Exponential Growth of Nonlinear Ballooning Instability[1]

A Nonlinear Benchmark for NIMROD Code

Ping Zhu

University of Wisconsin-Madison
Collaborators: C. R. Sovinec and C. C. Hegna

CEMM Meeting
November 16, 2008
Dallas, Texas

[1] Research supported by U.S. Department of Energy.

Most benchmark problems have been linear; nonlinear benchmark problems are emerging

- Benchmark for numerical codes
  - Code-experiment comparison (Validation)
  - Code-code comparison (Benchmark)
  - **Code-theory comparison (Verification)** –> this talk
- Recent linear benchmark:
  - Ideal and two-fluid *g*-mode

    [e.g. Schnack 2005; Ferraro and Jardin 2006; Zhu, Schnack, Ebrahimi *et al.* 2008]
  - Resistive and two-fluid tearing mode

    [e.g. King, Sovinec, Mirnov, Ramos 2008]
  - Ballooning and peeling instability [e.g. Kruger *et al.* 2007; Squires *et al.* 2008]
  - ......
- Recent nonlinear benchmark:
  - 2D resistive and fast reconnection

    [e.g. Bhattacharjee, Germaschewski, and Ng 2004; Chacon and Simakov 2007; 2008]
  - 1/1 kink-tearing and sawtooth (two-fluid)

    [e.g. Germaschewski and Bhattacharjee 2006; Lukin and Jardin 2007; Sovinec 2008]
  - **Nonlinear ballooning** [e.g. Zhu, Hegna, and Sovinec 2008] –> this talk
  - ......

Different nonlinear regimes of ballooning instability are characterized by the relative strength of the nonlinearity with powers of n^{-1}

- For $\varepsilon \ll n^{-1}$, linear ballooning mode theory [Connor, Hastie, and Taylor, 1979; Dewar and Glasser, 1983]
- For $\varepsilon \sim n^{-1}$, early nonlinear regime [Cowley and Artun, 1997; Hurricane, Fong, and Cowley, 1997; Wilson and Cowley, 2004]
- For $\varepsilon \sim n^{-1/2}$, **intermediate nonlinear regime** $\rightarrow$ this work [Zhu, Hegna, and Sovinec, 2006; Zhu *et al.* , 2007; Zhu and Hegna, 2008; Zhu, Hegna, and Sovinec, 2008;]
- For $\varepsilon \gg n^{-1/2}$, late nonlinear regime; analytic theory under development.

The local linear ballooning mode structure and growth continue to satisfy the nonlinear ballooning equations in Lagrangian space

- The nonlinear ballooning equations can be written in the compact form [Zhu and Hegna, 2008]

$$\left[\Psi + \xi^{\Psi}, \rho|\mathbf{e}_{\perp}|^2 \partial_t^2 \xi^{\Psi} - \mathcal{L}_{\perp}(\xi^{\Psi}, \xi^{\parallel})\right] = 0, \tag{1}$$

$$\rho B^2 \partial_t^2 \xi^{\parallel} - \mathcal{L}_{\parallel}(\xi^{\Psi}, \xi^{\parallel}) = 0. \tag{2}$$

- The general solution satisfies

$$\rho|\mathbf{e}_{\perp}|^2 \partial_t^2 \xi^{\Psi} = \mathcal{L}_{\perp}(\xi^{\Psi}, \xi^{\parallel}) + N(\Psi + \xi^{\Psi}, l, t), \tag{3}$$

$$\rho B^2 \partial_t^2 \xi^{\parallel} = \mathcal{L}_{\parallel}(\xi^{\Psi}, \xi^{\parallel}). \tag{4}$$

- A special solution to the nonlinear ballooning equations is the solution of the linear ballooning equations

$$N(\widetilde{\Psi}, l, t) = 0, \quad \text{where} \quad \widetilde{\Psi} = \Psi + \xi^{\Psi}. \tag{5}$$

An extra field equation is introduced in NIMROD code to advance ξ in simulations

In order to connect the Lagrangian and Eulerian frames,

$$\mathbf{r}(\mathbf{r}_0, t) = \mathbf{r}_0 + \boldsymbol{\xi}(\mathbf{r}_0, t) \tag{6}$$

an equation for ξ is used. In the Lagrangian frame

$$\frac{d\boldsymbol{\xi}(\mathbf{r}_0, t)}{dt} = \mathbf{u}(\mathbf{r}_0, t) \tag{7}$$

