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# DRIFT-KINETIC EQUATION FOR SLOW-DYNAMICS ELECTRON CLOSURES

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## SCOPE OF THIS WORK

**Drift-kinetic evaluation of the gyrotropic moments $p_{e\|} - p_{e\perp}$ and $q_{e\|}$ needed to close the electron fluid equations with applied RF sources (inertia and non-gyrotropic viscosity neglected), that is the generalized Ohm's law with RF current drive:**

$$\mathbf{E} \; + \; \mathbf{u}_e \times \mathbf{B} \; + \; \frac{1}{en}\Big\{ \nabla \cdot \Big[ p_e \mathbf{I} + (p_{e\|} - p_{e\perp})(\mathbf{bb} - \mathbf{I}/3) \Big] \; - \; \mathbf{F}_e^{coll} \; - \; \mathbf{F}_e^{RF} \Big\} \; = \; 0$$

**and the energy equation with RF heating:**

$$\frac{3}{2}\Big[ \frac{\partial p_e}{\partial t} + \nabla \cdot \Big( p_e \mathbf{u}_e \Big) \Big] \; + \; p_e \nabla \cdot \mathbf{u}_e \; + \; (p_{e\|} - p_{e\perp}) \Big\{ \mathbf{b} \cdot \Big[ (\mathbf{b} \cdot \nabla) \mathbf{u}_e \Big] - \nabla \cdot \mathbf{u}_e / 3 \Big\} \; +$$

$$+ \; \nabla \cdot \Big( q_{e\|} \mathbf{b} + \mathbf{q}_{e\perp} \Big) \; - \; G_e^{coll} \; - \; G_e^{RF} \; = \; 0 \; .$$

**Low-collisionality and slow-dynamics (diamagnetic drift scale) analysis.**

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## PROPOSED ORDERING SCHEME

$$\rho_\iota/L \sim k\rho_\iota \sim (m_e/m_\iota)^{1/2} \sim \delta \ll 1\ , \qquad \rho_e/L \sim k\rho_e \sim \delta^2$$

$$\frac{\|f_\iota - f_{\iota M}\|}{\|f_{\iota M}\|} \sim \delta\ , \qquad \frac{\|f_e - f_{eM}\|}{\|f_{eM}\|} \sim \delta^2$$

$$\nu_\iota \sim \delta\nu_e \sim \delta^2\Omega_{c\iota}$$

$$\partial/\partial t \sim \delta^2\Omega_{c\iota}\ , \qquad u_\iota \sim u_e \sim \delta v_{th\iota} \sim \delta^2 v_{the}$$

$$\epsilon \sim k_\parallel/k_\perp \sim \beta \sim ... \sim 1$$

**Hence, using $\Omega_{c\iota}$ as reference, the ordering of time scales is:**

$$O(\delta^{-2}): \ \ \Omega_{ce} = v_{the}/\rho_e$$

$$O(1): \quad \Omega_{c\iota} = v_{th\iota}/\rho_\iota \sim v_{the}/L$$

$$O(\delta): \quad \nu_e \sim \omega_A = kc_A \sim \omega_s = kc_s \sim v_{th\iota}/L$$

$$O(\delta^2): \quad \nu_\iota \sim \partial/\partial t \sim ku_{\iota,e} \sim \omega_{*\iota,e} = ku_{*\iota,e}$$

## ELECTRON KINETIC EQUATION IN THE MEAN FLOW REFERENCE FRAME AND THE CYLINDRICAL VELOCITY SPACE COORDINATES ALIGNED WITH THE MAGNETIC FIELD

$$\mathbf{v} \;=\; \mathbf{u}_e(\mathbf{x},t) \;+\; v'_\parallel \;\mathbf{b}(\mathbf{x},t) \;+\; v'_\perp \;[\cos\alpha\;\mathbf{e}_1(\mathbf{x},t) + \sin\alpha\;\mathbf{e}_2(\mathbf{x},t)]$$

$$\Omega_{ce}\;\frac{\partial f_e(v'_\parallel, v'_\perp, \alpha, \mathbf{x}, t)}{\partial \alpha} \;+\; \sum_{l=-2}^{2} e^{il\alpha}\left[\Lambda_l\;f_e \;+\; \lambda_l\;\frac{\partial f_e}{\partial \alpha}\right] \;=\; C_{ee}(f_e, f_e) \;+\; C_{e\iota}(f_e, f_\iota) \;+\; Q^{RF}(f_e)$$

