Discussion of Kinetic-MHD Formulation

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Theses:

1) Key next step for extended MHD computational models is to add multi-collisionality viscous forces that yield bootstrap current and poloidal flow.

2) A new approximate form for $\vec{\nabla}\cdot\overleftrightarrow{\pi}_{\parallel}$ that accomplishes this has been proposed — in UW-CPTC 09-6, available via `http://www.cptc.wisc.edu`.

3) Longer term we need a more exact kinetic-MHD hybrid model that obtains bootstrap current in neoclassical $\parallel$ Ohm's law & poloidal ion flow.

Outline:

Parallel viscous stress closure needed for bootstrap current, poloidal flow

New multi-collisionality parallel stress closure

What's needed in long term

Electron Parallel Force Balance Yields $\parallel$ Ohm's Law

- For times $t > 1/\nu_e \sim 10\,\mu$s, equilibrium electron $\parallel$ force balance becomes

$$0 = -n_e e\langle\vec{B}\cdot\vec{E}^A\rangle - \langle\vec{B}\cdot\vec{\nabla}\cdot\overleftrightarrow{\pi}_e\rangle + \langle\vec{B}\cdot\vec{R}_e\rangle + \langle\vec{B}\cdot\vec{S}_{em}\rangle - m_e n_{e0}\langle\vec{B}\cdot\overline{\tilde{\vec{V}}_e\cdot\vec{\nabla}\tilde{\vec{V}}_e}\,\rangle - n_{e0}e\langle\vec{B}\cdot\overline{\tilde{\vec{V}}_e\times\tilde{\vec{B}}_\perp}\rangle.$$

- Using the collisional friction relation $\vec{B}_0\cdot\bar{\vec{R}}_e = -\bar{\vec{B}}_0\cdot\vec{R}_i \simeq n_{e0}\,e\,\vec{B}_0\cdot\vec{J}/\sigma_\parallel$ and neoclassical closure $\langle\vec{B}_0\cdot\vec{\nabla}\cdot\overleftrightarrow{\pi}_{e\parallel}\rangle \simeq m_e n_{e0}\langle B_0^2\rangle(\mu_{e00}U_{e\theta} + \mu_{e01}Q_{e\theta})$, this equation yields an extended neoclassical-based parallel Ohm's law:[1]

$$\boxed{\underbrace{\langle\vec{B}_0\cdot\bar{\vec{E}}^A\rangle}_{\bar{E}_\parallel^A\ \text{field}} = \eta_\parallel^{\rm nc}\underbrace{\langle\vec{B}_0\cdot\vec{J}\rangle}_{\parallel\ \text{current}} - \frac{1}{\sigma_\parallel}\Big[\underbrace{\langle\vec{B}_0\cdot\vec{J}_{\rm bs}\rangle}_{\text{bootstrap}} + \underbrace{\langle\vec{B}_0\cdot\vec{J}_{\rm CD}\rangle}_{\text{current drive}} + \underbrace{\langle\vec{B}_0\cdot\vec{J}_{\rm dyn}\rangle}_{\text{dynamo}}\Big],} \qquad \eta_\parallel^{\rm nc} \simeq \frac{1}{\sigma_\parallel}\left(1 + \frac{\sigma_\parallel}{\sigma_\perp}\frac{\mu_{e00}}{\nu_e}\right).$$

- $\parallel$ currents are driven by $dP_0/d\psi_p$, $\parallel$ e momentum sources and fluctuations:

$$\langle\vec{B}_0\cdot\vec{J}_{\rm bs}\rangle \simeq -\frac{\sigma_\parallel}{\sigma_\perp}\frac{\mu_{e00}}{\nu_e}\left(I\frac{dP_0}{d\psi_p} - n_{e0}eU_{i\theta}\langle B_0^2\rangle\right), \qquad \text{bootstrap current,}$$

$$\langle\vec{B}_0\cdot\vec{J}_{\rm CD}\rangle \equiv -\frac{\sigma_\parallel}{n_{e0}e}\langle\vec{B}_0\cdot\left(\bar{\vec{S}}_{em} - m_e\bar{\vec{V}}_e\bar{S}_{en}\right)\rangle, \qquad \text{non-inductive current drive,}$$