In the Eulerian frame

$$\frac{d\boldsymbol{\xi}[\mathbf{r}(\mathbf{r}_0, t), t]}{dt} = \mathbf{u}[\mathbf{r}(\mathbf{r}_0, t), t] \tag{8}$$

ξ is advanced as an extra field in Eulerian coordinates using

$$\partial_t \boldsymbol{\xi}(\mathbf{r}, t) + \mathbf{u}(\mathbf{r}, t) \cdot \nabla \boldsymbol{\xi}(\mathbf{r}, t) = \mathbf{u}(\mathbf{r}, t) \tag{9}$$

where $\mathbf{u}(\mathbf{r}, t)$ is velocity field, $\partial_t = (\partial/\partial t)_{\mathbf{r}}$, and $\nabla = \partial/\partial \mathbf{r}$.

The prominence of Lagrangian compression $\nabla_0 \cdot \xi$ marks transitions in nonlinear regimes

Transforming from Lagrangian to Eulerian frames, one finds

$$\begin{aligned} \boldsymbol{\xi}(\mathbf{r}_0, t) &= \boldsymbol{\xi}[\mathbf{r} - \boldsymbol{\xi}(\mathbf{r}, t), t] \quad (10) \\ \nabla \boldsymbol{\xi} &= \frac{\partial \boldsymbol{\xi}}{\partial \mathbf{r}} \\ &= \left(\frac{\partial \mathbf{r}}{\partial \mathbf{r}} - \frac{\partial \boldsymbol{\xi}}{\partial \mathbf{r}} \right) \cdot \frac{\partial \boldsymbol{\xi}}{\partial \mathbf{r}_0} \\ &= (\mathbf{I} - \nabla \boldsymbol{\xi}) \cdot \nabla_0 \boldsymbol{\xi} \quad (11) \end{aligned}$$

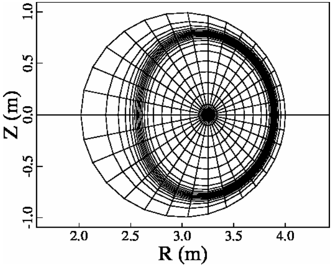
The Lagrangian compression $\nabla_0 \cdot \boldsymbol{\xi}$ is calculated from the Eulerian tensor $\nabla \boldsymbol{\xi}$ at each time step using

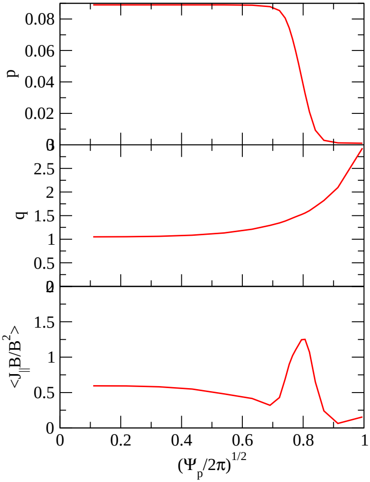
$$\nabla_0 \cdot \boldsymbol{\xi} = \mathrm{Tr}(\nabla_0 \boldsymbol{\xi}) = \mathrm{Tr}[(\mathbf{I} - \nabla \boldsymbol{\xi})^{-1} \cdot \nabla \boldsymbol{\xi}]. \quad (12)$$

Lagrangian compression $\nabla_0 \cdot \xi$ can be used to identify nonlinear regimes

- Linear regime
$$\nabla_0 \cdot \boldsymbol{\xi} = \nabla \cdot \boldsymbol{\xi} \tag{13}$$
- Early nonlinear regime
$$\begin{aligned} \nabla_0 \cdot \boldsymbol{\xi} &\sim \lambda_\Psi^{-1} \xi^\Psi + \lambda_\alpha^{-1} \xi^\alpha + \lambda_\parallel^{-1} \xi^\parallel \\ &\sim n^{1/2} n^{-1} + n^1 n^{-3/2} + n^0 n^{-1} \sim n^{-1/2} \ll 1. \end{aligned} \tag{14}$$
- Intermediate nonlinear regime
$$\nabla_0 \cdot \boldsymbol{\xi} \sim \lambda_\Psi^{-1} \xi^\Psi + \lambda_\alpha^{-1} \xi^\alpha + \lambda_\parallel^{-1} \xi^\parallel \sim n^{1/2} n^{-1/2} + n^1 n^{-1} + n^0 n^{-1/2} \sim 1. \tag{15}$$
- The Lagrangian compression is sensitive to nonlinearity: matrix $(\mathbf{I} - \nabla \boldsymbol{\xi})^{-1}$ could become singular passing beyond intermediate nonlinear regime.

