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ALGORITHM FOR THE ELECTRON NEOCLASSICAL SPITZER PROBLEM WITH FOKKER-PLANCK COLLISION OPERATORS AND GENERAL MAGNETIC GEOMETRY*

J.J. Ramos

M.I.T. Plasma Science and Fusion Center

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IN A LOW-COLLISIONALITY AXISYMMETRIC EQUILIBRIUM, AND IN TERMS OF THE RANDOM VELOCITY VARIABLES $(v'_\parallel, v'_\perp)$ IN THE REFERENCE FRAME OF ITS MEAN FLOW, THE PART OF THE GYROPHASE-AVERAGED NON-MAXWELLIAN ELECTRON DISTRIBUTION FUNCTION THAT IS ODD WITH RESPECT TO $v'_\parallel$ IS

$$\bar{f}^{odd}_{NMe}(\psi, \theta, v'_\parallel, v'_\perp) = -f^{(0)}_{Me}(\psi, v') \left\{ \frac{m_e U_e(\psi) B}{T^{(0)}_e(\psi)} + \frac{m_e I(\psi)}{2eBT^{(0)}_e(\psi)} \left[\frac{m_e v'^2}{T^{(0)}_e(\psi)} - 5 \right] \frac{dT^{(0)}_e(\psi)}{d\psi} \right\} v'_\parallel + h^{odd}(\psi, \theta, v'_\parallel, v'_\perp)$$

where

$$h^{odd}(\psi, \theta, v'_\parallel, v'_\perp) \; = \; \sigma(v_\parallel) \; H(1-\lambda) \; K(\psi, v', \lambda)$$

and

$$\lambda \; = \; \frac{v'^2_\perp}{v'^2} \; \frac{B_{max}(\psi)}{B(\psi, \theta)} \; = \; \sin^2 \chi \; \frac{B_{max}(\psi)}{B(\psi, \theta)}$$

THE SOLUTION FOR THE ABOVE ODD DISTRIBUTION FUNCTION WOULD SPECIFY

- **THE OHMIC AND BOOTSTRAP PARALLEL ELECTRON FLOW:**

$$u_{e\|}^{OH} + u_{e\|}^{BS} \;=\; U_e(\psi)B \qquad \textbf{with} \qquad U_e(\psi) \;=\; \frac{2\pi}{N^{(0)}(\psi)B_{max}(\psi)}\int_0^\infty dv' \; v'^3 \; \int_0^1 d\lambda \; K(\psi, v', \lambda) \; .$$

- **THE PARALLEL HEAT FLUX:**

$$q_{e\|} = -\frac{5N^{(0)}T_e^{(0)}I}{2eB} \; \frac{dT_e^{(0)}}{d\psi} + Q_e(\psi)B \quad \textbf{with} \quad Q_e(\psi) = \frac{\pi T_e^{(0)}}{B_{max}} \; \int_0^\infty dv' \; v'^3 \left(\frac{m_e v'^2}{T_e^{(0)}} - 5\right)\int_0^1 d\lambda \; K(\psi, v', \lambda) \; .$$

- **THE PARALLEL COLLISIONAL FRICTION FORCE:**

$$F_{e\|}^{coll} \;=\; \frac{2m_e\nu_e}{3(2\pi)^{1/2}}\left(\frac{j_\|}{e} + N^{(0)}U_eB - \frac{3N^{(0)}I}{2eB}\frac{dT_e^{(0)}}{d\psi}\right) \;-\; \frac{2\pi m_e\nu_e v_{the}^3 B}{B_{max}} \; \int_0^\infty dv' \; \int_0^1 d\lambda \; K(\psi, v', \lambda) \; .$$

- **THE MAGNETIC-SURFACE-AVERAGED NEOCLASSICAL PARALLEL VISCOSITY:**

$$-\oint_\psi dl \; (p_{e\|} - p_{e\perp}) \; \mathbf{b}\cdot\nabla\ln B \;=\; \oint_\psi dl\left(F_{e\|}^{coll} + \frac{eV_0N^{(0)}I}{BR^2}\right) \;=\; \mu_{e1}U_e(\psi) \;+\; \mu_{e3}Q_e(\psi) \;+\; \ldots$$

