i \mathbf{v} \, d\mathbf{v}, \quad u_{\parallel \mathrm{e}} = \int f_e \, v_\parallel \, d\mathbf{v}, \quad P_{\perp \mathrm{e}} = \int f_e \frac{1}{2} m_e v^2 \, d\mathbf{v}$$

Faraday's equation,

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E}$$

- Quasi-neutral
  - No displacement current in the Faraday's equation
- No transverse electron inertia (no electron polarization current). Electron FLR and polarization current can be added for reconnection problems.
- The magnetic field perturbation is 3-D, whereas in the $A_\parallel - \phi$ model $\delta \mathbf{B} = \nabla \times (A_\parallel \mathbf{b})$ is 2-D
- Unable to combine $A_\parallel - \phi$ field model with Vlasov ions. With GK ions ϕ is obtained from GK Poisson equation. With Vlasov ions the equation

$$n_i = n_e$$

does not determine ϕ! However, taking time derivative of this equation to the second order results in

$$\frac{n_0 q}{m_i} \nabla_\perp^2 \phi - \frac{q}{m_i} \nabla \phi \cdot \nabla n_0 + \frac{e}{m_e} \nabla_\parallel E_\parallel$$
$$= \nabla \cdot \left(\frac{1}{m_e} \nabla \cdot \mathbf{P}_i - \frac{q n_0}{m_i} \mathbf{V}_i \times \mathbf{B} \right) - \frac{1}{m_e} \nabla_\parallel \left(\frac{\delta \mathbf{B}}{B} \cdot \nabla (n_0 T_0) + \nabla_\parallel \delta P_{\parallel \mathrm{e}} \right) + \frac{\dot{\mathbf{E}} \times \mathbf{b}}{B} \cdot \nabla n_0$$

We have not been able to produce the Alfven waves solving this equation.

Frieman-Chen Electron GK Equation in $\mathbf{E}_1$ and $\mathbf{B}_1$

- Gyrokinetic equations are usually derived in terms of $\mathbf{A}$ and ϕ, to make explicit the ordering

$$\frac{\partial \mathbf{A}}{\partial t} \sim \epsilon_\delta \nabla_\perp \phi$$

- The Frieman-Chen gyrokinetic equation, assuming isotropy ($\partial F_0/\partial \mu = 0$),

$$\hat{L}_g \delta H_0 \equiv \left(\frac{\partial}{\partial t} + v_\parallel \mathbf{b} \cdot \nabla + \mathbf{v}_D \cdot \nabla \right) \delta H_0 = -\frac{q}{m} (S_L + \langle R_{\mathrm{NL}} \rangle),$$

where δH_0 is related to the perturbed distribution δF through $\delta F = \frac{q}{m} \phi \frac{\partial F_0}{\partial \epsilon} + \delta H_0$

$$S_L = \frac{\partial}{\partial t} \langle \phi - \mathbf{v} \cdot \mathbf{A} \rangle \frac{\partial F_0}{\partial \epsilon} - \nabla \langle \phi - \mathbf{v} \cdot \mathbf{A} \rangle \times \frac{\mathbf{b}}{\Omega} \cdot \nabla F_0,$$

$$\langle R_{\mathrm{NL}} \rangle = -\nabla \langle \phi - \mathbf{v} \cdot \mathbf{A} \rangle \times \frac{\mathbf{b}}{\Omega} \cdot \nabla \delta H_0.$$

- Define $\delta f = \frac{q}{m} \langle \phi \rangle \frac{\partial F_0}{\partial \epsilon} + \delta H_0$. The gyrokinetic equation for δf is, written in terms of $\mathbf{E}_1$ and $\mathbf{B}_1$

$$\frac{D}{Dt} \delta f = -\left(\frac{1}{B_0} \langle \mathbf{E}_1 \rangle \times \mathbf{b} + v_\parallel \frac{\langle \mathbf{B}_{1\perp} \rangle}{B_0} \right) \cdot \nabla F_0 + \frac{1}{m} \dot{\epsilon} \frac{\partial F_0}{\partial \epsilon}$$

$$\frac{D}{Dt} = \hat{L}_g + \left(\frac{1}{B_0} \langle \mathbf{E}_1 \rangle \times \mathbf{b} + v_\parallel \frac{\langle \mathbf{B}_{1\perp} \rangle}{B_0} \right) \cdot \nabla, \quad \dot{\epsilon} = q \left(v_\parallel \mathbf{b} + \mathbf{v}_D + v_\parallel \frac{\langle \mathbf{B}_{1\perp} \rangle}{B_0} \right) \cdot \langle \mathbf{E}_1 \rangle + q \langle \mathbf{v}_\perp \cdot \mathbf{E}_{1\perp} \rangle$$

- The perturbed electron diamagnetic flow comes from δf,

$$n_0 \mathbf{V}_D(\mathbf{x}) = \int (v_{\|}\mathbf{b} + \mathbf{v}_{\perp}(\mathbf{R}', \epsilon, \mu, \alpha))\delta f(\mathbf{R}', \epsilon, \mu)\delta(\mathbf{x} - \mathbf{R}' - \boldsymbol{\rho})\, J d\mathbf{R}'\, d\epsilon\, d\mu\, d\gamma$$

$n_0 \mathbf{V}_D$ is computed by depositing the particle current along the gyro-ring. In the drift-kinetic limit $\mathbf{V}_D$ reduces to the electron diamagnetic flow.