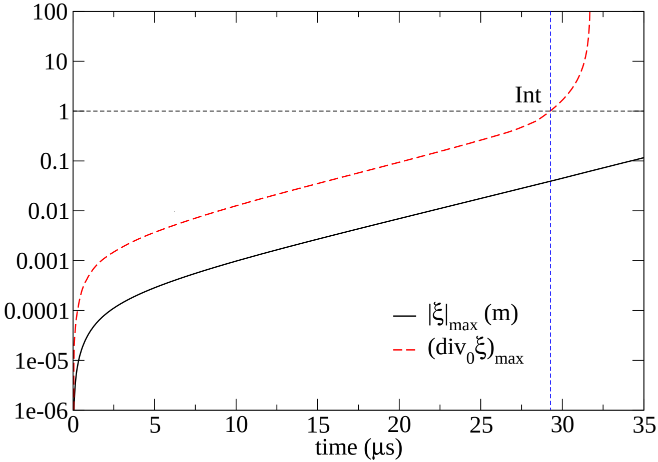
Simulations of ballooning instability are performed in a tokamak equilibrium with circular boundary and pedestal-like pressure





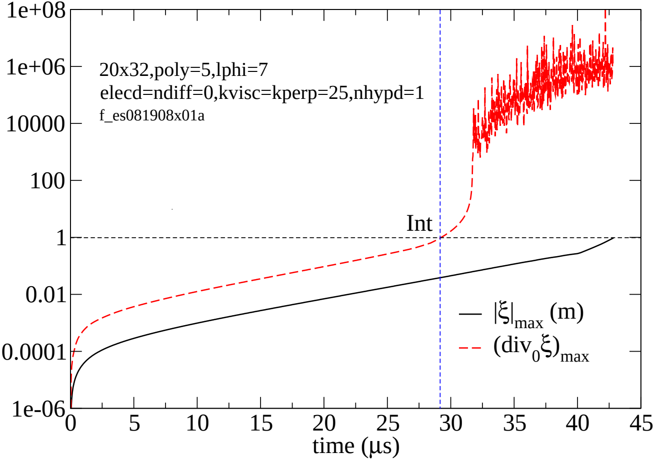
- Equilibrium from ESC solver [Zakharov and Pletzer,1999]
- Finite element mesh in NIMROD simulation.

Exponential linear growth persists in the intermediate nonlinear regime of tokamak ballooning instability [Zhu, Hegna, and Sovinec, 2008]



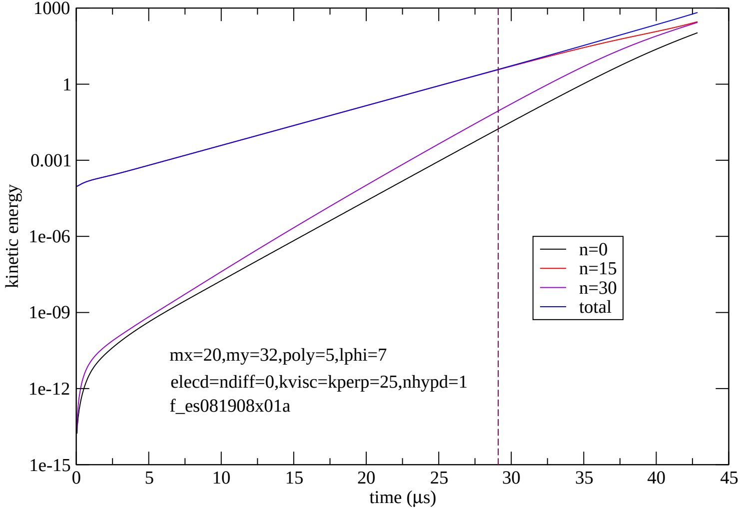
Dotted line indicates the transition to the intermediate nonlinear regime when $\nabla_0 \cdot \boldsymbol{\xi} \sim \mathcal{O}(1)$