**where**

$$\Lambda_l(\partial/\partial v'_\parallel, \partial/\partial v'_\perp, \partial/\partial \mathbf{x}, \partial/\partial t, v'_\parallel, v'_\perp, \mathbf{x}, t) \;=\; \Lambda^*_{-l} \;=\; \Lambda_l^{(0)} \;+\; \Lambda_l^{(2)} \;=\; O(v_{the}/L) \;+\; O(\delta^2 v_{the}/L)$$

$$\lambda_l(v'_\parallel, v'_\perp, \mathbf{x}, t) \;=\; \lambda^*_{-l} \;=\; \lambda_l^{(0)} \;+\; \lambda_l^{(2)} \;=\; O(v_{the}/L) \;+\; O(\delta^2 v_{the}/L)$$

$$C_{e\iota}(f_e, f_\iota) \sim Q^{RF}(f_e) \sim (\delta^2 v_{the}/L) f_e \;, \qquad C_{ee}(f_e, f_e) \sim (\delta^3 v_{the}/L) f_e$$

## RECURSIVE SOLUTION FOR THE DISTRIBUTION FUNCTION

$$f_e(v'_\|, v'_\perp, \alpha, \mathbf{x}, t) \;=\; f_{eM}(v'^2, \mathbf{x}, t) \;+\; f_e^{(2)}(v'_\|, v'_\perp, \alpha, \mathbf{x}, t) \;+\; ...$$

**where**

$$f_{eM}(v'^2, \mathbf{x}, t) \;=\; \left(\frac{m_e}{2\pi}\right)^{3/2} \frac{n(\mathbf{x}, t)}{T_e(\mathbf{x}, t)^{3/2}} \;\exp\left[-\;\frac{m_e v'^2}{2T_e(\mathbf{x}, t)}\right] \qquad \textbf{with} \qquad \mathbf{b}\cdot\nabla \ln T_e(\mathbf{x}, t) \;=\; O(\delta^2)$$

**satisfies**

$$\Lambda_0^{(0)} f_{eM} \;=\; O(\delta^2 v_{the}/L) f_{eM} \;.$$

**Then,**

$$f_e^{(2)} = \frac{i}{\Omega_{ce}}\left(e^{i\alpha}\Lambda_1^{(0)} f_{eM} - e^{-i\alpha}\Lambda_{-1}^{(0)} f_{eM}\right) \;+\; \bar{f}_e^{(2)}$$

**where** $\bar{f}_e^{(2)}(v'_\|, v'_\perp, \mathbf{x}, t)$ **satisfies the second order drift-kinetic equation:**

$$\Lambda_0^{(0)}\bar{f}_e^{(2)} \;+\; (\Lambda_0^{(0)} + \Lambda_0^{(2)}) f_{eM} \;+\; (\Lambda_{-1}^{(0)} + i\lambda_{-1}^{(0)})\left(\frac{i}{\Omega_{ce}}\Lambda_1^{(0)} f_{eM}\right) - (\Lambda_1^{(0)} - i\lambda_1^{(0)})\left(\frac{i}{\Omega_{ce}}\Lambda_{-1}^{(0)} f_{eM}\right) \;=$$

$$=\; \langle C_{e\iota}(f_{eM}, f_\iota) + Q^{RF}(f_{eM})\rangle^{(2)} \;\equiv\; \frac{1}{2\pi}\oint d\alpha \left[C_{e\iota}(f_{eM}, f_\iota) + Q^{RF}(f_{eM})\right]^{(2)}$$