- The electron $\mathbf{E} \times \mathbf{B}$ flow comes from the first term in δF,

$$n_0 \mathbf{V}_E(\mathbf{x}) = \frac{\mathbf{q}}{\mathbf{m}} \int \mathbf{v}\left(\phi(\mathbf{x}) - \langle \phi \rangle (\mathbf{x} - \boldsymbol{\rho}, \epsilon, \mu)\right) \frac{\partial \mathbf{F_0}}{\partial \epsilon} \mathbf{J}\, \mathbf{d}\epsilon\, \mathbf{d}\mu\, \mathbf{d}\gamma$$

in eikonal form,

$$n_0 \mathbf{V}_E = n_0 \frac{h}{B_0} \delta \mathbf{E}_k \times \mathbf{b}$$

with $b = k_\perp^2 v_T^2 / \Omega^2$ and

$$h(b) = -\frac{1}{b^2} \int_0^\infty e^{-x^2/2b} J_0(b) J_0'(b) x^2\, dx$$

In the limit of small $k\rho \ll 1$ the factor $h(b)$ become unity, so that $n_0 \mathbf{V}_E$ become the total guiding center $\mathbf{E} \times \mathbf{b}$ flow.

Convert Ampere's Equation into Ohm's Law

Starting with Ampere's equation

$$\nabla \times \mathbf{B} = \mu_0(\mathbf{J}_i - en_e(\mathbf{V}_{\mathrm{e}\perp} + u_{\parallel \mathrm{e}}\mathbf{b}))$$

Taking derivative w.r.t. time,

$$\nabla \times \frac{\partial \mathbf{B}}{\partial t} = \mu_0 \left(\frac{\partial \mathbf{J}_i}{\partial t} - en_e(\frac{\partial \mathbf{V}_{\mathrm{e}\perp}}{\partial t} + \frac{\partial u_{\parallel \mathrm{e}}}{\partial t}\mathbf{b}) \right)$$

Only use the parallel component of this equation! Use Faraday's equation for LHS, and electron momentum equation,

$$m_e n_e \frac{\partial u_{\parallel \mathrm{e}}}{\partial t} + \nabla_\parallel \delta P_{\parallel e} + \delta \mathbf{B} \cdot \nabla P_{\parallel \mathrm{e}0} + en_e E_\parallel = 0$$

And neglect $\mathbf{b} \cdot \dfrac{\partial J_i}{\partial t}$ (smaller by mass ratio), to obtain parallel Ohm's Law

$$enE_\parallel + \frac{m_e}{\mu_0 e}\mathbf{b} \cdot \nabla \times \nabla \times \mathbf{E} = -\nabla_\parallel \delta P_{\parallel \mathrm{e}} - \frac{\delta \mathbf{B}}{B} \cdot \nabla P_{\parallel \mathrm{e}0}$$

The remaining two components of the Ampere's equation are rewritten as

$$en\mathbf{E}_\perp = -\frac{1}{\mu_0}\mathbf{b} \times (\nabla \times \mathbf{B}) - \mathbf{J}_i \times \mathbf{b} - \nabla_\perp \delta P_{\perp \mathrm{e}}$$

Explicit time evolving is unstable at small $k_\perp$ due to the compressional Alfvén wave

Combine the momentum equation and the Maxwell equations to obtain Ohm's law:

$$
\begin{aligned}
en\mathbf{E}_\perp &= -\frac{1}{\mu_0}\mathbf{b}\times(\nabla\times\mathbf{B}) - \mathbf{J}_i\times\mathbf{b} - \nabla_\perp\delta P_{\perp\mathrm{e}} \\
enE_\parallel + \frac{m_e}{\mu_0 e}\mathbf{b}\cdot\nabla\times\nabla\times\mathbf{E} &= -\nabla_\parallel\delta P_{\parallel\mathrm{e}} \\
\frac{\partial\mathbf{B}}{\partial t} &= -\nabla\times\mathbf{E}
\end{aligned}
$$

δf method for ions and electrons

$$
\begin{aligned}
\frac{d}{dt}\delta f_i &= -q\mathbf{E}\cdot\mathbf{v}_i f_{0\mathrm{i}} \\
\frac{d}{dt}\delta f_e &= -\left(\mathbf{v}_E + v_\parallel\frac{\delta\mathbf{B}_\perp}{B_0}\right)\cdot\nabla f_{0\mathrm{e}} + \left(-eE_\parallel v_\parallel + \mu\frac{\partial B}{\partial t}\right) f_{0\mathrm{e}}
\end{aligned}
$$

For ρ_i scale instabilities $k_\perp\rho_i \sim 1$, $\beta \sim 1\%$, the compressional wave frequency $\omega/\Omega_i \geq 10$, $\Omega_i\triangle t << 0.01$ is needed! We would like to be able to use $\Omega_i\triangle t \sim 0.1$, i.e., just small enough to get the gyro-motion.

