Fluid And Transport Modeling Of Tokamak Plasmas

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Questions To Be Addressed:

1) Lessons learned from modeling RMP effects on pedestals?

2) What plasma models are applicable for ITER parameters?

3) Next steps for extended MHD modeling of tokamak plasmas?

Outline:

- **Modeling of small 3-D fields in tokamak plasmas (e.g., RMPs)**
- **Key plasma length and time scales in ITER**
- **Self-consistent modeling of plasmas on a hierarchy of time scales:**
  plasma kinetic equation with Fokker-Planck collision operator and sources, fluid moment equations when $t > 1/\nu$, Chapman-Enskog kinetic equation, extended MHD (ideal MHD for $t < 1/\nu$, dissipative MHD for $t > 1/\nu$), and comprehensive plasma transport equations.
- **Next steps for extended MHD $\Longrightarrow$ GUTS**

Major Themes Of My CEMRACS 2014 Lectures[1]

- **The fundamental 6-D ($\vec{x}, \vec{v}$) plasma kinetic equation (PKE) is mainly a first order hyperbolic (Vlasov) equation with a small parabolic (diffusive in $\vec{v}$) operator due to Coulomb collisions that is critical for temporal irreversibility and entropy production in kinetic, extended MHD and transport equations.**

- **Plasma kinetics is fundamental, but for time scales longer than species collision times (i.e., $t > 1/\nu_s$) 3-D ($\vec{x}$) fluid equations (extended MHD and transport) are feasible, appropriate, useful and needed. They are exact to extent that the relevant Chapman-Enskog kinetic equation (CEKE) can be solved for the kinetic distortion F_s which yields the needed collisional and closure moments.**

- **At present a GRAND UNIFIED TOKAMAK SIMULATION (GUTS)? — for developing "predictive capability" for ITER — seems to require us to:**

  use small gyroradius expansion to order various tokamak physics effects, especially those in the radial, parallel, and toroidal components of the species force balance equation,

  use extended MHD to check macrostability, obtain $\vec{B}$ field structure including plasma responses to 3-D fields, reconnecting regions and stochastic fields near separatrix X points,

  determine microturbulence by solving the CEKE in this "distorted" $\vec{B}$ field geometry and produce the collision- and microturbulence-induced closures and radial transport fluxes,

  solve resultant tokamak plasma transport equations simultaneously for n_e, Ω_t (E_ρ), p_s, and

  then iterate these extended MHD, CEKE, closures and transport steps for self-consistency.

[1] J.D. Callen, CEMRACS 2014 "Fluid and transport modeling of plasmas" lectures available via `http://homepages.cae.wisc.edu/~callen/plasmas`.

Plasma Response Reduces RMP-Fields At Rational Surfaces

- RMP-induced m/n fields from M3D-C1

  are reduced from vacuum values on m/n rational surfaces,

  by flow-screening factor $f_{\rm scr} \equiv \delta B^{\rm pl}/\delta B^{\rm vac}$,

  but grow $\sim$ linearly away from them.

- Parameters of the highlighted 11/3 RMP field are

  $f_{\rm scr} \sim 0.1$,

  $L_{\delta B} \sim 0.02\,a$
  ~ 1.6 cm.