# SECOND ORDER ELECTRON DRIFT-KINETIC EQUATION

$$v'_\parallel \mathbf{b}\cdot\frac{\partial \bar{f}_e^{(2)}}{\partial \mathbf{x}} \;+\; \left(\frac{T_e}{m_e}\mathbf{b}\cdot\nabla\ln n - \frac{v_\perp'^2}{2}\mathbf{b}\cdot\nabla\ln B\right)\frac{\partial \bar{f}_e^{(2)}}{\partial v'_\parallel} \;+\; \frac{v'_\perp v'_\parallel}{2}\mathbf{b}\cdot\nabla\ln B\;\frac{\partial \bar{f}_e^{(2)}}{\partial v'_\perp} \;+$$

$$+\;\Big\{\frac{v'_\parallel}{2T_e}\left[\frac{m_e}{T_e}(v_\parallel'^2+v_\perp'^2)-5\right]\mathbf{b}\cdot\nabla T_e \;-\; \frac{v'_\parallel}{nT_e}\;\mathbf{b}\cdot\left[\nabla p_{e\parallel}^{(2)} - \left(p_{e\parallel}^{(2)} - p_{e\perp}^{(2)}\right)\nabla\ln B - \mathbf{F}_e^{coll(2)} - \mathbf{F}_e^{RF(2)}\right] \;+$$

$$+\;\left(\frac{m_e v_\perp'^2}{2T_e}-1\right)\nabla\cdot\mathbf{u}_e \;+\; \frac{m_e}{2T_e}(2v_\parallel'^2 - v_\perp'^2)\;\mathbf{b}\cdot[(\mathbf{b}\cdot\nabla)\mathbf{u}_e] \;+\; \frac{1}{2T_e}\left[\frac{m_e}{T_e}(v_\parallel'^2+v_\perp'^2)-3\right]\left(\frac{\partial T_e}{\partial t}+\mathbf{u}_e\cdot\nabla T_e\right) \;-$$

$$-\;\frac{1}{2\Omega_{ce}T_e}\left[\frac{m_e}{T_e}(v_\parallel'^2+v_\perp'^2)-5\right]\left(v_\parallel'^2\mathbf{b}\times\boldsymbol{\kappa} + \frac{v_\perp'^2}{2}\mathbf{b}\times\nabla\ln B\right)\cdot\nabla T_e \;+$$

$$+\;\frac{1}{2m_e\Omega_{ce}n}\left[\frac{m_e}{T_e}(v_\parallel'^2+2v_\perp'^2)-5\right](\mathbf{b}\times\nabla n)\cdot\nabla T_e\;\Big\}f_{eM} \;=\; \langle C_{e\iota}(f_{eM},f_\iota) + Q^{RF}(f_{eM})\rangle^{(2)}$$

**with**

$$p_{e\parallel}^{(2)} \;=\; 2\pi m_e\int_{-\infty}^{\infty}dv'_\parallel\int_0^\infty dv'_\perp\; v'_\perp\; v_\parallel'^2\;\bar{f}_e^{(2)} \qquad \textbf{and} \qquad p_{e\perp}^{(2)} \;=\; \pi m_e\int_{-\infty}^{\infty}dv'_\parallel\int_0^\infty dv'_\perp\; v_\perp'^3\;\bar{f}_e^{(2)}$$

# MOMENTS OF THE SECOND ORDER ELECTRON DRIFT-KINETIC EQUATION

$1$ **and** $v'_{\|}$ **moments satisfied identically.**

$m_e(v'^2_{\|} + v'^2_{\perp})/2$ **moment:**

$$\frac{3n}{2}\left(\frac{\partial T_e}{\partial t} + \mathbf{u}_e \nabla \cdot T_e\right) \;+\; nT_e \nabla \cdot \mathbf{u}_e \;+\; \nabla \cdot \left(q_{e\|}\mathbf{b}\right) \;-\; \nabla \cdot \left(\frac{5nT_e}{2m_e\Omega_{ce}}\mathbf{b} \times \nabla T_e\right) \;=\; G_e^{coll(2)} \;+\; G_e^{RF(2)}$$

