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KINETIC-MHD AND ITS QUASINEUTRALITY CONDITION AS SPECIAL LIMIT TEST OF A GENERAL FLUID AND DRIFT-KINETIC SYSTEM*

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A HYBRID FLUID AND DRIFT-KINETIC SYSTEM [Phys. Plasmas 17, 082502 (2010) and 18, 102506 (2011)] IS BEING CONSIDERED FOR SIMULATION OF LONG-WAVELENGTH INSTABILITIES IN HIGH-TEMPERATURE PLASMAS (e.g. TOKAMAK CORE NTM's).

THIS SYSTEM IS VERY GENERAL:

- FULLY 3-DIMENSIONAL AND ELECTROMAGNETIC
- FINITE-LARMOR-RADIUS TO FIRST ORDER IN ρ_e/L AND SECOND ORDER IN ρ_i/L
- FOKKER-PLANCK-LANDAU COLLISION OPERATORS WITH $\nu_s L/v_{ths} \sim \rho_i/L$

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SPECIAL LIMIT REDUCTIONS ARE USEFUL FOR INCREMENTAL IMPLEMENTATION AND TESTING:

- THE FIRST ORDER IN ρ_i/L, STATIONARY AND AXISYMMETRIC LIMIT YIELDS THE RESULTS OF THE TOKAMAK NEOCLASSICAL THEORY IN THE BANANA REGIME

- THIS TALK WILL CONSIDER THE ZERO-LARMOR-RADIUS, COLLISIONLESS LIMIT

  THIS CORRESPONDS TO THE KINETIC-MHD THEORY OF KRUSKAL-OBERMAN, ROSENBLUTH-ROSTOKER AND KULSRUD

  THE PRESENT FORMALISM IS PARTICULARLY WELL SUITED FOR DEALING RIGOROUSLY WITH THE ISSUES OF CHARGE NEUTRALITY AND PARALLEL ELECTRIC FIELD CONTRIBUTION, THAT WERE NOT COMPLETELY RESOLVED IN THE CLASSIC PAPERS

ZERO-LARMOR-RADIUS, COLLISIONLESS, QUASINEUTRAL SYSTEM

Take the $\Omega_{cs} \to \infty$ and $\nu_s \to 0$ limit of the general quasineutral system, for $\beta \sim 1$ and $T_e \sim T_i$.

Then, $\rho_i \sim d_i \sim \rho_S \to 0$ and $\mathbf{u}_e \to \mathbf{u}_i \to \mathbf{u}$. Taking also $m_e/m_i \to 0$, the system becomes:

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}) , \qquad \mathbf{j} = \nabla \times \mathbf{B}$$

$$\frac{\partial n}{\partial t} + \nabla \cdot (n\mathbf{u}) = 0$$

$$m_i n \left[\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u}\right] - \mathbf{j} \times \mathbf{B} + \sum_{s=i,e} \nabla \cdot \left[nT_s \mathsf{I} + (p_{s\parallel} - p_{s\perp})(\mathbf{bb} - \mathsf{I}/3)\right] = 0$$

$$\frac{3n}{2}\left(\frac{\partial T_s}{\partial t} + \mathbf{u} \cdot \nabla T_s\right) + \left[nT_s \mathsf{I} + (p_{s\parallel} - p_{s\perp})(\mathbf{bb} - \mathsf{I}/3)\right] : (\nabla \mathbf{u}) + \nabla \cdot (q_{s\parallel} \mathbf{b}) = 0$$

$$(p_{s\parallel} - p_{s\perp}) \;=\; 2\pi m_s \int_0^\infty dv' \; v'^4 \int_0^\pi d\chi \;\; \sin\chi \;\; P_2(\cos\chi) \;\; \bar{f}_{NMs}(v', \chi, \mathbf{x}, t)$$

$$q_{s\parallel} = \; \pi m_s \int_0^\infty dv' \; v'^5 \int_0^\pi d\chi \;\; \sin\chi \;\; \cos\chi \;\; \bar{f}_{NMs}(v', \chi, \mathbf{x}, t)$$

where

$$\mathbf{v}' \;=\; \mathbf{v} \;-\; \mathbf{u}(\mathbf{x}, t) \;=\; v' \; \{\cos\chi \;\, \mathbf{b}(\mathbf{x}, t) \;+\; \sin\chi \;\, [\cos\alpha \;\, \mathbf{e}_1(\mathbf{x}, t) + \sin\alpha \;\, \mathbf{e}_2(\mathbf{x}, t)]\}$$

$$\bar{f}_{NMs}(v', \chi, \mathbf{x}, t) \;=\; (2\pi)^{-1} \oint d\alpha \; [f_s(v', \chi, \alpha, \mathbf{x}, t) - f_{Ms}(v', \mathbf{x}, t)]$$

$$f_{Ms}(v', \mathbf{x}, t) \;=\; \left(\frac{m_s}{2\pi}\right)^{3/2} \; \frac{n}{T_s^{3/2}} \;\; \exp\left(-\frac{m_s v'^2}{2T_s}\right)$$