Lagrangian compression is a good indentifer for entire nonlinear regimes

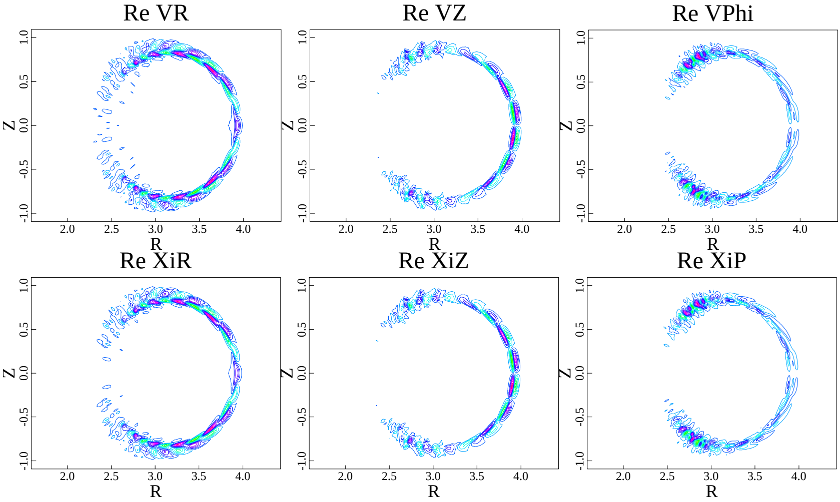


- Intermediate nonlinear regime is entered $\sim 28\mu s$.
- Large $\nabla_0 \cdot \xi$ indicates transition to nonlinear regimes.

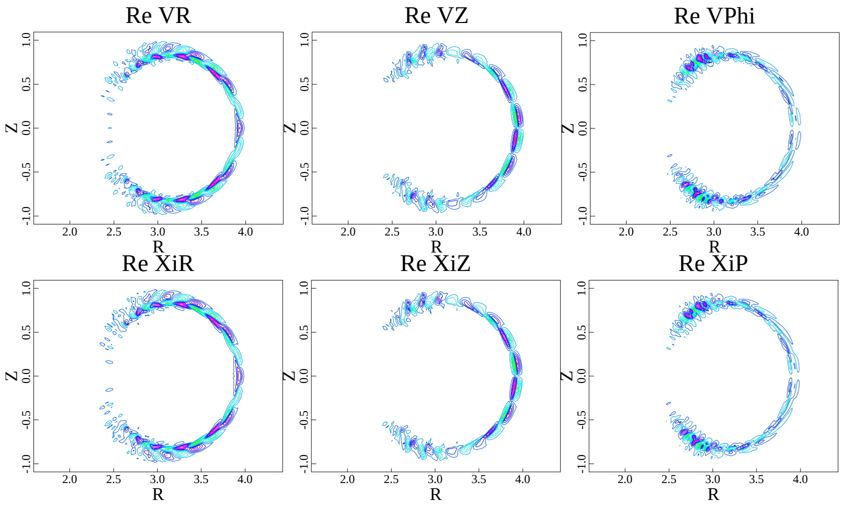
Perturbation energy grows with the linear growth rate into the intermediate nonlinear regime (vertical line)



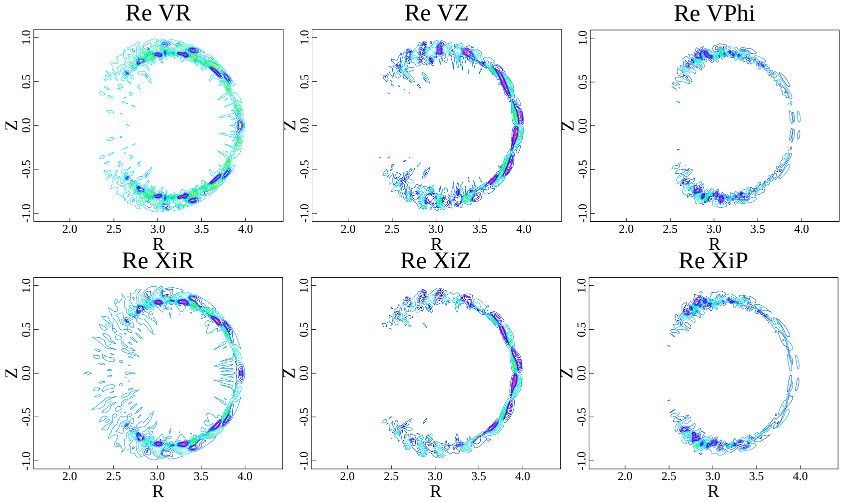
Coutours of plasma velocity and displacement ξ at $t = 5\mu s$, linear phase