$m_e(v'^2_{\|} - v'^2_{\perp}/2)$ **moment:**

$$nT_e\Big\{\mathbf{b} \cdot \Big[3(\mathbf{b} \cdot \nabla)\mathbf{u}_e\Big] - \nabla \cdot \mathbf{u}_e\Big\} \;+\; \nabla \cdot \Big[(2q_{eB\|} - q_{eT\|})\mathbf{b}\Big] \;+\; 3q_{eT\|}\mathbf{b} \cdot \nabla \ln B \;+$$

$$+\; \nabla \cdot \left(\frac{nT_e}{m_e\Omega_{ce}}\mathbf{b} \times \nabla T_e\right) \;+\; \left(\frac{3nT_e}{m_e\Omega_{ce}}\mathbf{b} \times \nabla T_e\right) \cdot \boldsymbol{\kappa} \;=\; 2G_{eB}^{coll(2)} - G_{eT}^{coll(2)} + 2G_{eB}^{RF(2)} - G_{eT}^{RF(2)}$$

**with**

$$q_{eB\|} = \pi m_e \int_{-\infty}^{\infty} dv'_{\|} \int_0^{\infty} dv'_{\perp}\; v'_{\perp}\; v'^3_{\|}\; \bar{f}_e^{(2)}\;, \qquad q_{eT\|} = \pi m_e \int_{-\infty}^{\infty} dv'_{\|} \int_0^{\infty} dv'_{\perp}\; v'^3_{\perp}\; v'_{\|} \bar{f}_e^{(2)} \quad \textbf{and} \quad q_{e\|} = q_{eB\|} + q_{eT\|}$$

## THE $v'_\parallel$-EVEN AND $v'_\parallel$-ODD PARTS OF $\bar{f}_e^{(2)}$ CAN BE SOLVED FOR INDEPENDENTLY

**Call** $$\bar{f}_e^{(2)}(v'_\parallel, v'_\perp, \mathbf{x}, t) = g_e^{(2)}(v'^2_\parallel, v'_\perp, \mathbf{x}, t) + m_e v'_\parallel h_e^{(2)}(v'^2_\parallel, v'_\perp, \mathbf{x}, t)$$

**and use the random kinetic energy $\varepsilon = m_e(v'^2_\parallel + v'^2_\perp)/2$ and magnetic moment $\mu = m_e v'^2_\perp/(2B)$ as phase-space coordinates.**

**Then,**

$$\mathbf{b} \cdot \frac{\partial g_e^{(2)}(\varepsilon, \mu, \mathbf{x}, t)}{\partial \mathbf{x}} \; + \; T_e \mathbf{b} \cdot \nabla \ln n \, \frac{\partial g_e^{(2)}(\varepsilon, \mu, \mathbf{x}, t)}{\partial \varepsilon} \; + \; \Big\{ \left( \frac{\varepsilon}{T_e} - \frac{5}{2} \right) \mathbf{b} \cdot \nabla \ln T_e \; -$$
$$- \; \frac{1}{nT_e} \, \mathbf{b} \cdot \left[ \nabla p_{e\parallel}^{(2)} - \left( p_{e\parallel}^{(2)} - p_{e\perp}^{(2)} \right) \nabla \ln B - \mathbf{F}_e^{RF(2)} \right] \; \Big\} f_{eM} \; = \; \frac{1}{v'_\parallel} \langle C_{e\iota}(f_{eM}, f_\iota) + Q^{RF}(f_{eM}) \rangle_{odd}^{(2)}$$

**with**

$$\int_0^\infty d\varepsilon \int_0^{\varepsilon/B} d\mu \;\, B \;\, [2(\varepsilon - \mu B)]^{-1/2} \;\, g_e^{(2)} \;\; = \;\; 0$$

$$p_{e\parallel}^{(2)} \;\; = \;\; 4\pi m_e^{-3/2} \int_0^\infty d\varepsilon \int_0^{\varepsilon/B} d\mu \;\, B \;\, [2(\varepsilon - \mu B)]^{1/2} \;\, g_e^{(2)}$$

$$p_{e\perp}^{(2)} \;\; = \;\; 4\pi m_e^{-3/2} \int_0^\infty d\varepsilon \int_0^{\varepsilon/B} d\mu \;\, B \;\, [2(\varepsilon - \mu B)]^{-1/2} \;\, \mu B \;\, g_e^{(2)}$$