Implicit Scheme

$$\frac{\delta\mathbf{B}^{n+1} - \delta\mathbf{B}^{n}}{\triangle t} = -\nabla \times \mathbf{E}^{n+1}$$

$$\mathbf{E}_{\perp}^{n+1} - \frac{\triangle t}{\beta}\mathbf{b} \times (\nabla \times \nabla \times \mathbf{E}^{n+1}) = -\nabla_{\perp}\delta P_{\perp \mathrm{e}}^{n+1} - \frac{1}{\beta}\mathbf{b} \times (\nabla \times \delta\mathbf{B}^{n}) - \mathbf{J}_{\mathrm{i}\perp}^{*} \times \mathbf{b} - \delta\mathbf{J}_{\perp \mathrm{i}} \times \mathbf{b}$$

$$E_{\parallel}^{n+1} + \frac{m_e}{m_i}\frac{1}{\beta}\mathbf{b} \cdot \nabla \times \nabla \times \mathbf{E}^{n+1} = -\nabla_{\parallel}\delta P_{\parallel \mathrm{e}}^{n+1}$$

- Particle coordinates and electron weights (hence pressure) explicitly advanced.
- Ions weights first advanced without $\mathbf{E}_{\perp}$ then used to gather $\mathbf{J}_{\mathrm{i}\perp}^{*}$

$$w_i^{*} = w_i^{n} + qE_{\parallel}^{n} v_{\parallel} \triangle t$$

- After fields solved, update ion weights

$$w_i^{\mathrm{n+1}} = w_i^{*} + qE_{\perp}^{\mathrm{n+1}} \cdot \mathbf{v}_{\perp} \triangle t$$

Implicit Scheme (cont'd)

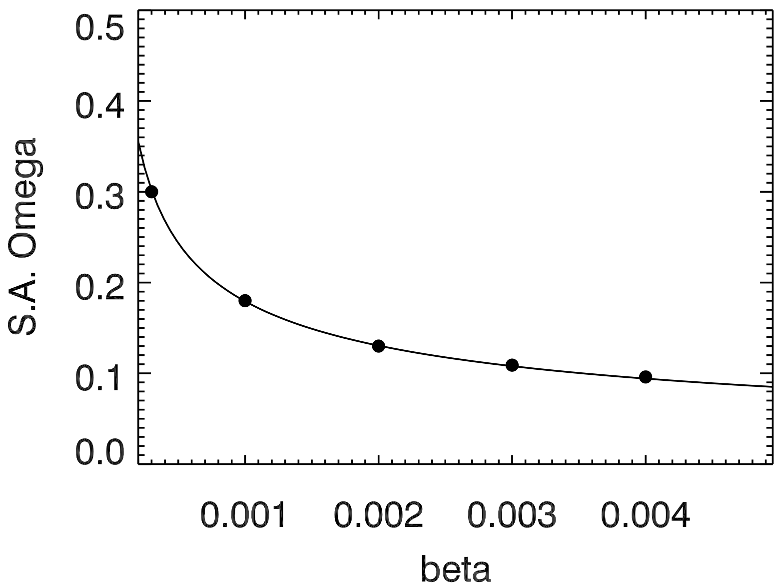
- It turns out necessary to treat the increment to $\mathbf{J}_{\perp \mathrm{i}}$ due to $\mathbf{E}_{\perp}^{\mathrm{n+1}}$ fully implicitly

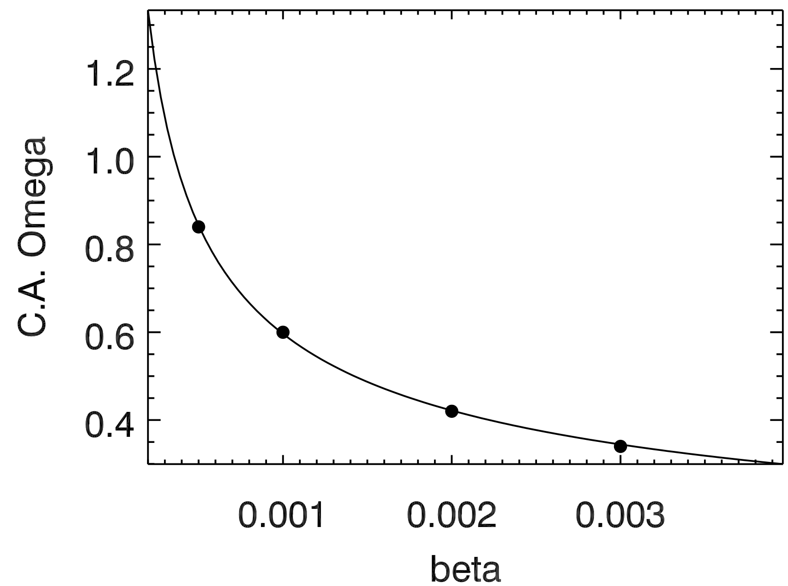
$$\delta \mathbf{J}_{\perp \mathrm{i}}(\mathbf{x}) = \triangle t \sum_{j} q \mathbf{E}_{\perp}^{\mathrm{n+1}}(\mathbf{x}_{j}^{\mathrm{n+1}}) \cdot \mathbf{v}_{j}^{\mathrm{n+1}} S(\mathbf{x} - \mathbf{x}_{j}^{\mathrm{n+1}})$$

$$\delta \mathbf{J}_{\perp \mathrm{i}}(\mathbf{x}) \approx q\, n_i\, \triangle t\, \mathbf{E}_{\perp}^{\mathrm{n+1}}(\mathbf{x}) \equiv \delta \mathbf{J}'_{\perp \mathrm{i}}$$