$$\langle\vec{B}_0\cdot\vec{J}_{\rm dyn}\rangle = \underbrace{\frac{\sigma_\parallel}{n_{e0}e}\langle\vec{B}_0\cdot\left(m_e n_{e0}\overline{\tilde{\vec{V}}_e\cdot\vec{\nabla}\tilde{\vec{V}}_e} + \overline{\vec{\nabla}\cdot\overleftrightarrow{\pi}_{e\wedge}}\right)\rangle}_{\parallel\ \text{Reynolds stress}} + \underbrace{\sigma_\parallel\langle\vec{B}_0\cdot\overline{\tilde{\vec{V}}_e\times\tilde{\vec{B}}_\perp}\,\rangle}_{\parallel\ \text{Maxwell stress}}, \qquad \text{dynamo.}$$

[1] For illustrative purposes the equations here are simplified versions where the effects of the poloidal electron heat flow $Q_{e\theta}$ have been neglected.

Poloidal Flow Is Obtained From Plasma ∥ Force Balance

- **Summing the parallel force balances over species yields (for $\bar{S}_n = 0$)**

$$m_i n_{i0} \frac{\partial \langle B_0 V_{i\|} \rangle}{\partial t} \simeq -\langle \vec{B}_0 \cdot \vec{\nabla} \cdot \overleftrightarrow{\pi}_i \rangle - m_i n_0 \langle \vec{B}_0 \cdot \overline{\tilde{\vec{V}}_i \cdot \vec{\nabla} \tilde{\vec{V}}_i} \rangle + \langle \vec{B}_0 \cdot \overline{\tilde{\vec{J}}_\wedge \times \tilde{\vec{B}}_\perp} \rangle + \langle \vec{B}_0 \cdot \textstyle\sum_s \bar{\vec{S}}_{sm} \rangle.$$

- **The poloidal flow is determined mainly by the parallel ion viscous force:**

$$\langle \vec{B} \cdot \vec{\nabla} \cdot \overleftrightarrow{\pi}_{i\|} \rangle \simeq m_i n_{i0} \left[\mu_{i00} U_{i\theta} + \mu_{i01} \frac{-2}{5 n_i T_i} Q_{i\theta} + \cdots \right] \langle B^2 \rangle, \qquad \mu_{i00}, \mu_{i01} \sim \sqrt{\epsilon}\, \nu_i.$$

- **For $t > 1/\nu_i \sim 1$ ms, poloidal flow from NCLASS, or $\langle \vec{B} \cdot \vec{\nabla} \cdot \overleftrightarrow{\pi}_{i\|} \rangle \simeq 0$ is:**

$$U^0_{i\theta}(\psi_p) \equiv \frac{\vec{V} \cdot \vec{\nabla} \theta}{\vec{B} \cdot \vec{\nabla} \theta} \simeq -\frac{\mu_{i01}}{\mu_{i00}} \frac{-2}{5 n_i T_i} Q_{i\theta} \simeq \frac{c_p I}{q_i \langle B^2 \rangle} \frac{dT_{i0}}{d\psi_p} \quad \Longrightarrow \quad \boxed{V_p \simeq \frac{1.17}{q_i B} \frac{dT_{i0}}{dr} + \mathcal{O}\{\delta^2\}.}$$

- **Including all the drives in the parallel plasma force balance above yields**

$$\boxed{U_{i\theta}(\psi_p) \simeq \underbrace{U^0_{i\theta}(\psi_p)}_{\text{neoclassical}} - \underbrace{\frac{\langle \vec{B}_0 \cdot (m_i n_{i0} \overline{\tilde{\vec{V}}_i \cdot \vec{\nabla} \tilde{\vec{V}}_i} + \overline{\vec{\nabla} \cdot \overleftrightarrow{\pi}_{i\wedge}}) \rangle}{m_i n_{i0} \mu_{i00} \langle B_0^2 \rangle}}_{\|\ \text{Reynolds stress}} + \underbrace{\frac{\langle \vec{B}_0 \cdot \overline{\tilde{\vec{J}}_\wedge \times \tilde{\vec{B}}_\perp} \rangle + \langle \vec{B}_0 \cdot \sum_s \bar{\vec{S}}_{sm} \rangle}{m_i n_{i0} \mu_{i00} \langle B_0^2 \rangle}}_{\|\ \text{Maxwell stress + flow sources}}.}$$