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and

$$\frac{\partial \bar{f}_{NMs}}{\partial t} + \cos\chi \left(v' \mathbf{b} \cdot \frac{\partial \bar{f}_{NMs}}{\partial \mathbf{x}} + \frac{T_s}{m_s} \mathbf{b} \cdot \nabla \ln n \;\frac{\partial \bar{f}_{NMs}}{\partial v'} \right) - \frac{\sin\chi}{v'} \left(\frac{T_s}{m_s} \mathbf{b} \cdot \nabla \ln n - \frac{v'^2}{2} \mathbf{b} \cdot \nabla \ln B \right) \frac{\partial \bar{f}_{NMs}}{\partial \chi} =$$

$$= \Big\{ \cos\chi \;\frac{v'}{2T_s} \left(5 - \frac{m_s v'^2}{T_s} \right) \mathbf{b} \cdot \nabla T_s \;+\; \cos\chi \;\frac{v'}{nT_s} \;\mathbf{b} \cdot \left[\frac{2}{3} \nabla (p_{s\parallel} - p_{s\perp}) - \left(p_{s\parallel} - p_{s\perp} \right) \nabla \ln B \right] +$$

$$+ \;P_2(\cos\chi) \;\frac{m_s v'^2}{3T_s} \big(\nabla \cdot \mathbf{u} - 3\mathbf{b} \cdot [(\mathbf{b} \cdot \nabla)\mathbf{u}] \big) \;+\; \frac{1}{3nT_s} \left(\frac{m_s v'^2}{T_s} - 3 \right) \nabla \cdot (q_{s\parallel} \mathbf{b}) \Big\} \;f_{Ms}$$

NOTEWORTHY FEATURES OF THIS DRIFT-KINETIC EQUATION:

- **THE PHASE-SPACE VELOCITY VARIABLE $\mathbf{v}'$ IS THE RANDOM VELOCITY IN THE REFERENCE FRAME OF THE MACROSCOPIC FLOW $\mathbf{u}(\mathbf{x}, t)$. DIRECT EVALUATION OF THE FLUID CLOSURE MOMENTS $(p_{s\|} - p_{s\perp})$ AND $q_{s\|}$.**

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- WORKING IN THE REFERENCE FRAME OF THE MACROSCOPIC FLOW, THE ELECTRIC FIELD HAS BEEN ELIMINATED ALGEBRAICALLY USING THE EXACT MOMENTUM CONSERVATION EQUATION OF EACH SPECIES. ITS PARALLEL COMPONENT IS ACCOUNTED FOR AUTOMATICALLY.

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- THE DYNAMIC EVOLUTION OF $\bar{f}_{NMs}$ PRESERVES THE CONSTRAINTS THAT $\int d^3\mathbf{v}'(1, v'_\|, v'^2)\bar{f}_{NMs} = 0$. QUASINEUTRALITY IS ALWAYS SATISFIED AND THE REDUNDANCY OF THE DRIFT-KINETIC EQUATION WITH PARTS OF THE FLUID SYSTEM IS AVOIDED.

LINEARIZATION ABOUT A MAXWELLIAN EQUILIBRIUM WITHOUT FLOW

$$\mathbf{u}_0 = 0 \ , \qquad \mathbf{b}_0 \cdot \nabla n_0 = 0 \ , \qquad \mathbf{b}_0 \cdot \nabla T_{s0} = 0 \ , \qquad \partial/\partial t = -i\omega \ , \qquad \mathbf{u}_1 = -i\omega \ \boldsymbol{\xi}$$

$$\mathbf{B}_1 \ = \ \nabla \times (\boldsymbol{\xi} \times \mathbf{B}_0) \ , \qquad \mathbf{j}_1 \ = \ \nabla \times \mathbf{B}_1$$

$$n_1 \ = \ -\boldsymbol{\xi} \cdot \nabla n_0 - n_0 \nabla \cdot \boldsymbol{\xi}$$

$$\omega^2 m_i n_0 \ \boldsymbol{\xi} = -\mathbf{j}_0 \times \mathbf{B}_1 - \mathbf{j}_1 \times \mathbf{B}_0 + \sum_{s=i,e} \left\{ \nabla (n_1 T_{s0} + n_0 T_{s1}) + \nabla \cdot \left[(p_{s\|} - p_{s\perp}) \left(\mathbf{b}_0 \mathbf{b}_0 - \mathbf{I}/3 \right) \right] \right\}$$

$$T_{s1} \ = \ -\boldsymbol{\xi} \cdot \nabla T_{s0} - \frac{2}{3} T_{s0} \nabla \cdot \boldsymbol{\xi} - \frac{2i}{3\omega n_0} \nabla \cdot (q_{s\|} \mathbf{b}_0)$$

$$-i\omega \bar{f}_{NMs} \ + \ v' \cos\chi \ \mathbf{b}_0 \cdot \frac{\partial \bar{f}_{NMs}}{\partial \mathbf{x}} \ + \ \frac{v'}{2} \sin\chi \ \mathbf{b}_0 \cdot \nabla \ln B_0 \ \frac{\partial \bar{f}_{NMs}}{\partial \chi} \ =$$

$$= \left\{ \cos\chi \ \frac{v'}{2T_{s0}} \left(5 - \frac{m_s v'^2}{T_{s0}} \right) (\mathbf{b}_0 \cdot \nabla T_{s1} + \mathbf{b}_1 \cdot \nabla T_{s0}) + \cos\chi \frac{v'}{nT_{s0}} \mathbf{b}_0 \cdot \left[\frac{2}{3} \nabla (p_{s\|} - p_{s\perp}) - \left(p_{s\|} - p_{s\perp} \right) \nabla \ln B_0 \right] - \right.$$

$$\left. -i\omega \ P_2(\cos\chi) \ \frac{m_s v'^2}{3T_{s0}} \Big(\nabla \cdot \boldsymbol{\xi} - 3\mathbf{b}_0 \cdot [(\mathbf{b}_0 \cdot \nabla)\boldsymbol{\xi}] \Big) \ + \ \frac{1}{3n_0 T_{s0}} \left(\frac{m_s v'^2}{T_{s0}} - 3 \right) \nabla \cdot (q_{s\|} \mathbf{b}_0) \right\} \ f_{Ms0}$$