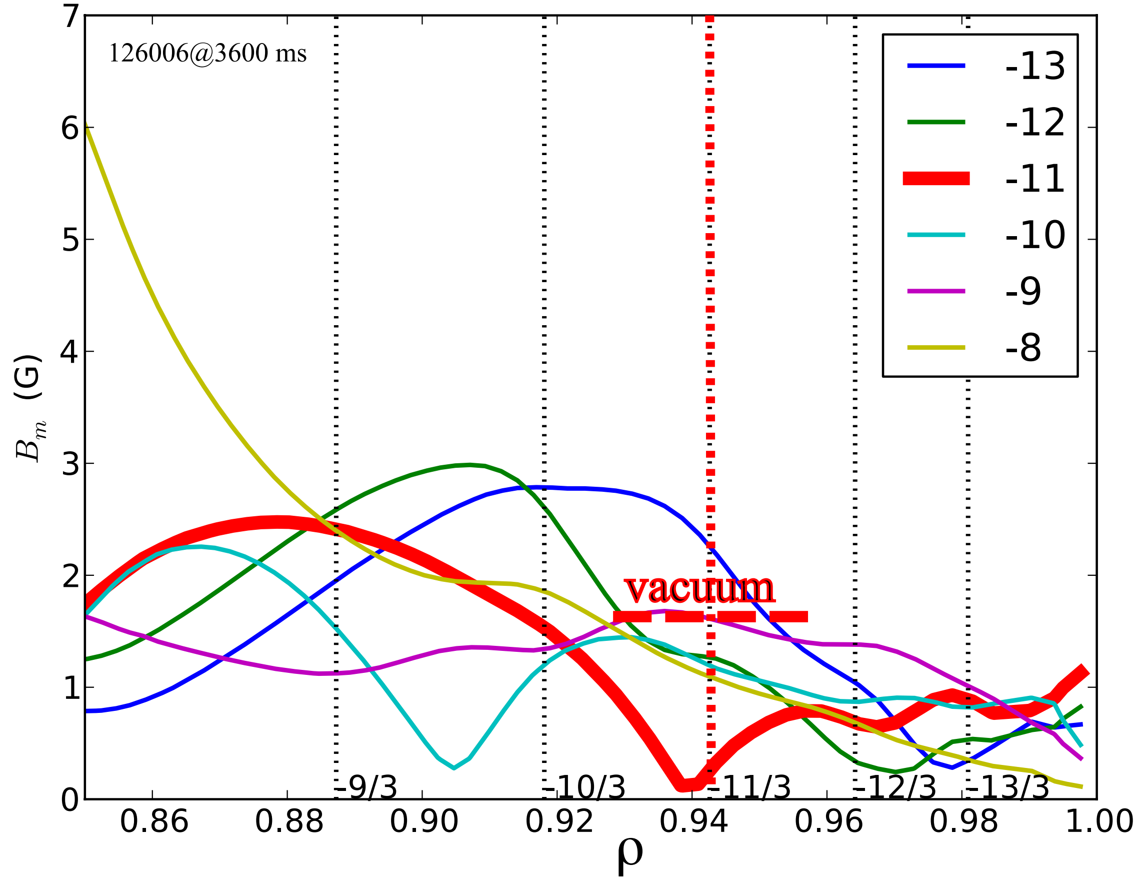

Figure 1: **Flow-screened RMP-induced $\langle \hat{B}^{\rm pl}_{\rho\, m/n} \rangle$. Courtesy of N.M. Ferraro, O. Meneghini & S.P. Smith 2012.**

Flutter Model χ_e and T_e Profiles Using M3D-C1 RMP Fields Matches 5.2 kA 126440 Data Reasonably Well[2]

- **"Diffusivity hill" at pedestal top reduces $|\vec{\nabla} T_e|$, limits pedestal expansion $\Longrightarrow$?stabilizes P-B instabilities, suppresses ELMs?**
- **Island model predicts too much transport and too flat T_e profile.**

[2]P.T. Raum, S.P. Smith, J.D. Callen, N.M. Ferraro and O. Meneghini, "Modeling of Magnetic Flutter-Induced Transport In DIII-D," GA-A27559, July 2013. S.P. Smith, invited talk NI2.05 on "Magnetic Flutter Plasma Transport Induced by 3D Fields in DIII-D," APS-DPP mtg., Nov. 11-15, 2013.

