Large-guide-field diamagnetic-drift tearing with extended-MHD in slab geometry

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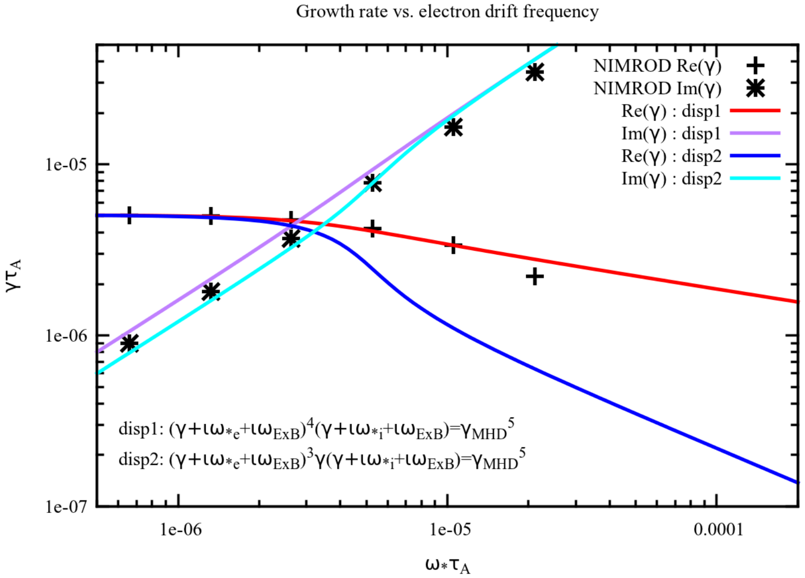
The NIMROD numerical dispersion relation is not the expected result.



Now that I may have your attention, a shameless advertisement.

- Tuesday at 3:30pm, JI2 #4: “First-order FLR effects on magnetic tearing and relaxation in pinch configurations”

Figure: Billy Mays

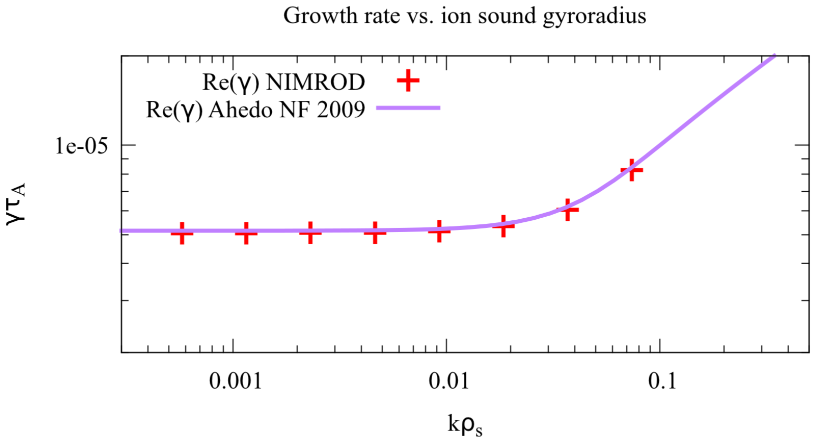
Back to diamagnetic-drift tearing: motivation.

- Connect modeling of drift-tearing with full extended-MHD equations to 'drift' reduced models.
- The analytics are often in slab geometry, computation provides the ability to analyze experimental configurations after model verification.
- Tokamak RMP theory: important effects from Waelbroeck [PoP 2003]
  - KAW/semi-collisional effects: covered through NIMROD benchmark with Ahedo & Ramos [NF 2009].
  - Diamagnetic-drift tearing, this work: Coppi [PoF 1964/65], Drake & Lee [PoF 1977], Biskamp [NF 1978], Hassam [PoF 1980], Drake et al. [PoF 1983] etc.
  - Viscosity and RMP interaction: future work.

Disclaimer: this work is still preliminary and subject to change.

(Under construction - like ITER)

We choose a small ion skin depth to avoid KAW/whistler effects.



- Our cases are based off the benchmark with Ahedo and Ramos [2009].
- Cases use $k\rho_s = 0.0006$ and $kd_i = 0.002$ and will vary n_0' to vary ω_*.