CHANGE VARIABLES:

$$\textbf{from} \qquad \bar{f}_{NMs} \qquad \textbf{to} \qquad \hat{f}_s = \bar{f}_{NMs} - \frac{1}{3}\left[\frac{m_s v'^2}{T_{s0}}\nabla\cdot\boldsymbol{\xi} \;+\; \frac{i}{\omega n_0 T_{s0}}\left(\frac{m_s v'^2}{T_{s0}} - 3\right)\nabla\cdot(q_{s\|}\mathbf{b}_0)\right] f_{Ms0}$$

$$\textbf{from} \qquad (\mathbf{x}, v', \chi) \qquad \textbf{to} \qquad (\mathbf{x}, v', \lambda), \quad \textbf{with} \;\; \lambda = \sin^2\chi/B_0(\mathbf{x})$$

THE LINEARIZED DRIFT-KINETIC EQUATION BECOMES

$$-i\omega\hat{f}_s \;+\; v'\varsigma\;(1-\lambda B_0)^{1/2}\;\mathbf{b}_0\cdot\frac{\partial\hat{f}_s}{\partial\mathbf{x}} \;=\; \left[\frac{v'\varsigma}{n_0 T_{s0}}\;(1-\lambda B_0)^{1/2}\hat{F}_{s\|} \;+\; \frac{i\omega m_s v'^2}{T_{s0}}\;Q\right] f_{Ms0}$$

where

$$(1-\lambda B_0)^{1/2} \geq 0\;, \qquad \varsigma = \mathrm{sign}(\cos\chi) = \mathrm{sign}(v'_\|)$$

$$\hat{F}_{s\|}(\mathbf{x}) \;=\; \mathbf{b}_0\cdot\left\{\nabla\left[-\frac{5}{3}n_0 T_{s0}\nabla\cdot\boldsymbol{\xi} - \frac{2i}{3\omega}\nabla\cdot(q_{s\|}\mathbf{b}_0) + \frac{2}{3}(p_{s\|} - p_{s\perp})\right] - \left(p_{s\|} - p_{s\perp}\right)\nabla\ln B_0\right\}$$

$$Q(\mathbf{x},\lambda) \;=\; \frac{1}{2}\lambda B_0\nabla\cdot\boldsymbol{\xi} \;+\; \left(1 - \frac{3}{2}\lambda B_0\right)\mathbf{b}_0\cdot[(\mathbf{b}_0\cdot\nabla)\boldsymbol{\xi}]$$

THE NEW VARIABLES HAVE THE FOLLOWING MEANING:

- $\hat{f}_s$ **IS THE NON-CONVECTIVE PART OF THE TOTAL PERTURBED DISTRIBUTION FUNCTION**

$$\hat{f}_s = \bar{f}_{NMs} + f_{Ms1} + \boldsymbol{\xi} \cdot \frac{\partial f_{Ms0}}{\partial \mathbf{x}}$$

- λ **IS THE RATIO OF THE MAGNETIC MOMENT TO THE KINETIC ENERGY IN THE MOVING FRAME**

$$\lambda = \frac{v_\perp'^2}{v'^2 B_0}$$

- **THE PARALLEL COMPONENT OF THE LINEAR FLUID MOMENTUM EQUATION IS**

$$\omega^2 m_i n_0 \xi_\parallel \;=\; \hat{F}_{i\parallel} + \hat{F}_{e\parallel}$$

- **SEPARATING THE CONTRIBUTIONS OF** $\xi_\parallel$ **AND** $\boldsymbol{\xi}_\perp$ **TO** Q

$$Q \;=\; (1-\lambda B_0)^{1/2} \mathbf{b}_0 \cdot \frac{\partial}{\partial \mathbf{x}} \left[(1-\lambda B_0)^{1/2} \xi_\parallel \right] \;+\; \frac{1}{2} \lambda B_0 \nabla \cdot \boldsymbol{\xi}_\perp \;-\; \left(1 - \frac{3}{2} \lambda B_0 \right) \boldsymbol{\xi}_\perp \cdot \boldsymbol{\kappa}_0$$