- Iterate on the difference between $\delta \mathbf{J}_{\perp \mathrm{i}}$ and $\delta \mathbf{J}'_{\perp \mathrm{i}}$
- $3 \sim 4$ iterations are accurate enough

3-D Shearless Slab Alfven Wave Simulation





$32 \times 32 \times 32$ grids, $1,048,576$ particles per species
For shear Alfvén wave, $k_\perp = 0$, $k_\parallel \rho_i = 0.00626$, initialize with $\delta \mathbf{B}_\perp$.
For compressional Alfven wave, $k_\parallel = 0$, $k_\perp \rho_i = 0.019$, initiallize with $\delta \mathbf{B}_\parallel$.
Shear Alfvén cold plasma dispersion relation obtained with the Hall term in the Ohm's law

$$\frac{\omega^2}{k_\parallel^2} = \left(1 - \frac{\omega}{\omega_{\rm ci}}\right) v_A^2$$

∇T_e Driven Kinetic Alfven Instability

$$\frac{\partial f_1}{\partial t} + v_\parallel \nabla_\parallel f_1 = \kappa(E_y + v_\parallel B_x) f_0 + (-E_\parallel v_\parallel + \mu \frac{\partial B_\parallel}{\partial t}) f_0$$

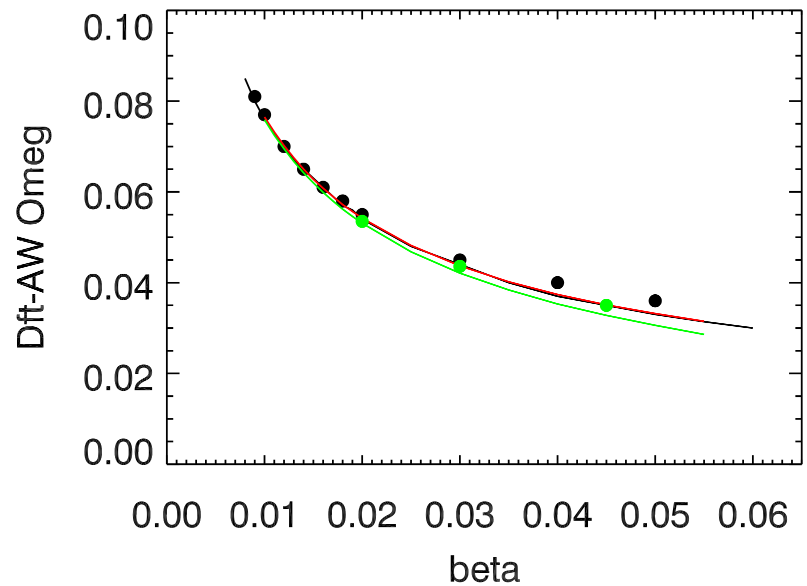
$$E_\parallel + \frac{m_e}{m_i}\frac{1}{\beta}\mathbf{b}\cdot\nabla\times\nabla\times\mathbf{E} - \kappa_T B_x = -\nabla_\parallel \delta P_{\parallel \mathrm{e}}$$

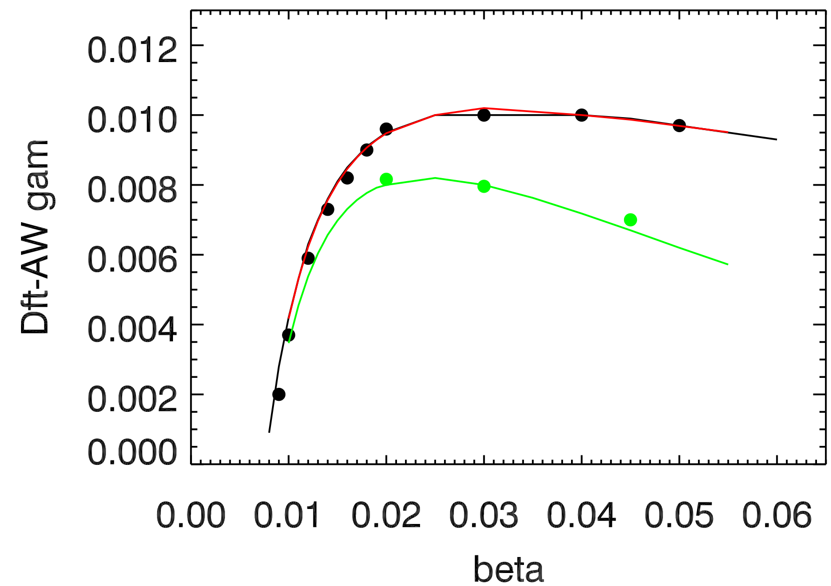
$$\mathbf{E}_\perp + \frac{1}{\beta}\mathbf{b}\times(\nabla\times\mathbf{B}_1) = -\mathbf{J}_i\times\mathbf{B} - \nabla_\perp \delta P_\perp$$

$$\frac{\partial \mathbf{B}_1}{\partial t} = -\nabla\times\mathbf{E}$$

$\kappa = \kappa_T(mv^2/2 - 3/2)$, $\kappa_T = -\frac{1}{T}\frac{\partial T_e}{\partial x}$

∇T_e Driven Kinetic Alfven Wave





- $k_x = 0$, $k_y \rho_s = 3.5$, $k_\| \rho_s = 0.00284$. $\delta p_{\perp \mathrm{e}} = 0$ for simulation data points and dispersion replation (solid black line).
- Green line from dispersion relation with $\delta p_{\perp \mathrm{e}}$, which has strong stabilizing effect at high beta.
- Red line from $A_\| / \phi$ gyrokinetic dispersion relation. Good agreement with $\delta p_{\perp \mathrm{e}} = 0$ Vlasov ion DR, because in GK model only parallel Ampere's equation is used.