- **Given the poloidal flow ($\Omega_{*p} \equiv I\, U_{i\theta}/R^2$), relation of toroidal flow to E_r is:**

$$\Omega_t \equiv \vec{V} \cdot \vec{\nabla} \zeta = -\left(\frac{d\Phi}{d\psi_p} + \frac{1}{n_i q_i} \frac{dp_i}{d\psi_p} \right) + \Omega_{*p} \quad \Longrightarrow \quad \boxed{V_t \simeq \frac{E_r}{B_p} - \frac{1}{n_i q_i B_p} \frac{dp_i}{dr} + \frac{1.17}{q_i B_p} \frac{dT_i}{dr}.}$$

How Can We Add These $\parallel$ Flow Damping Effects To Codes?

- **Braginskii viscous force due to CGL form for parallel stresses is**

$$\overleftrightarrow{\boldsymbol{\pi}}_\parallel \equiv \pi_\parallel \left(\frac{\vec{B}\vec{B}}{B^2} - \frac{\overleftrightarrow{\mathbf{I}}}{3} \right), \qquad \pi_\parallel \equiv -\frac{3}{2}\eta_0 \frac{\vec{B}\cdot\overleftrightarrow{\mathbf{W}}\cdot\vec{B}}{B^2}, \qquad \overleftrightarrow{\mathbf{W}} \equiv \vec{\nabla}\vec{V} + (\vec{\nabla}\vec{V})^{\mathsf{T}} - \frac{2}{3}\overleftrightarrow{\mathbf{I}}(\vec{\nabla}\cdot\vec{V}).$$

- **Parallel component of parallel rate of strain has a couple of forms:**

$$\boxed{\vec{B}\cdot\overleftrightarrow{\mathbf{W}}\cdot\vec{B}/2} = B(\vec{B}\cdot\vec{\nabla})(\vec{V}\cdot\vec{B}/B) + [\vec{B}\times(\vec{B}\times\vec{V})]\cdot\vec{\kappa} - (B^2/3)\vec{\nabla}\cdot\vec{V}$$

$$\boxed{= B^2\vec{V}\cdot\vec{\nabla}\ln B + \vec{B}\cdot\vec{\nabla}\times(\vec{V}\times\vec{B}) + (2B^2/3)\vec{\nabla}\cdot\vec{V} - (\vec{B}\cdot\vec{V})(\vec{\nabla}\cdot\vec{B}).}$$

- **For $\vec{\nabla}\cdot\vec{B} = 0$, $\vec{\nabla}\cdot\vec{V} = 0$ and $\vec{V}_\perp = (1/B^2)\vec{B}\times\vec{\nabla}f$, the last form yields**

$$\pi_\parallel = -3\eta_0\,(\vec{V}\cdot\vec{\nabla}\ln B) + \Delta\pi_\parallel, \quad \text{where } \Delta\pi_\parallel \equiv -(3\eta_0/B^3)(\vec{B}\cdot\vec{\nabla}f)[\vec{B}\cdot\vec{\nabla}\times(\vec{B}/B)] \text{ is small.}$$

- **Viscous force for the Braginskii viscous stress is ($\vec{\kappa}$ is curvature vector)**

$$\vec{\nabla}\cdot\overleftrightarrow{\boldsymbol{\pi}}_\parallel = \pi_\parallel\,[\vec{\kappa} - \vec{B}(\vec{B}\cdot\vec{\nabla}\ln B)/B^2] + (1/B^2)\vec{B}(\vec{B}\cdot\vec{\nabla})\pi_\parallel - (1/3)\vec{\nabla}\pi_\parallel$$

$$\Longrightarrow \qquad \vec{B}_0\cdot\vec{\nabla}\cdot\overleftrightarrow{\boldsymbol{\pi}}_\parallel = -\pi_\parallel\,(\vec{B}_0\cdot\vec{\nabla}\ln B_0) + (2/3)(\vec{B}_0\cdot\vec{\nabla})\pi_\parallel.$$