SPLIT $\hat{f}_s$ INTO ITS EVEN AND ODD PARTS WITH RESPECT TO $v'_\parallel$:

$$-i\omega \hat{f}_s^{even} \; + \; v'\varsigma \; (1-\lambda B_0)^{1/2} \; \mathbf{b}_0 \cdot \frac{\partial \hat{f}_s^{odd}}{\partial \mathbf{x}} \; = \; \frac{i\omega m_s v'^2}{T_{s0}} \; Q f_{Ms0}$$

$$-i\omega \hat{f}_s^{odd} \; + \; v'\varsigma \; (1-\lambda B_0)^{1/2} \; \mathbf{b}_0 \cdot \frac{\partial \hat{f}_s^{even}}{\partial \mathbf{x}} \; = \; \frac{v'\varsigma}{n_0 T_{s0}} \; (1-\lambda B_0)^{1/2} \hat{F}_{s\parallel} f_{Ms0}$$

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MOMENTS OF THIS SYSTEM YIELD:

$$\nabla \cdot \boldsymbol{\xi} \;\; = \;\; -\frac{2\pi}{n_0} B_0 \int_0^\infty dv' \;\; v'^2 \int_0^{1/B_0} d\lambda \;\; (1-\lambda B_0)^{-1/2} \hat{f}_s^{even}$$

$$\hat{F}_{s\|} \;\; = \;\; 2\pi m_s B_0 \int_0^\infty dv' \;\; v'^4 \int_0^{1/B_0} d\lambda \;\; (1-\lambda B_0)^{1/2} \mathbf{b}_0 \cdot \frac{\partial \hat{f}_s^{even}}{\partial \mathbf{x}}$$

$$\frac{i}{\omega} \nabla \cdot (q_{s\|} \mathbf{b}_0) \;\; = \;\; \pi B_0 \int_0^\infty dv' \;\; v'^2 \;\; (5T_{s0} - m_s v'^2) \int_0^{1/B_0} d\lambda \;\; (1-\lambda B_0)^{-1/2} \hat{f}_s^{even}$$

WHICH IMPLY

$$\int d^3\mathbf{v}' \;\; (1, v'_{\|}, v'^2) \;\; \bar{f}_{NMs} \;\; = \;\; 0$$

NOW, THE VARIABLES NEEDED TO CLOSE THE FLUID MOMENTUM EQUATION ARE

$$T_{s1} = -\boldsymbol{\xi} \cdot \nabla T_{s0} - \frac{2\pi}{3n_0} B_0 \int_0^\infty dv' \; v'^2 \; (3T_{s0} - m_s v'^2) \int_0^{1/B_0} d\lambda \; (1 - \lambda B_0)^{-1/2} \hat{f}_s^{even}$$

$$(p_{s\parallel} - p_{s\perp}) = 2\pi m_s B_0 \int_0^\infty dv' \; v'^4 \int_0^{1/B_0} d\lambda \; (1 - \lambda B_0)^{-1/2} \left(1 - \frac{3}{2}\lambda B_0\right) \hat{f}_s^{even}$$

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THEREFORE, IN ORDER TO CLOSE THE FLUID SYSTEM, ONE NEEDS TO SOLVE ONLY FOR THE EVEN PART OF THE PERTURBED DISTRIBUTION FUNCTION.

ELIMINATING $\hat{f}_s^{odd}$, THE SECOND-ORDER EQUATION FOR $\hat{f}_s^{even}$ IS

$$\omega^2\hat{f}_s^{even} \;+\; v'^2\;(1-\lambda B_0)^{1/2}\;\mathbf{b}_0\cdot\frac{\partial}{\partial\mathbf{x}}\left[(1-\lambda B_0)^{1/2}\;\mathbf{b}_0\cdot\frac{\partial\hat{f}_s^{even}}{\partial\mathbf{x}}\right] \;=$$

$$=\;\left\{\frac{v'^2}{n_0T_{s0}}(1-\lambda B_0)^{1/2}\;\mathbf{b}_0\cdot\frac{\partial}{\partial\mathbf{x}}\left[(1-\lambda B_0)^{1/2}\;\hat{F}_{s\|}\right]\;-\;\frac{\omega^2 m_s v'^2}{T_{s0}}\;Q\right\}f_{Ms0}$$

TO BE SPECIFIC, CONSIDER AN EQUILIBRIUM WITH TOKAMAK-LIKE GEOMETRY:

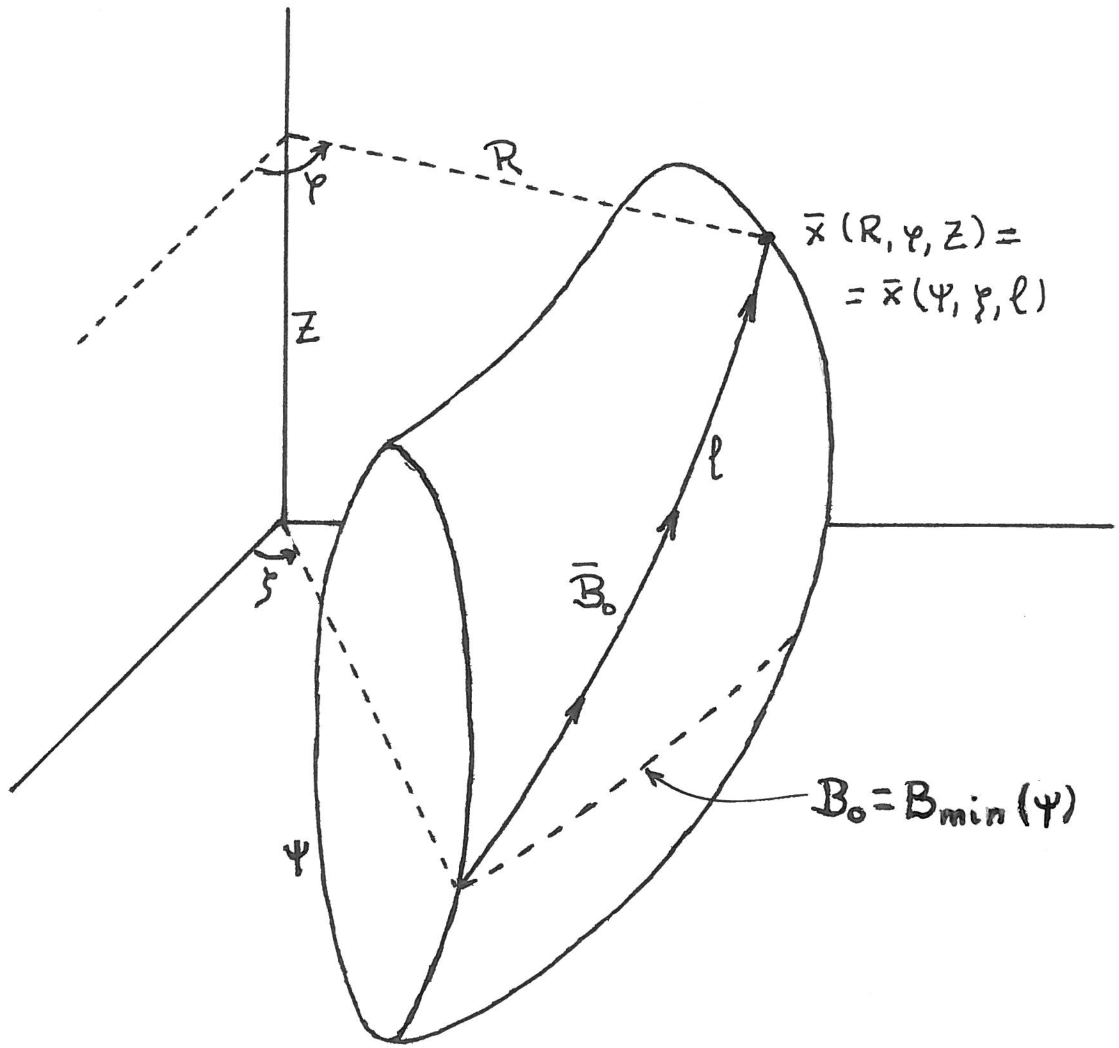
- **AXISYMMETRIC WITH NESTED FLUX SURFACES**
- **MOST FLUX SURFACES COVERED ERGODICALLY BY A MAGNETIC LINE**
- **ONE MAGNETIC WELL ON EACH FLUX SURFACE**

use magnetic coordinates $\mathbf{x}(\psi, \zeta, l)$

$$\mathbf{B}_0 = \nabla\psi \times \nabla\zeta$$

$$\zeta = \varphi - \textstyle\int_0^l dl' \, B_{0\varphi}/(RB_0)$$

the origin of l is chosen at the point where $B_0 = B_{min}(\psi)$; l and ζ have a cut at the point where $B_0 = B_{max}(\psi)$ and the jump of ζ is $2\pi q(\psi)$