SUMMARY

- We proposed a kinetic simulation model with Vlasov ions/Drift kinetic electrons which is
  - Quasi-neutral and fully electromagnetic
  - suitable for MHD scales, edge or ITB plasmas with strong $\mathbf{E} \times \mathbf{B}$ flows
- The time step for explicit integration limited by the compressional Alfvén wave
- Semi-implicit scheme allows $\Omega_i \triangle t \geq 0.1$
  - Treat Faraday's law and $\mathbf{E}_\perp \cdot \mathbf{v}_\perp$ in the ion weight equation implicitly
- Demonstrated 3-D shearless slab simulation for compressional and shear Alfvén waves, and whistler and Kinetic Alfvén instabilities driven by electron temperature gradient, and the ion accoustic waves.

The MHD equations for the shear Alfvén wave

- **Quasi-neutrality**

$$\boxed{-\nabla\cdot\frac{n_0 m_i}{B(\vec{r})^2}\nabla_\perp\phi=\delta n_i-\delta n_e}$$

- **Continuity equations**

$$\left.\begin{aligned}\frac{\partial\delta n_i}{\partial t}+n_0\vec{B}\cdot\nabla\frac{u_{\|i}}{B}+\vec{E}\times\hat{b}\cdot\nabla n_0&=0\\ \frac{\partial\delta n_e}{\partial t}+n_0\vec{B}\cdot\nabla\frac{u_{\|e}}{B}+\vec{E}\times\hat{b}\cdot\nabla n_0&=0\end{aligned}\right\}$$

$$\boxed{\frac{\partial}{\partial t}(\delta n_i-\delta n_e)+n_0\vec{B}\cdot\nabla\frac{(u_{\|i}-u_{\|e})}{B}=0}$$

- **Ampere's law**

$$\boxed{-\nabla_\perp{}^2A_\|=\mu_0 q n_0(u_{\|i}-u_{\|e})}$$

- **Faraday's law**

$$\boxed{\frac{\partial A_\|}{\partial t}+\hat{b}\cdot\nabla\phi=0}$$

- **MHD TAE equation**

$$\boxed{\frac{\partial^2}{\partial t^2}\nabla\cdot\frac{1}{V_A(\vec{r})^2}\nabla_\perp\phi=\vec{B}\cdot\nabla\frac{1}{B}\nabla_\perp{}^2\hat{b}\cdot\nabla\phi}$$

Simulation parameters

- Basic parameters:

$$B_0 = 1.91\,\mathrm{T},\ R_0 = 1.67\,m,\ a = 0.36 R_0$$

- Profile:

$$q(r) = 1.3\left(\frac{r}{r_0}\right)^{0.3},\quad r_0 = a/2,\quad \hat{s}(r) = 0.3\left(\frac{r_0}{r}\right)^{0.7}$$

- Plasma to magnetic pressure ratio and mass ratio:

$$\beta = 3.0\times 10^{-3},\ m_i / m_p = 2$$

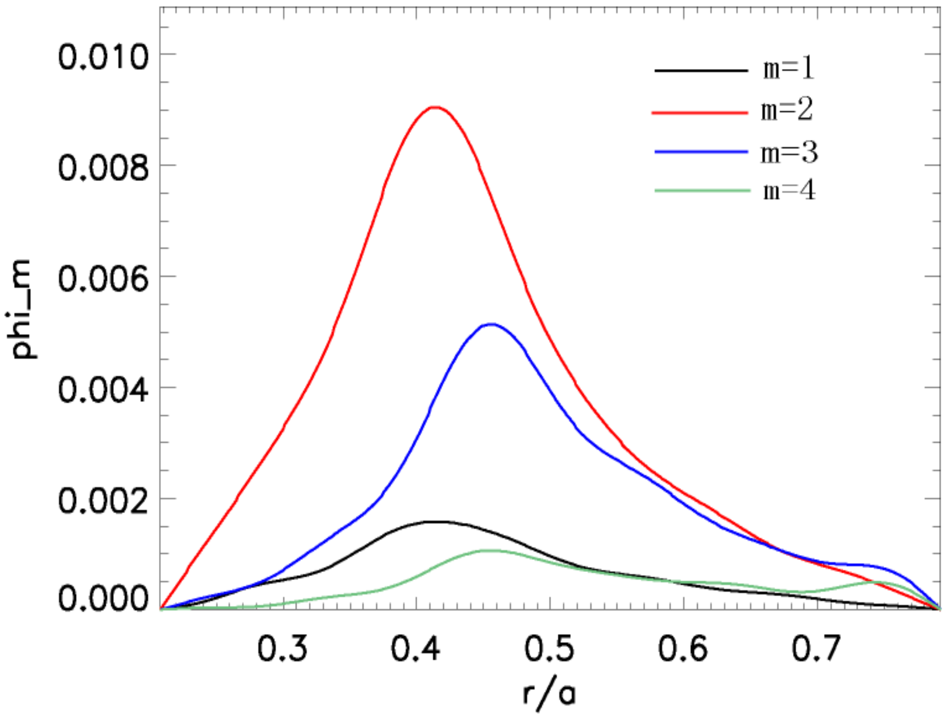
- Simulation domain:

$$[0.2a, 0.8a]$$

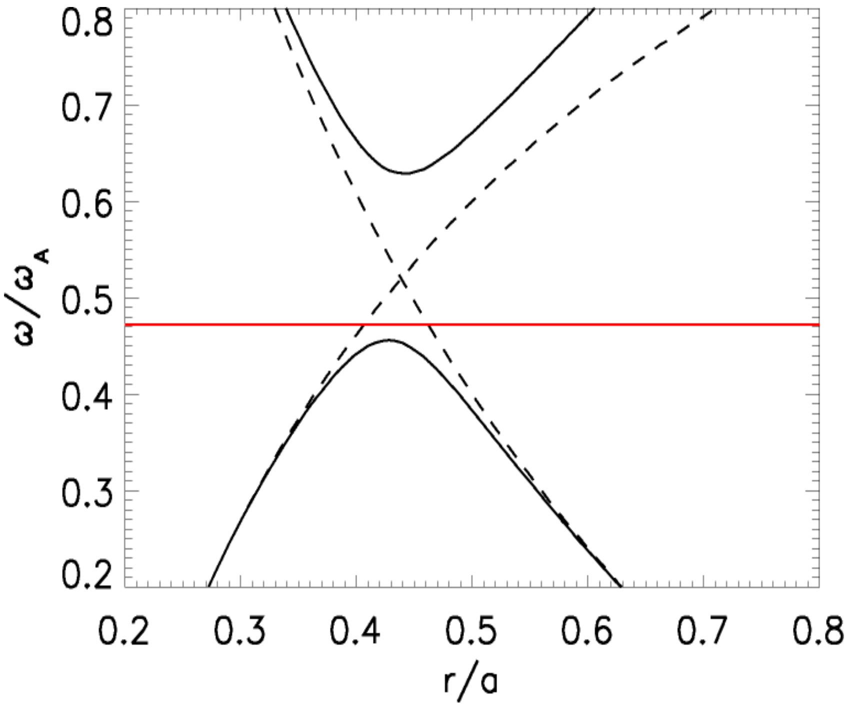
- External drive:

Add external n=2 current for 200 steps, then observe the subsequent oscillation and mode structure

Simulation results with 2-fluid model



Different poloidal harmonics of the electric potential



Comparison with theoretical calculation

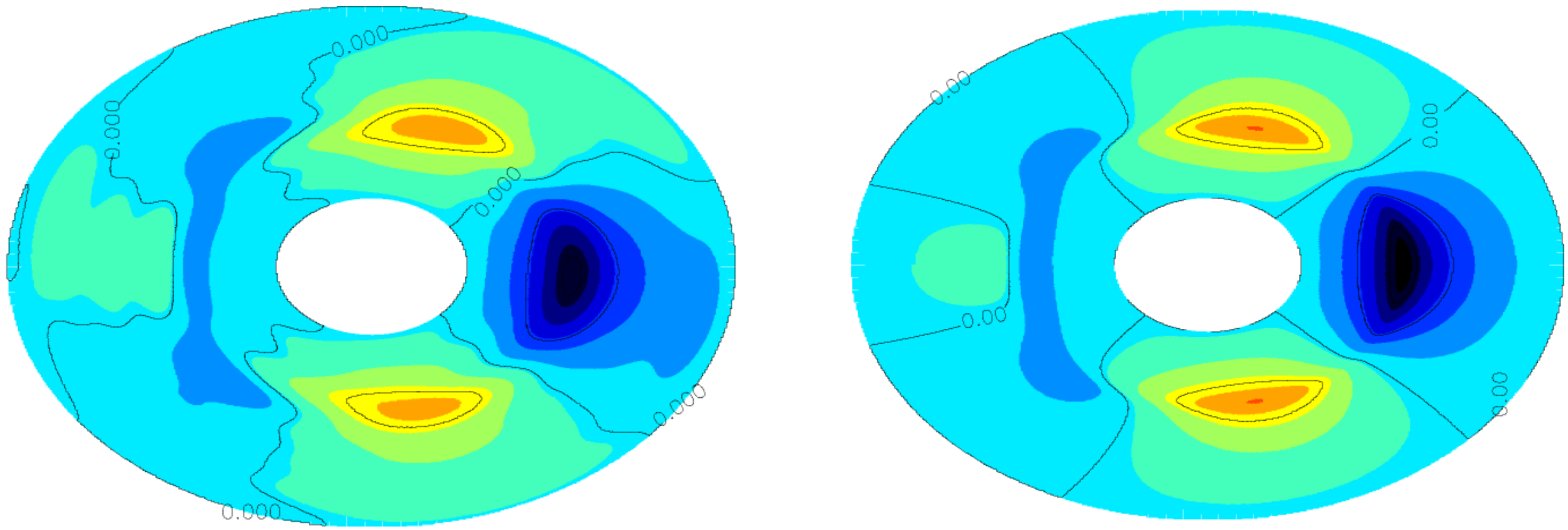
- The global mode structure is observed
- The mode frequency falls in the gap predicted by Fu and Van dam (1989)
- The mode frequency is well above the lower continuum branch