- **FSA neglecting $\Delta\pi_\parallel$ & using $\vec{V}\cdot\vec{\nabla}\ln B_0 = (\vec{B}_0\cdot\vec{\nabla}\ln B_0)\,U_\theta(\psi_p)$ yields**

$$\boxed{\langle\vec{B}_0\cdot\vec{\nabla}\cdot\overleftrightarrow{\boldsymbol{\pi}}_\parallel\rangle = 3\eta_0\,\langle(\vec{B}_0\cdot\vec{\nabla}\ln B_0)^2\rangle\,U_\theta,} \text{ with } U_\theta(\psi) \equiv \frac{\vec{V}\cdot\vec{\nabla}\theta}{\vec{B}_0\cdot\vec{\nabla}\theta} \text{ from } \vec{\nabla}\cdot\vec{V} = 0.$$

Adding Parallel Flow Damping Effects To Codes (continued)

- **A multi-collisionality parallel stress that yields the Braginskii and flux-surface-averaged neoclassical closures has been proposed**[2] **($\hat{b} \equiv \vec{B}_0/B_0$):**

$$\pi_\| \;=\; -\, mn\,\mu\,\langle B_0^2\rangle \frac{\hat{b}\cdot\vec{\nabla}B_0}{\langle\,(\hat{b}\cdot\vec{\nabla}B_0)^2\rangle}\,(U_\theta - U_\theta^0)$$

$$-\,3\,\eta_0\left(\frac{\vec{B}\cdot\vec{\nabla}\times(\vec{V}\times\vec{B})}{B^2} + \frac{2}{3}\vec{\nabla}\cdot\vec{V} - \frac{(\vec{B}\cdot\vec{V})(\vec{\nabla}\cdot\vec{B})}{B^2}\right).$$

- **Electron neoclassical poloidal flow damping frequency μ is of the form**

$$\mu_e \simeq \frac{2.3\sqrt{\epsilon}\,\nu_e}{(1+\nu_{*e}^{1/2}+\nu_{*e})(1+\epsilon^{3/2}\nu_{*e})},\qquad \text{for collisionality parameter } \nu_{*e} \equiv \frac{\nu_e}{\epsilon^{3/2}\omega_{te}} = \frac{Rq}{\epsilon^{3/2}\lambda_e}$$

$\Longrightarrow$ **banana regime for $\nu_{*e} \ll 1$, plateau for $1 \ll \nu_{*e} \ll \epsilon^{-3/2}$, Braginskii for $\nu_{*e} \gg \epsilon^{-3/2}$.**

- **The "offset" poloidal flow velocity for electrons or ions is given by**

$$U_{s\theta}^0(\psi_p) \equiv k_s\,\frac{I(\psi_p)}{q_s\langle B_0^2\rangle}\,\frac{dT_{s0}}{d\psi_p},\qquad \text{in which } k_s \text{ has a form similar to } \mu_e.$$

[2] J.D. Callen, "Viscous Forces Due To Collisional Parallel Stress For Extended MHD Codes," UW-CPTC 09-6 (via `http://www.cptc.wisc.edu`).

Caveats And Long Term Needs

- **PROPOSAL: Implement Braginskii operator with neoclassical viscous damping frequencies μ_e, μ_i and flow offsets $U_{e\theta}$, $U_{i\theta}$ in M3D and NIMROD[3]? Some issues for such a proposition:**

  Best form of $\pi_\parallel$ to use? $\vec{V}_e \to -\vec{J}/n_e e \sim \nabla^2 \vec{B}$ yields 4th order operator in $\partial \vec{B}/\partial t$ eqn.

  Poloidal variation of viscous force for $\nu_* \ll 1$ not properly captured — but do we care?

  Long parallel scale variations should still be relaxed with Braginskii coefficient η_0?

  Need heat flow offsets (U_θ^0) to damp flows to nonzero values.

  Need $Z_{\rm eff}$ effects on μ for realistic tokamak plasma situations.

- **Ultimate test of procedure is via more fundamental kinetic-based approach that solves drift-kinetic equation and uses result to obtain $\pi_\parallel$ closure:**

  Held — finite element continuum solution of drift-kinetic equation,

  Ramos — ordered, closed fluid and electron formulation for slow-MHD.

- **Challenge for kinetic-based approaches before it would be appropriate to implement them in M3D and NIMROD would be demonstration that they can obtain the axisymmetric neoclassical parallel Ohm's law including the bootstrap current and the neoclassical poloidal ion flow damping.**

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[3] C.R. Sovinec, `www.cptc.wisc.edu/sovinec_research/notes/e_viscosity2.pdf`; C.R. Sovinec et al., 2007 Sherwood Conf., Annapolis, MD.