THE FUNCTION h^{odd} IS DETERMINED AS THE SOLUTION OF THE GENERALIZED SPITZER PROBLEM:

$$\oint_{\psi,v',\lambda} dl\ v_\parallel'^{-1}\ \mathcal{C}_e[h^{odd}] \ = \ -\oint_{\psi,v'} dl\ \mathcal{S}_e(\psi,\theta,v') \ \equiv -\bar{\mathcal{S}}_e(\psi,v') \qquad \textbf{for} \qquad \lambda < 1 \ ,$$

WHERE $\mathcal{C}_e$ IS THE GYROAVERAGED LINEAR ELECTRON COLLISION OPERATOR AND

$$\mathcal{S}_e(\psi,\theta,v') \ = \ \Big\{\frac{eV_0 I}{T_e^{(0)} B R^2} \ + \ \nu_e \left(U_\iota B \ + \ \frac{I}{eN^{(0)}B}\frac{dP^{(0)}}{d\psi}\right) \frac{v_{the}}{v_{th\iota}^2 v'}\ \xi\left(\frac{v'}{v_{th\iota}}\right) \ +$$

$$+ \ \frac{\nu_e m_e I}{eBT_e^{(0)}}\frac{dT_e^{(0)}}{d\psi}\ \frac{v_{the}}{v'}\left[2\varphi\left(\frac{v'}{v_{the}}\right) - 10\xi\left(\frac{v'}{v_{the}}\right) + \frac{1}{2}\varphi\left(\frac{v'}{v_{th\iota}}\right) - \frac{5v_{the}^2}{2v_{th\iota}^2}\xi\left(\frac{v'}{v_{th\iota}}\right)\right]\Big\}\ f_{Me}^{(0)} \ .$$

IT SHOULD BE DESIRABLE TO SOLVE THIS PROBLEM USING THE EXACT FOKKER-PLANCK COLLISION OPERATOR AND A MAGNETIC GEOMETRY CONSISTENT WITH AN ACTUAL GRAD-SHAFRANOV EQUILIBRIUM.

USING THE POLAR VELOCITY COORDINATES (v', χ)**,**

$$h^{odd}(\psi, \theta, v', \chi) = \sigma(\cos\chi)\ H(1 - \sin^2\chi B_{max}/B)\ K(\psi, v', \sin^2\chi B_{max}/B)$$

CAN BE EXPANDED IN THE LEGENDRE POLYNOMIAL SERIES

$$h^{odd}(\psi, \theta, v', \chi) = \sum_{odd\ \ l=1}^{\infty} h_l(\psi, \theta, v')\ P_l(\cos\chi)\ .$$

THEN,

$$h_l(\psi, \theta, v') \ = \ (2l+1) \int_0^{\pi/2} d\chi\ \ \sin\chi\ h^{odd}(\psi, \theta, v', \chi)\ P_l(\cos\chi) \ =$$

$$= \ (l + 1/2) \sum_{m=0}^{(l-1)/2} a_{lm}\ (B/B_{max})^{m+1}\ K_m(\psi, v'),$$

WHERE

$$K_m(\psi, v') \equiv \int_0^1 d\lambda\ K(\psi, v', \lambda)\ \lambda^m$$

AND a_{lm} **IS THE MATRIX OF CONSTANTS DEFINED BY**

$$\frac{P_l(\cos\chi)}{\cos\chi} \ = \ \sum_{m=0}^{(l-1)/2} a_{lm}\ \sin^{2m}\chi \qquad \textbf{for}\ \ \textbf{odd}\ \ l\ .$$

SPLIT THE LINEAR FOKKER-PLANCK COLLISION OPERATOR $\mathcal{C}_e$ INTO ITS LORENTZ PART $\nu_{eD}(\psi, v')\mathcal{L}$ AND THE REMAINDER, TO BE DENOTED BY $\hat{\mathcal{C}}_e$:

$$\mathcal{C}_e = \nu_{eD}(\psi, v')\mathcal{L} \; + \; \hat{\mathcal{C}}_e$$

$$\nu_{eD}(\psi, v') \equiv \frac{\nu_e v_{the}^3}{v'^3}\left[\varphi\left(\frac{v'}{v_{the}}\right) - \xi\left(\frac{v'}{v_{the}}\right) + \varphi\left(\frac{v'}{v_{th\iota}}\right) - \xi\left(\frac{v'}{v_{th\iota}}\right)\right] \; .$$