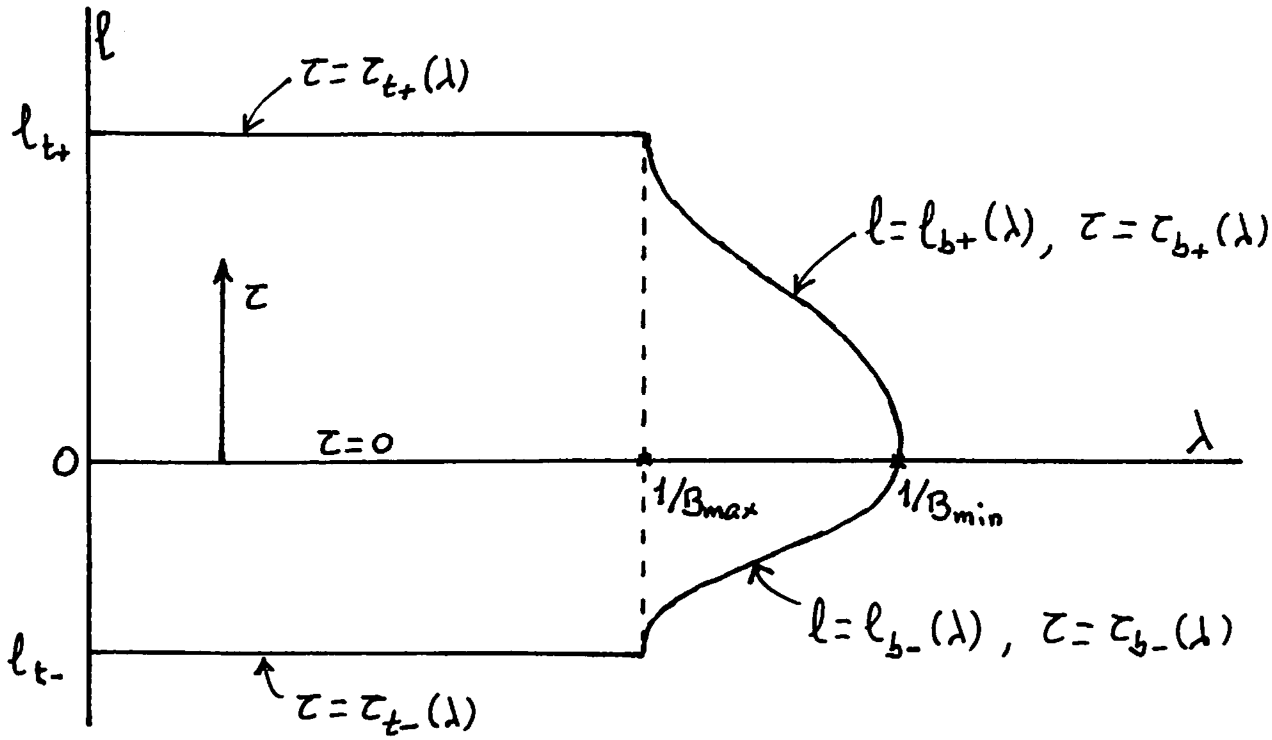


MAKE ONE LAST CHANGE OF VARIABLE:

$$\textbf{from} \qquad (\psi, \zeta, l, v', \lambda) \qquad \textbf{to} \qquad (\psi, \zeta, \tau, v', \lambda), \quad \textbf{with} \;\; \tau = \frac{1}{v'} \int_0^l dl' \;\; [1 - \lambda B_0(\psi, l')]^{-1/2}$$

$$\textbf{so that} \;\; v' \, (1 - \lambda B_0)^{1/2} \, \mathbf{b}_0 \cdot \frac{\partial}{\partial \mathbf{x}} \; = \; \frac{\partial}{\partial \tau}$$

PHASE-SPACE SECTION AT CONSTANT (ψ, ζ, v'):



IN THE τ VARIABLE, THE LINEARIZED DRIFT-KINETIC EQUATION BECOMES

$$\omega^2 \hat{f}_s^{even} + \frac{\partial^2 \hat{f}_s^{even}}{\partial \tau^2} = \left\{ \frac{v'}{n_0 T_{s0}} \frac{\partial}{\partial \tau} \left[(1 - \lambda B_0)^{1/2} \hat{F}_{s\|} \right] - \frac{\omega^2 m_s v'^2}{T_{s0}} Q \right\} f_{Ms0}$$

WHICH HAS THE GENERAL SOLUTION

$$\hat{f}_s^{even} = \frac{v'}{n_0 T_{s0}} f_{Ms0} \int_0^\tau d\tau' \cos[\omega(\tau - \tau')] \left[(1 - \lambda B_0)^{1/2} \hat{F}_{s\|} \right] (\tau', \ldots) -$$

$$- \frac{\omega^2 m_s v'^2}{T_{s0}} f_{Ms0} \int_0^\tau d\tau' \exp[-i\omega(\tau - \tau')] \int_0^{\tau'} d\tau'' \exp[i\omega(\tau' - \tau'')] \, Q(\tau'', \ldots) +$$

$$+ C_s^\omega(\psi, \zeta, v', \lambda) \cos \omega\tau + D_s^\omega(\psi, \zeta, v', \lambda) \sin \omega\tau$$

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$$+ \; C_s^\omega(\psi, \zeta, v', \lambda) \; \cos \omega\tau \; + \; D_s^\omega(\psi, \zeta, v', \lambda) \; \sin \omega\tau$$

IN THE $\omega \longrightarrow 0$ LIMIT

$$\hat{f}_s^{even}(\omega \longrightarrow 0) \; = \; \frac{f_{Ms0}}{n_0 T_{s0}} \; \left[\hat{G}_s(\psi, \zeta, l) - \hat{G}_s(\psi, \zeta, 0) \right] \; + \; C_s^0(\psi, \zeta, v', \lambda)$$

where

$$\mathbf{b}_0 \cdot \nabla \hat{G}_s(\mathbf{x}) \; = \; \hat{F}_{s\|}(\mathbf{x})$$

THE INTEGRATION CONSTANTS C_s^ω AND D_s^ω ARE SPECIFIED BY IMPOSING APPROPRIATE BOUNDARY CONDITIONS ON $\hat{f}_s^{even}$

IN THE TRAPPED DOMAIN, $1/B_{max} < \lambda < 1/B_{min}$, THE BOUNDARY CONDITIONS ARE

$$\frac{\partial \hat{f}_s^{even}(\tau = \tau_{b+})}{\partial \tau} = \frac{\partial \hat{f}_s^{even}(\tau = \tau_{b-})}{\partial \tau} = 0$$

SO THAT, WHEN EXPRESSED BACK IN (v', χ) COORDINATES, $\hat{f}_s^{even}$ HAS CONTINUOUS DERIVATIVE AT $\chi = \pi/2$, i.e. $v'_{\parallel} = 0$

IN THE PASSING DOMAIN, $0 < \lambda < 1/B_{max}$, THE BOUNDARY CONDITIONS ARE DICTATED BY CONTINUITY AT THE BRANCH CUT OF ζ AND l:

$$\hat{f}_s^{even}(\zeta = \varphi_0, l = l_{t+}) = \hat{f}_s^{even}(\zeta = \varphi_0 + 2\pi q, l = l_{t-}) \ , \quad \hat{f}_s^{even}(\zeta = \varphi_0, l = l_{t-}) = \hat{f}_s^{even}(\zeta = \varphi_0 - 2\pi q, l = l_{t+})$$