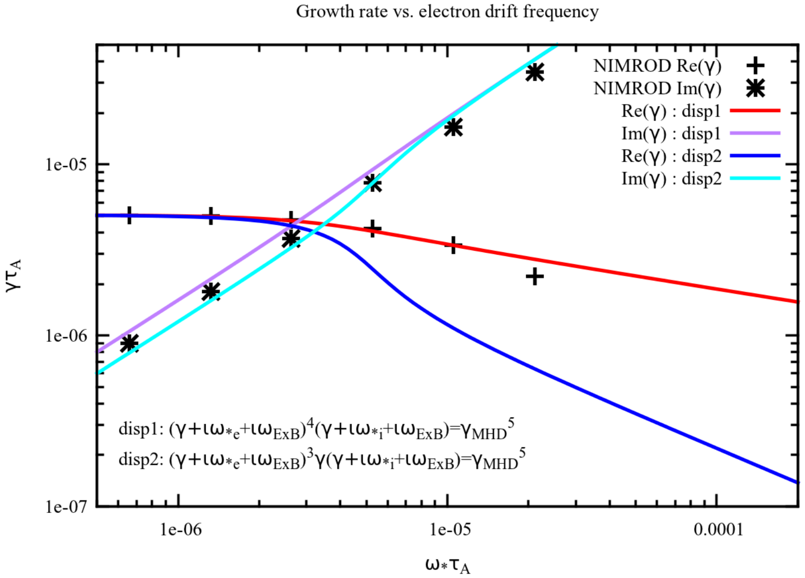
We use large-guide field sheared slab equilibria, but add a pressure profile.

- The parameters are chosen based off the benchmark with Ahedo and Ramos.
  - $\epsilon_B = B_{y\infty}/B_{z0} = 0.02$
  - $kL_B = 0.76$, $\Delta'/k = 1.46$, $a/L_B = 6$, $S_k = 3.5 \times 10^7$, $P_m = 0$, $\beta = 0.1$
- The pressure profile gradient is the result of a density gradient and a flat temperature profile to avoid ITG-like modes.

$$n_0\left(x\right) = n_0\left(1 + \frac{n_1}{n_0}\tanh\left[\frac{x}{L_B}\right]\right)$$

- Each case is run twice: once with a single-fluid Ohm's law to ensure the outer solution (Δ') is not significantly modified by the pressure profile, and again with the generalized Ohm's law.
- No gyroviscosity or mean flow (for now).

The NIMROD numerical dispersion relation is not the expected drift-tearing relation (disp2).



Single-fluid cases with a pressure gradient exhibit a numerical instability without small particle/thermal diffusion.

- Small particle and thermal diffusions are used to avoid these numerical problems, $D_n \mu_0 / \eta = \chi_\perp \mu_0 / \eta = 0.266$.
- With the small diffusion, the expected MHD growth rate is observed for density gradient values up to $L_B n_0' / n_0 = 0.16$.
- Also tried (without success): density hyper-diffusivity, increased resolution/smaller time-step, no density equation evolution, small viscosity.
  - Results indicate numerical instability is sensitive to the pressure evolution.

Reduced models typically consider the parallel-vorticity equation and the parallel Ohm's law.

- The goal is to capture slow instability dynamics after elimination of the fast waves. See, for example, Strauss [PoF 1976/77] or Hazeltine et al. [PoF 1985]
- Parallel vorticity equation is straight forward, $\hat{\mathbf{b}}_0 \cdot \nabla \times [momentum\ eqn]$.
- Parallel Ohm's law: Using a flux representation, $\mathbf{B} = B_{\|}\hat{\mathbf{b}}_0 + \mathbf{B}_0 \times \nabla\psi$, where $\psi \sim A_{\|}$.
  - $\hat{\mathbf{b}}_0 \cdot [\mathbf{E} = -\nabla\Phi - \partial\mathbf{A}/\partial t] \rightarrow E_{\|} = -ik_{\|}\Phi - B_0\partial\psi/\partial t$.
  - Provides and equation for the evolution of the flux function.
  - Generalized Ohm's law provides $E_{\|}$.