$\hat{\mathcal{C}}_e$ IS DIAGONAL IN THE LEGENDRE REPRESENTATION:

$$\hat{\mathcal{C}}_e\left[\sum_{l=0}^{\infty} f_l(v')P_l(\cos\chi)\right] \; = \; \sum_{l=0}^{\infty} P_l(\cos\chi) \; \hat{\mathcal{C}}_{e,l}[f_l](v') \; ,$$

with

$$\hat{\mathcal{C}}_{e,l}[f_l](v') \; = \; \frac{\nu_e v_{the}}{n} f_{Me}(v')\left\{4\pi v_{the}^2 f_l(v') \; - \; \Phi_l[f_l](v') \; + \; \frac{v'^2}{v_{the}^2}\frac{d^2\Psi_l[f_l](v')}{dv'^2}\right\} \; +$$

$$+ \; \frac{\nu_e v_{the}^3}{v'^2}\frac{d}{dv'}\left\{\xi\left(\frac{v'}{v_{the}}\right)\left[v'\frac{df_l(v')}{dv'} + \frac{v'^2}{v_{the}^2} \; f_l(v')\right] \; + \; \xi\left(\frac{v'}{v_{th\iota}}\right)\left[v'\frac{df_l(v')}{dv'} + \frac{m_e v'^2}{m_\iota v_{th\iota}^2} \; f_l(v')\right]\right\} \; ,$$

$$\frac{1}{v'^2}\frac{d}{dv'}\left\{v'^2\frac{d\Phi_l[f_l](v')}{dv'}\right\} \; - \; \frac{l(l+1)}{v'^2}\Phi_l[f_l](v') \; = \; -4\pi f_l(v') \; ,$$

$$\frac{1}{v'^2}\frac{d}{dv'}\left\{v'^2\frac{d\Psi_l[f_l](v')}{dv'}\right\} \; - \; \frac{l(l+1)}{v'^2}\Psi_l[f_l](v') \; = \; \Phi_l[f_l](v') \; .$$

FOR $\lambda < 1$, USING THE EXPRESSION OF $\mathcal{L}$ IN THE λ VARIABLE AND THE LEGENDRE SERIES EXPRESSION OF $\hat{\mathcal{C}}_e$ IN THE χ VARIABLE:

$$\mathcal{C}_e[h^{odd}] \;=\; \nu_{eD}(\psi, v')\mathcal{L}[h^{odd}] \;+\; \hat{\mathcal{C}}_e \left[\sum_{odd\;\; l=1}^{\infty} h_l(\psi, \theta, v') \;\; P_l(\cos\chi) \right] \;=$$

$$=\; 2\;\nu_{eD}(\psi, v') \left(1 - \frac{\lambda B}{B_{max}} \right)^{1/2} \frac{\partial}{\partial\lambda} \left[\frac{\lambda B_{max}}{B} \left(1 - \frac{\lambda B}{B_{max}} \right)^{1/2} \frac{\partial K(\psi, v', \lambda)}{\partial \lambda} \right] \;+$$

$$+\;\; \sum_{odd\;\; l=1}^{\infty} (l + 1/2) \sum_{m=0}^{(l-1)/2} a_{lm}\;\, (B/B_{max})^{m+1}\;\hat{\mathcal{C}}_{e,l}[K_m](\psi, v')\;\; P_l(\cos\chi)$$

AND, AFTER DIVIDING BY $v'_{\|} = v' \cos\chi = v'(1 - \lambda B/B_{max})^{1/2}$:

$$v_{\|}'^{-1}\;\mathcal{C}_e[h^{odd}] \;=\; 2\;v'^{-1}\;\nu_{eD}(\psi, v')\;\frac{\partial}{\partial\lambda} \left[\frac{\lambda B_{max}}{B} \left(1 - \frac{\lambda B}{B_{max}} \right)^{1/2} \frac{\partial K(\psi, v', \lambda)}{\partial \lambda} \right] \;+$$

$$+\;v'^{-1} \sum_{odd\;\; l=1}^{\infty} (l + 1/2) \sum_{m=0}^{(l-1)/2} \quad \sum_{n=0}^{(l-1)/2} a_{lm}\;\, a_{ln}\;\, (B/B_{max})^{m+n+1}\;\hat{\mathcal{C}}_{e,l}[K_m](\psi, v')\;\;\lambda^n\;\;.$$

NOW, CALLING

$$W(\psi,\lambda) \;\equiv \oint_{\psi,\lambda} dl \; [B_{max}(\psi)/B(\psi,\theta)] \; [1-\lambda B(\psi,\theta)/B_{max}(\psi)]^{1/2}$$

and

$$b_m(\psi) \;\equiv \oint_{\psi} dl \; [B(\psi,\theta)/B_{max}(\psi)]^{m+1} \; ,$$