**and**

$$
\begin{aligned}
&2(\varepsilon-\mu B)\left[\mathbf{b}\cdot\frac{\partial h_e^{(2)}(\varepsilon,\mu,\mathbf{x},t)}{\partial\mathbf{x}} \;+\; T_e\mathbf{b}\cdot\nabla\ln n\;\frac{\partial h_e^{(2)}(\varepsilon,\mu,\mathbf{x},t)}{\partial\varepsilon}\right] \;+\; \left(T_e\mathbf{b}\cdot\nabla\ln n-\mu\mathbf{b}\cdot\nabla B\right)\,h_e^{(2)} \;+\\
&+\Big\{\;\left(\frac{2\varepsilon-3\mu B}{3T_e}\right)\left(3\mathbf{b}\cdot[(\mathbf{b}\cdot\nabla)\mathbf{u}_e]-\nabla\cdot\mathbf{u}_e\right) \;-\; \frac{1}{nT_e}\left(\frac{2\varepsilon}{3T_e}-1\right)\left[\nabla\cdot(q_{e\|}\mathbf{b})-G_e^{RF(2)}\right] \;-\\
&-\;\frac{1}{m_e\Omega_{ce}}\left[\;2\left(\frac{\varepsilon^2-\varepsilon\mu B}{T_e^2}\right)\;-\;\frac{5}{3}\left(\frac{4\varepsilon-3\mu B}{T_e}\right)\;+\;\frac{5}{2}\;\right](\mathbf{b}\times\boldsymbol{\kappa})\cdot\nabla T_e \;-\\
&-\;\frac{1}{m_e\Omega_{ce}}\left[\;\frac{\varepsilon\mu B}{T_e^2}\;-\;\frac{5}{6}\left(\frac{2\varepsilon+3\mu B}{T_e}\right)\;+\;\frac{5}{2}\;\right](\mathbf{b}\times\nabla\ln B)\cdot\nabla T_e \;-\\
&-\;\frac{1}{m_e\Omega_{ce}}\left(\frac{2\varepsilon-3\mu B}{3T_e}\right)(\mathbf{b}\times\nabla\ln n)\cdot\nabla T_e\;\Big\}f_{eM} \;=\; \left\langle C_{e\iota}(f_{eM},f_\iota)+Q^{RF}(f_{eM})\right\rangle_{even}^{(2)}
\end{aligned}
$$

**with**

$$\int_0^\infty d\varepsilon\int_0^{\varepsilon/B}d\mu\;B\;[2(\varepsilon-\mu B)]^{1/2}\;h_e^{(2)} \;=\; 0$$

$$q_{e\|} \;=\; 4\pi m_e^{-3/2}\int_0^\infty d\varepsilon\int_0^{\varepsilon/B}d\mu\;B\;[2(\varepsilon-\mu B)]^{1/2}\;\varepsilon\;h_e^{(2)}$$

# SUMMARY

We have derived a drift-kinetic equation suitable to evaluate the non-Maxwellian contributions to the gyrotropic pressure tensor and the parallel heat flux for electrons (and therefore close their fluid equations), in the second order of a low-colisionality and slow-dynamics expansion.

With the adopted low-collisionality ordering scheme (which is realistic for fusion-relevant plasmas), the main collisional transport effects as well as the explicit collisional terms (i.e. moments of the collision operator) in the generalized Ohm's law and the energy equation are third order.

ECCD RF sources have been consistently incorporated in the system. Near term applications of interest to the CSWIM project will be the study of current drive effects in systems of increasing geometrical complexity: slab, bumpy cylinder, axisymmetric tokamak, full three-dimensional.

The analysis will proceed to the third order so as to include properly the transport-scale physics, as needed in particular for the study of neoclassical tearing modes.