Benchmark with eigenmode analysis

- The simplified form: $\vec{B}\cdot\nabla\frac{1}{B} \sim \hat{b}\cdot\nabla \;,\quad \hat{b}\cdot\nabla \sim \frac{\partial}{\partial z}$

  In simulations, continuity equation becomes: $$\frac{\partial}{\partial t}(\delta n_i - \delta n_e) + n_0\hat{b}\cdot\nabla(u_{\|i} - u_{\|e}) = 0$$

- Eigenmode calculation: $$\frac{1}{V_A^2}\nabla\cdot\frac{\partial^2}{\partial t^2}\nabla_\perp\phi = \hat{b}\cdot\nabla\nabla_\perp^2\hat{b}\cdot\nabla\phi$$



contour plot of electric potential from simulations

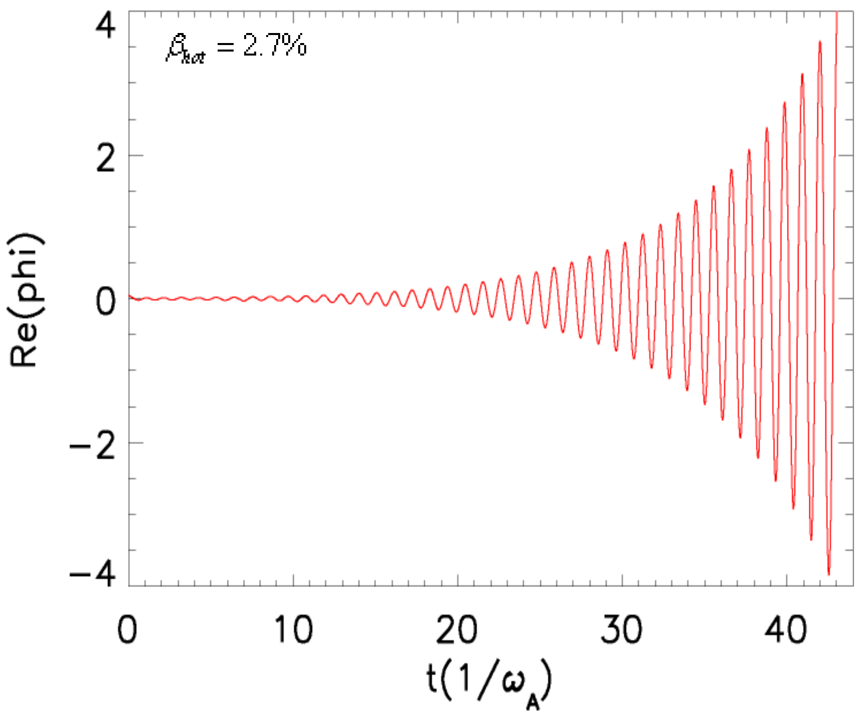
contour plot of electric potential from eigenmode analysis

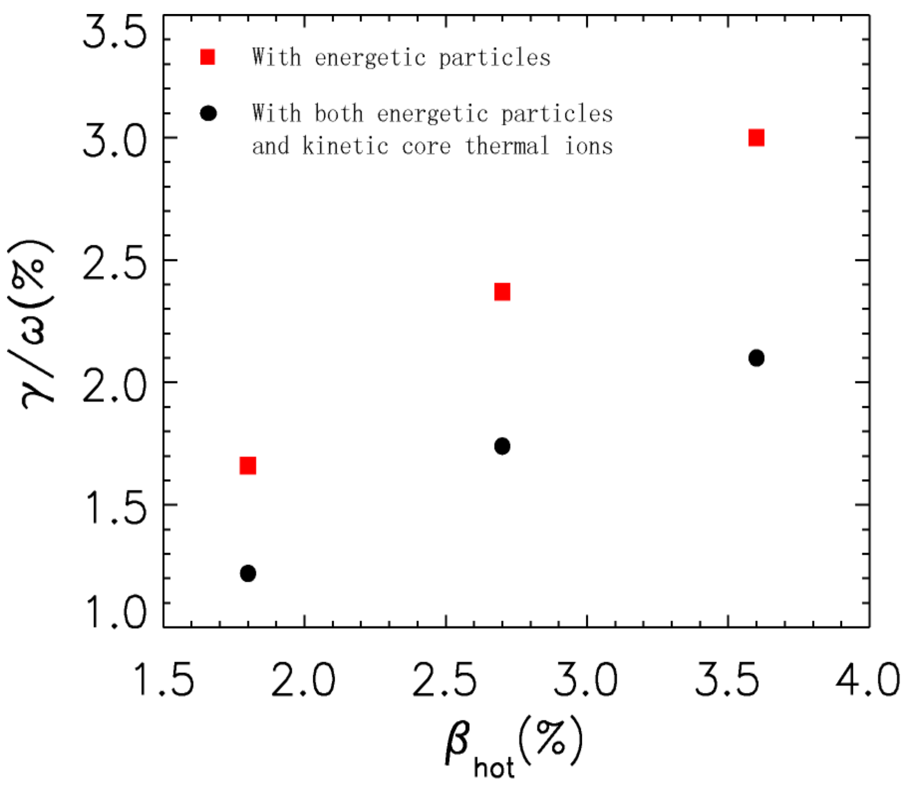
$$\frac{\Delta\omega}{\omega_0} < 0.5\%$$