A straight-forward analogue to the parallel Ohm's law is the radial induction equation.

- With flux rep., $\tilde{B}_r = ik_\perp B_0 \tilde{\psi}$.
- Thus $\hat{r} \cdot \left[\frac{\partial \tilde{\mathbf{B}}}{\partial t} = -\nabla \times \mathbf{E} \right] \to \gamma B_0 ik_\perp \tilde{\psi} = -ik_\perp \tilde{E}_\parallel + ik_\parallel \tilde{E}_\perp$ or

$$\gamma B_0 \tilde{\psi} = -\tilde{E}_\parallel + \frac{k_\parallel}{k_\perp} \tilde{E}_\perp$$

- Should produce the same result as the parallel Ohm's law:

$$\gamma B_0 \tilde{\psi} = -\tilde{E}_\parallel - ik_\parallel \tilde{\Phi}$$

- The relevant comparison is between $-\tilde{E}_\perp / ik_\perp$ and $\tilde{\Phi}$.

The flux equations agree term by term with resistive MHD.

- Using tearing layer ordering, $\tilde{\psi}'' >> k\tilde{\psi}' \sim k^2\tilde{\psi}$ for $\Delta'/k \sim \mathcal{O}(1)$ and $k_{\|} \simeq k_{\|}' x$ where $kx << 1$, and a stream-function for ion velocity $\tilde{\mathbf{v}} = \hat{\mathbf{b}}_0 \times \nabla\tilde{\phi} + \tilde{v}_{\|}\hat{\mathbf{b}}_0$.

$$\tilde{E}_{\|} = \eta\tilde{J}_{\|} \simeq -B_0\eta\tilde{\psi}''$$

$$\tilde{E}_{\perp} = -ik_{\perp}\tilde{\phi}B_0 + \eta\tilde{J}_{\perp}$$

$$\gamma B_0\tilde{\psi} = -\tilde{E}_{\|} + \frac{k_{\|}}{k_{\perp}}\tilde{E}_{\perp} \simeq B_0\eta\tilde{\psi}'' - ik_{\|}\tilde{\phi}B_0$$

$$\Rightarrow \ \gamma\tilde{\psi} = -ik_{\|}\tilde{\phi} + \eta\tilde{\psi}''$$

- The electrostatic potential is related to the stream-function through the $\mathbf{E} \times \mathbf{B}$ flow via $\tilde{E}_{\perp}$:

$$\tilde{\mathbf{v}}_{E\times B} = \frac{\tilde{\mathbf{E}} \times \mathbf{B}_0}{B_0^2} \simeq -\frac{\nabla\tilde{\Phi}}{B_0} \times \hat{\mathbf{b}}_0 \simeq -\nabla\tilde{\phi} \times \hat{\mathbf{b}}_0 \ .$$

The comparison with a generalized Ohm's law is not as clear.

$$\tilde{E}_{\parallel} = v_{e0\perp}\tilde{B}_r - \frac{ik_{\parallel}\tilde{p}_e}{n_0 e} + \eta\tilde{J}_{\parallel}$$

$$\tilde{E}_{\perp} = -ik_{\perp}\tilde{\phi}B_0 - v_{e0\parallel}\tilde{B}_r - \frac{\tilde{J}_r B_0}{n_0 e} - \frac{ik_{\perp}\tilde{p}_e}{n_0 e} + \eta\tilde{J}_{\perp}$$

- Radial induction equation:

$$\gamma B_0\tilde{\psi} = -\tilde{E}_{\parallel} + \frac{k_{\parallel}}{k_{\perp}}\tilde{E}_{\perp}$$

$$\Rightarrow \ (\gamma + ik_{\perp}v_{e0\perp})\,\tilde{\psi} = -ik_{\parallel}\tilde{\phi} - ik_{\parallel}\frac{\tilde{B}_{\parallel}}{\mu_0 n_0 e} + \eta\tilde{\psi}''$$