THE TRANSIT AVERAGE INTEGRAL OF $v_{\|}'^{-1} \; \mathcal{C}_e[h^{odd}]$ **IS**

$$\oint_{\psi,v',\lambda} dl \; v_{\|}'^{-1} \; \mathcal{C}_e[h^{odd}] \;=\; 2 \; v'^{-1} \; \nu_{eD}(\psi,v') \; \frac{\partial}{\partial\lambda}\left[\lambda \; W(\psi,\lambda) \; \frac{\partial K(\psi,v',\lambda)}{\partial\lambda}\right] \; +$$

$$+ \; v'^{-1} \sum_{odd \;\; l=1}^{\infty} (l+1/2) \sum_{m=0}^{(l-1)/2} \;\; \sum_{n=0}^{(l-1)/2} a_{lm} \; a_{ln} \; b_{m+n}(\psi) \; \hat{\mathcal{C}}_{e,l}[K_m](\psi,v') \; \lambda^n$$

AND THE SPITZER PROBLEM BECOMES THE FOLLOWING EQUATION FOR $K(\psi,v',\lambda)$**:**

$$\frac{\partial}{\partial\lambda}\left[\lambda \; W(\psi,\lambda) \; \frac{\partial K(\psi,v',\lambda)}{\partial\lambda}\right] \;=$$

$$= \; - \; \frac{1}{2\nu_{eD}(\psi,v')}\left\{v' \; \bar{\mathcal{S}}_e(\psi,v') + \sum_{odd \;\; l=1}^{\infty} (l+1/2) \sum_{m=0}^{(l-1)/2} \;\; \sum_{n=0}^{(l-1)/2} a_{lm} \; a_{ln} \; b_{m+n}(\psi) \; \hat{\mathcal{C}}_{e,l}[K_m](\psi,v') \; \lambda^n\right\} \; .$$

INTEGRATING TWICE, WITH THE BOUNDARY CONDITION $K(\psi, v', 1) = 0$ **:**

$$K(\psi, v', \lambda) \; = \; \frac{1}{2\nu_{eD}(\psi, v')} \left\{ v' \; \bar{\mathcal{S}}_e(\psi, v') + \frac{3}{2} \; b_0(\psi) \; \hat{\mathcal{C}}_1[K_0](\psi, v') \right\} \int_\lambda^1 \frac{d\lambda'}{W(\psi, \lambda')} \; +$$

$$+ \; \frac{1}{2\nu_{eD}(\psi, v')} \; \sum_{odd \;\; l=3}^{\infty} (l + 1/2) \sum_{m=0}^{(l-1)/2} \; \sum_{n=0}^{(l-1)/2} \frac{a_{lm} \; a_{ln}}{n+1} \; b_{m+n}(\psi) \; \hat{\mathcal{C}}_{e,l}[K_m](\psi, v') \int_\lambda^1 \frac{d\lambda' \; \lambda'^n}{W(\psi, \lambda')} \; .$$

THIS YIELDS THE FOLLOWING SYSTEM FOR THE MOMENTS $K_m(\psi, v') = \int_0^1 d\lambda \; K(\psi, v', \lambda) \lambda^m$**:**

$$K_m(\psi, v') \; = \; \frac{1}{\nu_{eD}(\psi, v')} \left\{ \frac{2v \; \bar{\mathcal{S}}_e(\psi, v')}{3 \; b_0(\psi)} \; + \; \hat{\mathcal{C}}_{e,1}[K_0](\psi, v') \right\} \gamma_{0m}^1(\psi) \; +$$

$$+ \; \frac{1}{\nu_{eD}(\psi, v')} \; \sum_{odd \;\; l=3}^{\infty} \; \sum_{m'=0}^{(l-1)/2} \; \gamma_{m'm}^l(\psi) \; \hat{\mathcal{C}}_{e,l}[K_{m'}](\psi, v') \; ,$$

WHERE THE PARAMETERS $\gamma_{m'm}^l(\psi)$**, THAT CONTAIN ALL THE INFORMATION NEEDED ABOUT THE MAGNETIC EQUILIBRIUM GEOMETRY, ARE**

$$\gamma_{m'm}^l(\psi) \; \equiv \; \frac{a_{lm'}(2l+1)}{4(m+1)} \; \sum_{n=0}^{(l-1)/2} \frac{a_{ln}}{n+1} \; b_{m'+n}(\psi) \; \int_0^1 \frac{d\lambda \; \lambda^{m+n+1}}{W(\psi, \lambda)} \; .$$

AN ITERATIVE SOLUTION CAN BE DEVISED IF THE $\hat{\mathcal{C}}_{e,l}$ TERMS ARE CONSIDERED TO BE SUBDOMINANT FOR $l \geq 3$ (AS IS THE CASE IN THE LIMIT OF LARGE l).