Driving effect from energetic particles and damping effect from kinetic core thermal ions

- There is a linear scaling between the mode growth rate and the center energetic particle pressure

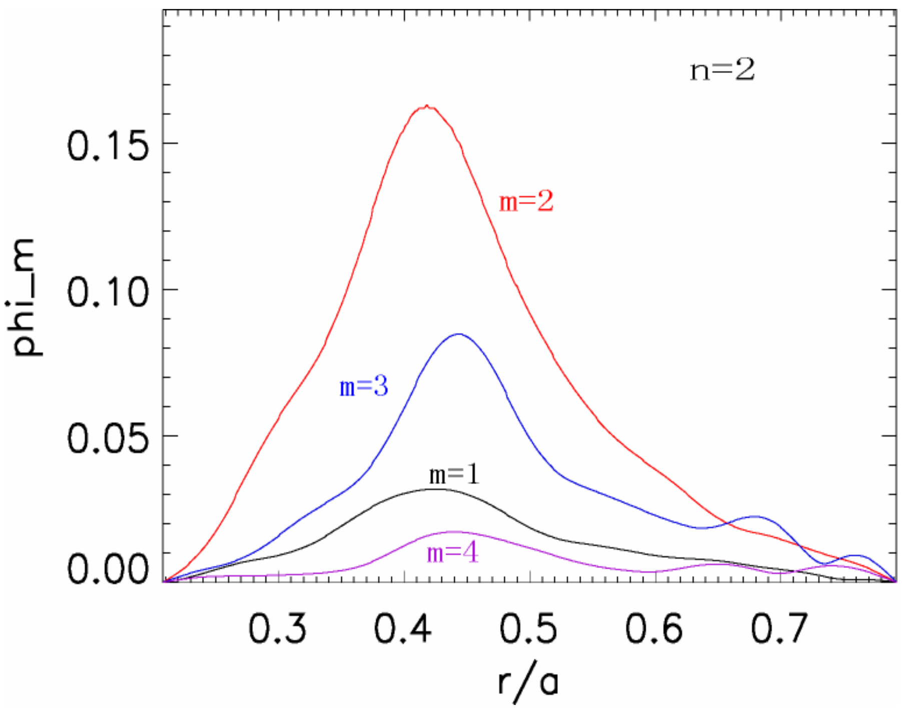
- The damping effect from kinetic core thermal ions is observed



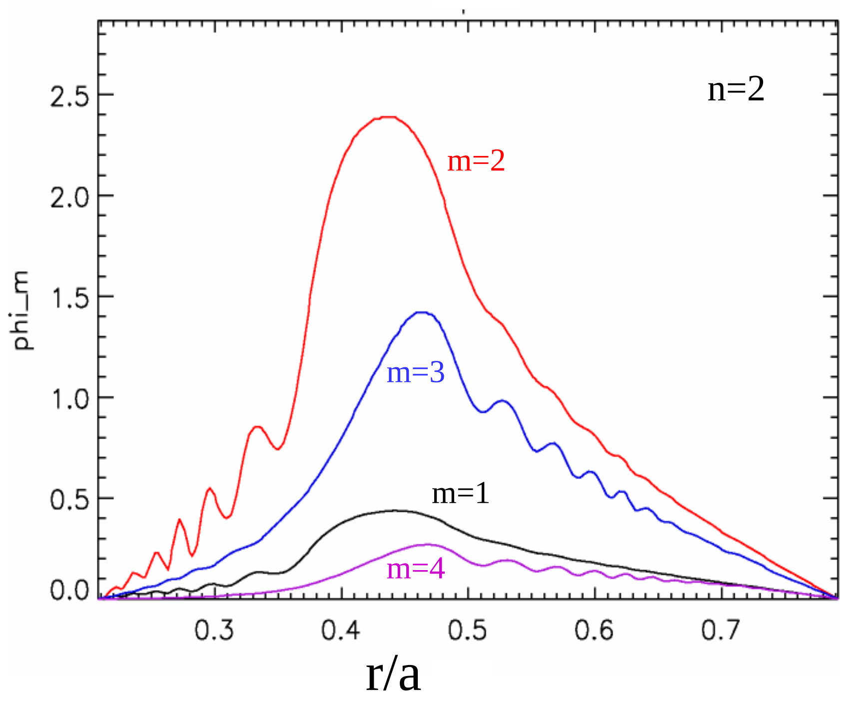


The damping of TAEs

- Thermal ion Landau damping (very small)
- Trapped electron collisional damping (neglected by fluid electron approximation)
- Radiative damping due to coupling to the kinetic Alfvén waves



Mode structure with MHD operator



Mode structure with gyrokinetic operator

Radiative damping

Analytical calculation of the thermal ion radiative damping (Fu. et al., Phys. Plasmas, 1996)

$$\frac{\gamma_d}{\omega} \propto e^{-\frac{2}{\sqrt{\eta}} I_1(h)},$$

where $\eta = 8n^2 \hat{s}^2 q^2 \rho_i^2 / \varepsilon^3 / r_m^2$

$$I_1(h) = \int_0^{\sqrt{1+h}} \sqrt{1-(h-x^2)^2}\, dx,$$

where $h = (\omega^2 / \omega_{TAE}^2 - 1) / \varepsilon$

Damping rate increases with thermal ion larmor radius