- Parallel Ohm's law:

$$(\gamma + ik_{\perp}v_{e0\perp})\,\tilde{\psi} = -ik_{\parallel}\frac{\tilde{\Phi}}{B_0} + \frac{ik_{\parallel}\tilde{p}_e}{B_0 n_0 e} + \eta\tilde{\psi}''$$

Traditional drift-tearing analysis manipulates the parallel Ohm's law into a simplified form.

$$(\gamma + i\hat{\omega}_{*e})\,\tilde{\psi} = -ik_{\|}\frac{\tilde{\Phi}}{B_0} + \frac{ik_{\|}\tilde{p}_e}{B_0 n_0 e} + \eta\tilde{\psi}''$$

- After finding expressions for $\tilde{\Phi}$ and $\tilde{p}_e$,

$$\Rightarrow\ (\gamma + i\hat{\omega}_{*e})\,\tilde{\psi} = -\left(1 + \frac{i\hat{\omega}_{*e}}{\gamma}\right)\tilde{\phi} + \eta\tilde{\psi}''$$

$$\Rightarrow\ (\gamma + i\hat{\omega}_{*e})^3\,\gamma\,(\gamma + i\hat{\omega}_{*i}) = \gamma_{MHD}^5$$

- We use $\hat{\omega}_{*e} = k_\perp v_{e0\perp}$ and $\hat{\omega}_{*i} = k_\perp v_{i0\perp}$.

What is the contribution from $\tilde{B}_\parallel$ to the radial induction equation?

$$\left(\gamma + i\hat{\omega}_{*e}\right)\tilde{\psi} = -ik_\parallel\tilde{\phi} - ik_\parallel\frac{\tilde{B}_\parallel}{\mu_0 n_0 e} + \eta\tilde{\psi}''$$

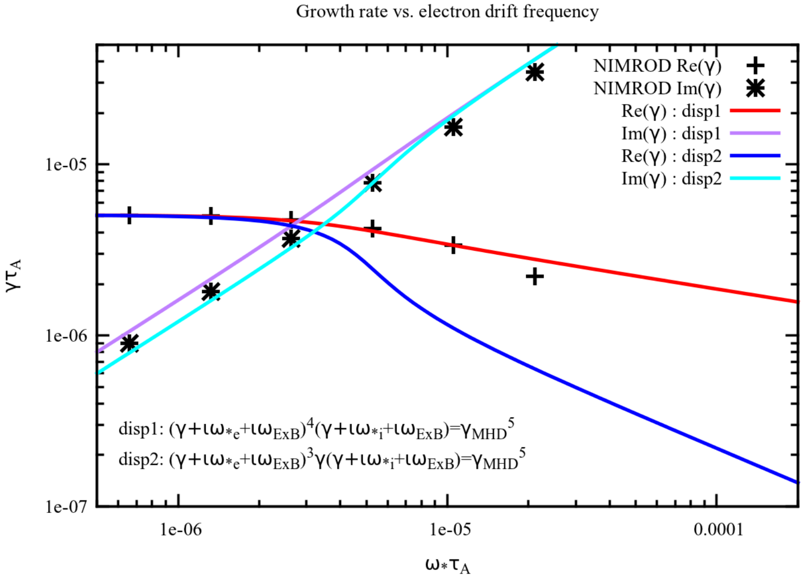
- In general, let the contribution be $\sim M\tilde{\phi}$:

$$\left(\gamma + i\hat{\omega}_{*e}\right)\tilde{\psi} = -ik_\parallel\left(1 + M\right)\tilde{\phi} + \eta\tilde{\psi}''$$

$$\Rightarrow \;\frac{\left(\gamma + i\hat{\omega}_{*e}\right)^4\left(\gamma + i\hat{\omega}_{*i}\right)}{\left(1 + M\right)} = \gamma_{MHD}^5$$

- Standard drift tearing: $M = i\hat{\omega}_{*e}/\gamma$.
- Another limit, $M = 0 \Rightarrow \left(\gamma + i\hat{\omega}_{*e}\right)^4\left(\gamma + i\hat{\omega}_{*i}\right) = \gamma_{MHD}^5$.