THEN, PIECING TOGETHER THE $\hat{f}_s^{even}$ SOLUTIONS WITH DIFFERENT VALUES OF ζ SHIFTED BY MULTIPLES OF $2\pi q$, AN EXTENDED FUNCTION CAN BE CONSTRUCTED

$$\hat{f}_{sX}^{even}(\psi, \tau, v', \lambda) \qquad \textbf{with} \quad -\infty < \tau < +\infty \quad \textbf{for an ergodic flux surface}$$

WHICH MUST BE BOUNDED AS $\tau \to \pm\infty$

ASSUMING AN UNSTABLE MODE, $-i\omega = \gamma$, $\mathrm{Re}\ \gamma > 0$, THE CONDITION THAT THE EXPONENTIALLY GROWING TERMS OF $\hat{f}_{sX}^{even}(\tau \to \pm\infty)$ VANISH, SPECIFIES C_s^ω AND D_s^ω

AFTER IMPOSING THE BOUNDARY CONDITIONS AND TAKING THE $\gamma = -i\omega \to 0$ LIMIT, THE MARGINALLY STABLE DRIFT-KINETIC SOLUTION IS FOUND TO BE

$$\hat{f}_s^{even}(\gamma \to 0) \ = \ \frac{f_{Ms0}}{n_0 T_{s0}} \left[\hat{G}_s \ - \ \langle \hat{G}_s \rangle_\tau \ - \ m_s n_0 v'^2 \langle Q \rangle_\tau \right]$$

where the $\langle ... \rangle_\tau$ average is defined as

$$\langle X \rangle_\tau \equiv \frac{1}{\tau_{b+} - \tau_{b-}} \ \int_{\tau_{b-}}^{\tau_{b+}} d\tau \ X(\tau, ...) \qquad \textbf{for} \qquad 1/B_{max} < \lambda < 1/B_{min}$$

$$\langle X \rangle_\tau \equiv \frac{1}{2} \lim_{\gamma \to 0} \left[\gamma \int_{-\infty}^{+\infty} d\tau \ \exp(-\gamma |\tau|) \ X(\tau, ...) \right] \qquad \textbf{for} \qquad 0 < \lambda < 1/B_{max}$$

Note that $\langle ... \rangle_\tau$ is continuous at the trapped-passing boundary $\lambda = 1/B_{max}$

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Note that $\langle ... \rangle_\tau$ is continuous at the trapped-passing boundary $\lambda = 1/B_{max}$

THE PARALLEL DISPLACEMENT $\xi_\parallel$ DOES NOT CONTRIBUTE TO $\langle Q \rangle_\tau$:

$$\langle Q \rangle_\tau \;=\; \Big\langle \; \frac{1}{2} \lambda B_0 \nabla \cdot \boldsymbol{\xi}_\perp - \left(1 - \frac{3}{2} \lambda B_0\right) \boldsymbol{\xi}_\perp \cdot \boldsymbol{\kappa}_0 \; \Big\rangle_\tau$$

AT MARGINAL STABILITY ($\gamma = -i\omega \to 0$), THE $\int d^3\mathbf{v}'\ \bar{f}_{NMs} = 0$ CONDITION AND THE PARALLEL COMPONENT OF THE LINEAR FLUID MOMENTUM EQUATION YIELD

$$\nabla \cdot \boldsymbol{\xi} \;=\; -\,\frac{\hat{G}_s}{n_0 T_{s0}} \;+\; \frac{B_0}{2 n_0 T_{s0}} \int_0^{1/B_0} d\lambda\ (1-\lambda B_0)^{-1/2} \langle \hat{G}_s \rangle_\tau \;+\; \frac{3 B_0}{2} \int_0^{1/B_0} d\lambda\ (1-\lambda B_0)^{-1/2} \langle Q \rangle_\tau$$

AND

$$\hat{G}_i \;=\; -\,\hat{G}_e$$

THESE IMPLY

$$\hat{G}_s \;=\; \frac{B_0}{2} \int_0^{1/B_0} d\lambda\ (1-\lambda B_0)^{-1/2} \langle \hat{G}_s \rangle_\tau$$

WHICH HAS THE SOLUTION $\hat{G}_s = \hat{G}_s(\psi, \zeta) = \langle \hat{G}_s \rangle_\tau$**. THEREFORE,** $\hat{F}_{s\|} = \mathbf{b}_0 \cdot \nabla \hat{G}_s = 0$ **AND ONE CAN TAKE** $\hat{G}_s = 0$ **WITHOUT LOSS OF GENERALITY**

IN SUMMARY, THE MARGINAL STABILITY ($\gamma = -i\omega \to 0$) KINETIC SOLUTION IS

$$\hat{f}_s^{even}(\gamma \to 0) \;=\; -\,\frac{m_s v'^2}{T_{s0}}\,\langle Q\rangle_\tau\; f_{Ms0}$$

where

$$\langle Q\rangle_\tau \;=\; \Big\langle\, \frac{1}{2}\lambda B_0 \nabla\cdot\boldsymbol{\xi}_\perp - \left(1-\frac{3}{2}\lambda B_0\right)\boldsymbol{\xi}_\perp\cdot\boldsymbol{\kappa}_0 \,\Big\rangle_\tau$$