ACCORDINGLY, THE LOWEST APPROXIMATION WOULD BE

$$\hat{\mathcal{C}}_{e,1}[K_0^{\{0\}}](\psi, v') \; - \; \frac{\nu_{eD}(\psi, v') K_0^{\{0\}}(\psi, v')}{\gamma_{00}^1(\psi)} \; = \; - \; \frac{2v' \; \bar{\mathcal{S}}_e(\psi, v')}{3 \; b_0(\psi)}$$

and

$$K_m^{\{0\}}(\psi, v') \; = \; \frac{\gamma_{0m}^1(\psi)}{\gamma_{00}^1(\psi)} \; K_0^{\{0\}}(\psi, v') \qquad \textbf{for} \qquad m > 0 \; .$$

THEN, AFTER N ITERATIONS AND WITH A FINITE LEGENDRE SERIES, $l \leq L$:

$$\hat{\mathcal{C}}_{e,1}[K_0^{\{N+1\}}](\psi, v') - \frac{\nu_{eD}(\psi, v') K_0^{\{N+1\}}(\psi, v')}{\gamma_{00}^1(\psi)} \; = \; - \; \frac{2v \; \bar{\mathcal{S}}_e(\psi, v')}{3 \; b_0(\psi)} - \sum_{odd \;\; l=3}^{L} \; \sum_{m'=0}^{(l-1)/2} \; \frac{\gamma_{m'0}^l(\psi)}{\gamma_{00}^1(\psi)} \; \hat{\mathcal{C}}_{e,l}[K_{m'}^{\{N\}}](\psi, v')$$

and

$$K_m^{\{N+1\}}(\psi, v') \; = \; \frac{\gamma_{0m}^1(\psi)}{\gamma_{00}^1(\psi)} \; K_0^{\{N+1\}}(\psi, v') \; +$$

$$+ \; \frac{1}{\nu_{eD}(\psi, v')} \; \sum_{odd \;\; l=3}^{L} \; \sum_{m'=0}^{(l-1)/2} \left[\gamma_{m'm}^l(\psi) - \frac{\gamma_{m'0}^l(\psi) \;\; \gamma_{0m}^1(\psi)}{\gamma_{00}^1(\psi)} \right] \hat{\mathcal{C}}_{e,l}[K_{m'}^{\{N\}}](\psi, v') \qquad \textbf{for} \qquad m > 0 \; .$$

THE PROPOSED ITERATIVE PROCEDURE REQUIRES ONLY TO INVERT THE 1-D OPERATOR $\hat{\mathcal{C}}_{e,1} - \nu_{eD}(\psi, v')/\gamma^1_{00}(\psi)$ AND TO APPLY THE 1-D OPERATORS $\hat{\mathcal{C}}_{e,l}$.

THE OPERATOR $\hat{\mathcal{C}}_{e,1} - \nu_{eD}(\psi, v')/\gamma^1_{00}(\psi)$ DIFFERS FROM THE $l = 1$ COMPONENT OF THE COMPLETE COLLISION OPERATOR, $\mathcal{C}_{e,1} = \hat{\mathcal{C}}_{e,1} - \nu_{eD}(\psi, v')$, ONLY BY $\gamma^1_{00}(\psi) \neq 1$ WHICH REFLECTS THE NEOCLASSICAL TRAPPING EFFECTS IN A NON-UNIFORM B.

THE COMPLETE FUNCTION $K(\psi, v', \lambda)$ COULD BE RECONSTRUCTED FROM ITS MOMENTS, $K_m(\psi, v')$, ONCE THESE HAVE BEEN SOLVED FOR. HOWEVER, FOR THE PURPOSE OF OBTAINING THE RELEVANT KINETIC CLOSURE VARIABLES, ONLY THE LOWEST MOMENT, $K_0(\psi, v')$, IS NEEDED.