The relations disp1 and disp2 correspond to $M = 0$ (no $\tilde{B}_\parallel$ contribution) and $M = i\hat{\omega}_{*e}/\gamma$ (standard D-T).



Consideration of the parallel induction equation shows drift-like terms ($\sim n_0'\tilde{\phi}$).

$$
\begin{aligned}
\left(\gamma + i\mathbf{k}\cdot\mathbf{v}_{e0}\right)\tilde{B}_\parallel = -B_0\left(\nabla_\perp\cdot\tilde{\mathbf{v}}\right) + \frac{\left(i\mathbf{k}\cdot\mathbf{B}_0\right)}{\mu_0 n_0 e}B_0\tilde{\psi}'' + \frac{\eta}{\mu_0}\nabla^2\tilde{B}_\parallel \\
+ \left(\hat{b}_0\cdot\mathbf{B}_0'\right) ik_\perp\left[\tilde{\phi} + \frac{\tilde{B}_\parallel}{\mu_0 n_0 e} + \frac{k_\parallel}{k_\perp}\frac{B_0\tilde{\psi}'}{\mu_0 n_0 e}\right] \\
-\left(\frac{n_0'}{n_0}\right)^2\frac{\left(ik_\perp\right)^2\left(T_{i0}+T_{e0}\right)\tilde{\phi}}{e\left(\gamma + i\mathbf{k}\cdot\mathbf{v}_0\right)} + \frac{ik_\perp n_0'\left(\Gamma T_{e0}+T_{i0}\right)}{n_0 e\left(\gamma + i\mathbf{k}\cdot\mathbf{v}_0\right)}\nabla\cdot\tilde{\mathbf{v}}
\end{aligned}
$$

- The $\hat{b}_0\cdot\mathbf{B}_0' \simeq -p_0'\mu_0/B_0$ through equilibrium considerations.
- We have used the linear continuity and energy equations without diffusion to eliminate $\tilde{n}$ and $\tilde{p}_e$.
- We ignore the compressive ($\nabla\cdot\tilde{\mathbf{v}}$), KAW/Whistler $\sim\tilde{\psi}''$, and diffusive terms for now.

Including only 'drift' contributions gives an algebraic expression for $\tilde{B}_{\parallel}\left(\tilde{\phi}\right)$.

$$
\begin{aligned}
&\left(\gamma + i\mathbf{k}\cdot\mathbf{v}_{e0} + \omega_{*e} + \omega_{*i}\right)\tilde{B}_{\parallel} \\
&\qquad \simeq -\left[ik_{\perp}\frac{p_0'\mu_0}{B_0} + \left(\frac{n_0'}{n_0}\right)^2 \frac{\left(ik_{\perp}\right)^2\left(T_{i0}+T_{e0}\right)}{e\left(\gamma + i\mathbf{k}\cdot\mathbf{v}_0\right)}\right]\tilde{\phi}
\end{aligned}
$$

- Plugging this into the radial induction equation we find

$$M = \frac{i\omega_*}{\gamma} - \frac{2}{\beta}\frac{\omega_*^2}{\gamma^2}$$

- Where $\omega_* = \omega_{*i} + \omega_{*e}$, the $i\omega_*/\gamma$ term is from $\tilde{\mathbf{v}}_e \times \mathbf{B}_0$, and the term $\sim \omega_*^2$ results from $\nabla\tilde{p}_e$ and $\tilde{n}/n_0^2 e$ contributions.

$$\frac{\left(\gamma + i\hat{\omega}_{*e}\right)^4\left(\gamma + i\hat{\omega}_{*i}\right)}{\left(1+M\right)} = \gamma_{MHD}^5 \tag{1}$$

This dispersion relation with $\tilde{B}_{\|}$ contributions $\sim n_0'$ is not descriptive of the computations.

$$\frac{(\gamma + i\hat{\omega}_{*e})^4 (\gamma + i\hat{\omega}_{*i})}{\left(1 + \frac{i\omega_*}{\gamma} - \frac{2}{\beta}\frac{\omega_*^2}{\gamma^2}\right)} = \gamma_{MHD}^5$$

- There are now seven roots (spurious roots?) and the most-unstable value of $\gamma\tau_A$ becomes large relative to $\gamma_{MHD}\tau_A$ at large values of $\omega_*\tau_A$.