THIS SOLUTION IS CONSISTENT WITH THE $\int d^3\mathbf{v}'\;\bar{f}_{NMs} = 0$ CONDITION (HENCE QUASINEUTRALITY) AND THE PARALLEL COMPONENT OF THE FLUID MOMENTUM EQUATION, WITH A FLUID COMPRESSIBILITY GIVEN BY

$$\nabla\cdot\boldsymbol{\xi} \;=\; \frac{3B_0}{2}\int_0^{1/B_0} d\lambda\;(1-\lambda B_0)^{-1/2}\langle Q\rangle_\tau$$

EVALUATING WITH IT THE PRESSURE TENSOR CLOSURE VARIABLES:

$$T_{s1} \;=\; -\boldsymbol{\xi}\cdot\nabla T_{s0} \;-\; T_{s0}\;B_0\int_0^{1/B_0} d\lambda\;(1-\lambda B_0)^{-1/2}\langle Q\rangle_\tau$$

$$(p_{s\|}-p_{s\perp}) \;=\; -\,\frac{15}{2}\,n_0 T_{s0}\;B_0\int_0^{1/B_0} d\lambda\;(1-\lambda B_0)^{-1/2}\left(1-\frac{3}{2}\lambda B_0\right)\langle Q\rangle_\tau$$

ALL OTHER MOMENT RELATIONS ARE SATISFIED IDENTICALLY

ALTERNATIVELY, ONE COULD USE AS CLOSURE VARIABLES THE PERTURBED PARALLEL AND PERPENDICULAR PRESSURES:

$$p_{s\|1} \;=\; n_1 T_{s0} + n_0 T_{s1} + \frac{2}{3}(p_{s\|} - p_{s\perp}) \;=\; -\boldsymbol{\xi}\cdot\nabla(n_0 T_{s0}) \;-\; \frac{15}{2} n_0 T_{s0} B_0 \int_0^{1/B_0} d\lambda\; (1-\lambda B_0)^{1/2} \langle Q\rangle_\tau$$

$$p_{s\perp 1} \;=\; n_1 T_{s0} + n_0 T_{s1} - \frac{1}{3}(p_{s\|} - p_{s\perp}) \;=\; -\boldsymbol{\xi}\cdot\nabla(n_0 T_{s0}) \;-\; \frac{15}{4} n_0 T_{s0} B_0^2 \int_0^{1/B_0} d\lambda\; (1-\lambda B_0)^{-1/2} \lambda \langle Q\rangle_\tau$$

IN AGREEMENT WITH THE ROSENBLUTH-ROSTOKER RESULT (THAT WAS DERIVED ASSUMING $\xi_\| = 0$ AND NOT ENFORCING QUASINEUTRALITY)

ELECTRIC FIELD

THE ELECTRIC FIELD WAS ELIMINATED FROM THE SYSTEM AND THE ANALYSIS WAS CARRIED OUT WITHOUT ANY REFERENCE TO IT. AFTER THE SOLUTION HAS BEEN OBTAINED, THE ELECTRIC FIELD CAN BE INFERRED.

- **FROM FARADAY'S LAW:**

$$\mathbf{E} \;=\; i\omega\mathbf{A}_1 \;-\; \nabla\phi$$

- **FROM THE PARALLEL MOMENTUM EQUATION OF EACH SPECIES:**

Equilibrium: $\mathbf{b}_0 \cdot \nabla\phi_0 \;=\; 0$

Linearized at $\omega = 0$: $\hat{F}_{s\|} \;-\; e_s n_0 E_{\|} \;=\; 0$ **or** $\hat{G}_s \;+\; e_s n_0(\phi_1 + \boldsymbol{\xi}\cdot\nabla\phi_0) \;=\; 0$

- **FOR THE $\omega = 0,\ \hat{G}_s = 0$ SOLUTION:**

$$\phi_1 \;=\; -\boldsymbol{\xi}\cdot\nabla\phi_0 \qquad \textbf{or} \qquad E_{\|} \;=\; \mathbf{b}_0\cdot\nabla\phi_1 + \mathbf{b}_1\cdot\nabla\phi_0 \;=\; 0$$

CONCLUSIONS

- THE ZERO-LARMOR-RADIUS, COLLISIONLESS LIMIT OF A DRIFT-KINETIC FORMULATION BASED ON THE MACROSCOPIC FLOW REFERENCE FRAME, THAT INCORPORATES NATURALLY THE QUASINEUTRALITY CONDITION, YIELDS THE ZERO-FREQUENCY PRESSURE TENSOR CLOSURE RELATIONS OF ROSENBLUTH AND ROSTOKER'S.

- AT ZERO-FREQUENCY, THIS SOLUTION SATISFIES QUASINEUTRALITY AND PARALLEL FORCE BALANCE WITH A VANISHING PARALLEL ELECTRIC FIELD AND A NON-ZERO PARALLEL FLUID DISPLACEMENT (DETERMINED FROM A COMPRESSIBILITY CONDITION) THAT THE PRESSURE TENSOR CLOSURES DO NOT DEPEND ON.