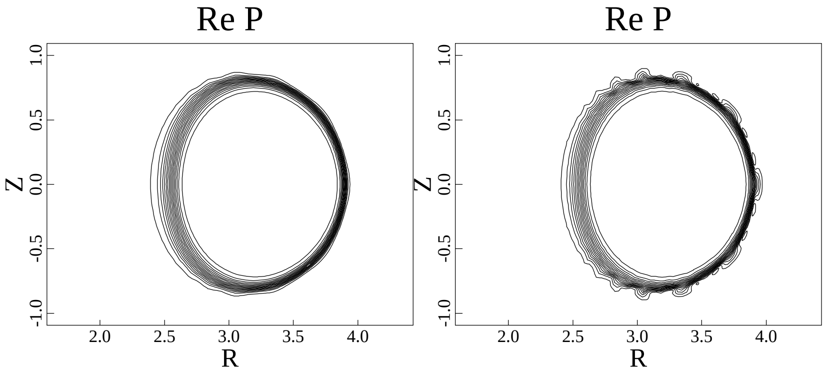
Coutours of plasma velocity and displacement ξ at $t = 30\mu s$, intermediate nonlinear phase



Coutours of plasma velocity and displacement ξ at $t = 40\mu s$, late nonlinear phase

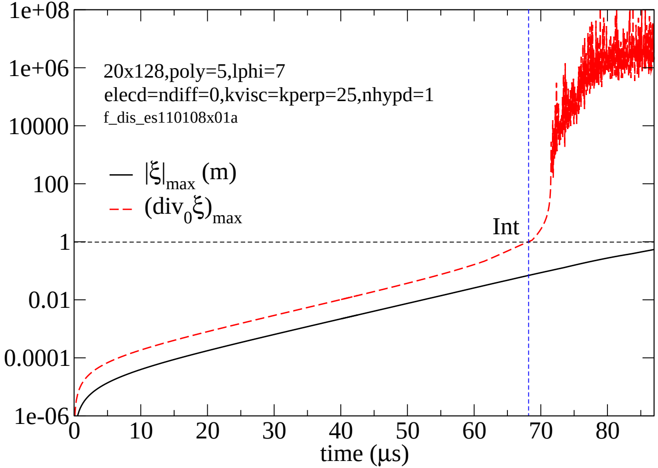


Distortions of flux surface are consistent with plasma displacement



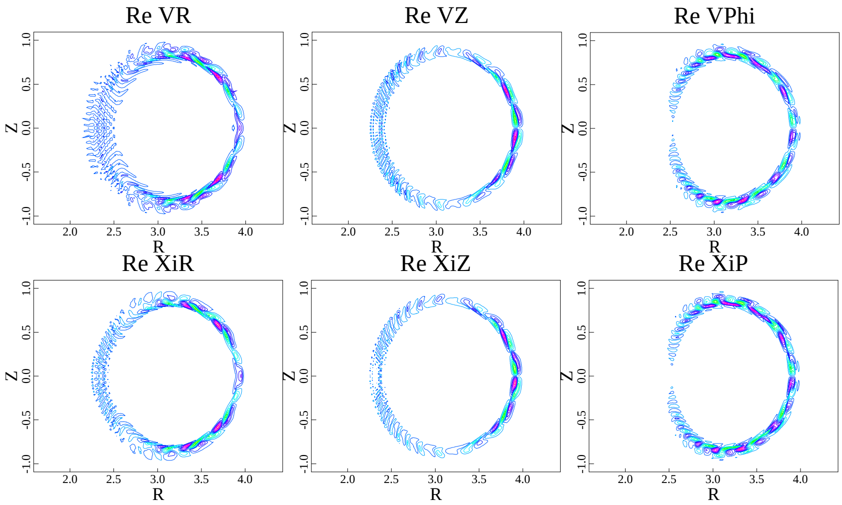
- Above: Pressure contours.
- Left: $t = 30\mu s$; Right: $t = 40\mu s$.

Agreement converges at higher resolutions



- `my=128`, linear growth slower.
- Intermediate nonlinear regime is entered $\sim 68\mu s$.
- When $\nabla_0 \cdot \boldsymbol{\xi} \sim 1$, $\xi \sim L_p \sim 0.05 - 0.1$.

Coutours of plasma velocity and displacement ξ at $t = 80\mu s$, late nonlinear phase (higher resolution)



Summary

- Nonlinear ballooning theory predicts exponential global growth in intermediate nonlinear regime [Zhu and Hegna, 2008]
- NIMROD simulations of tokamak ballooning instability agree with the theory prediction [Zhu, Hegna, and Sovinec, 2008]
- Nonlinear ballooning instability can serve as a benchmark problem for a MHD code.