Grasso et al. [2001] includes the effect of particle diffusivity in the D-T parallel Ohm's law analysis.

- They find three regimes:
  - standard D-T: $D\mu_0/\eta << (\omega_*\tau_A\Delta'/k\epsilon_B)^{2/3}$
  - intermediate particle diffusion: $(\omega_*\tau_A\Delta'/k\epsilon_B)^{2/3} << D\mu_0/\eta << S^{1/2}(\omega_*\tau_A)^{2/3}\epsilon_B^{-1/6}$
  - diffusive regime: $S^{1/2}(\omega_*\tau_A)^{2/3}\epsilon_B^{-1/6} << D\mu_0/\eta$
    - $\gamma_{MHD} = \gamma^{1/5}(\gamma + i\omega_*)^{4/5}(\gamma/i\omega_*)^{1/4}$
    - similar to solution with $M = 0$.
- For our cases:
  - $(\omega_*\tau_A\Delta'/k\epsilon_B)^{2/3} \simeq 1.3\times10^{-3} - 1.3\times10^{-2}$
  - $D\mu_0/\eta \simeq 0.27$
  - $S^{1/2}(\omega_*\tau_A)^{2/3}\epsilon_B^{-1/6} \simeq 0.86 - 8.6$
- Even small diffusion matters! Are our cases beginning to transition from a diffusive to standard-DT regime?

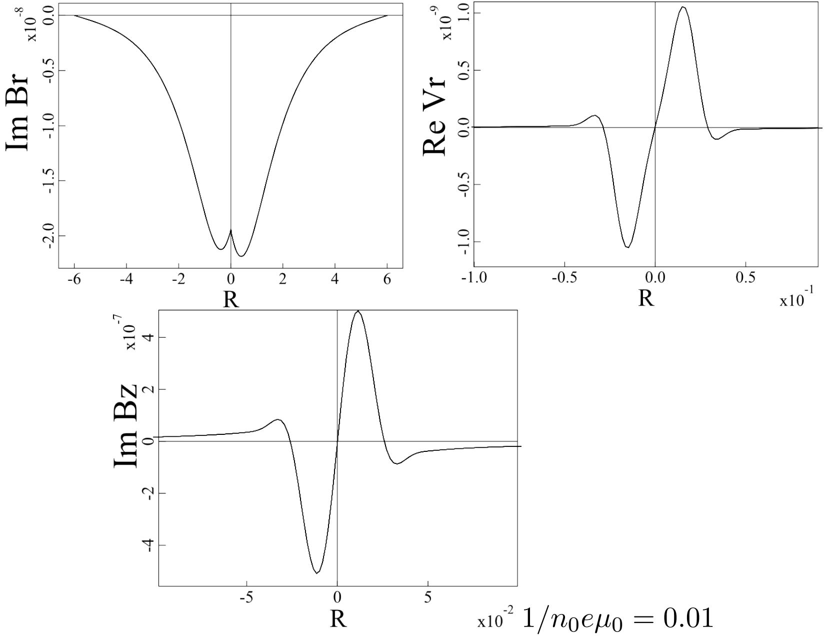
Future drift-tearing directions

- Plot contributions to the parallel and radial induction equations by term.
- Investigate the effects of diffusion and compression in $\tilde{B}_{\parallel}$.
- Run cases with Hall-MHD and no contribution from $\nabla \tilde{p}_e$.
- Add gyroviscosity and equilibrium flow.
- Investigate the single-fluid numerical instability with pressure gradient: is there a method alternative to particle and thermal diffusion for mitigation?

Conclusions

- Drift tearing computations with extended-MHD and small thermal and particle diffusion do not reproduce the standard drift-tearing behavior.
- It appears even small amounts of diffusion has a large effect.

Eigenfuction - $\omega_* \tau_A = 1.6 \times 10^{-5}$



$1/n_0 e \mu_0 = 0.01$