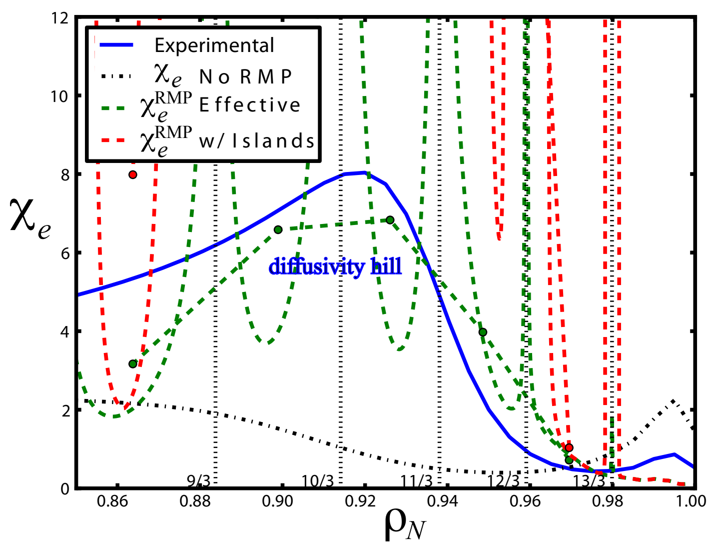

Figure 2: **Electron thermal diffusivity profiles for flutter and island models in DIII-D RMP discharge 126440.**

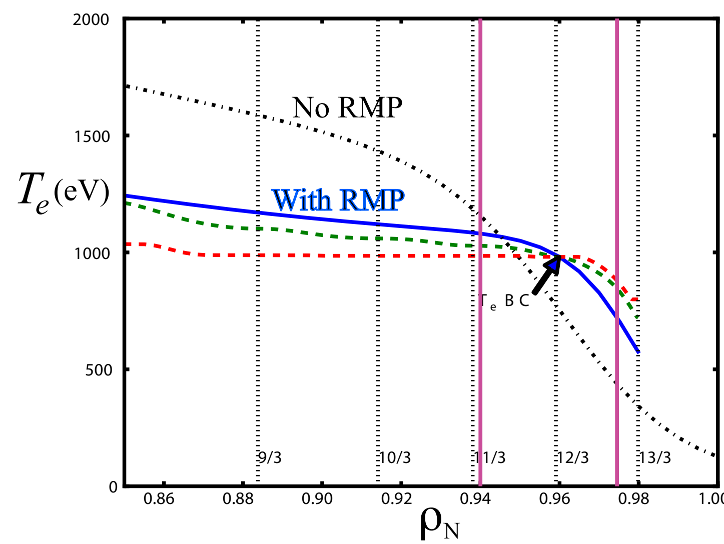

Figure 3: **Corresponding T_e profiles for flutter and island models in the edge of DIII-D RMP discharge 126440.**

RMP Flutter Model Validation Studies Status, Future

- **Present status of flutter transport model validation studies:**
  it matches power balance (ONETWO) $\chi_e^{\rm RMP}$ reasonably well for 2 cases;[2]
  need more cases, with low $\beta_e^{\rm ped}$ where linear M3D-C1 might be applicable?
- **Next level RMP-induced flutter predictions[3] need to be tested:**
  increases in electric field E_ρ in edge region and at the pedestal top,
  increase in density diffusivity ($\sim \Delta E_\rho$?) at pedestal top, which may be critical for ELM suppression because density growth there precipitates ELM;[4]
  near-separatrix X-point flutter transport model that is being developed.[5]
- **Critical elements of flutter plasma transport modeling are:**
  toroidal rotation profile $\Omega_{\rm t}$ — for flow-screening plasma response of RMPs,
  $\vec{B}$ field induced in plasma by RMPs — from M3D-C1 extended MHD modeling,
  analytic model of RMP-induced radial plasma transport – for prediction, and
  ONETWO modeling using diagnostic data — for experimental transport.
- **This validation study needs integrated, self-consistent modeling that couples extended MHD for $\vec{B}$ including plasma response to $\Omega_{\rm t}$, kinetic determination of flutter transport, and transport modeling of n_e, p_s AND $\Omega_{\rm t}$.**

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[3] J.D. Callen, C.C. Hegna and A.J. Cole, "Magnetic-flutter-induced pedestal plasma transport," Nucl. Fusion **53**, 113015 (2013).
[4] R.J. Groebner, T.H. Osborne, A.W. Leonard and M.E. Fenstermacher, "Temporal evolution of H-mode pedestal in DIII-D," NF **49** 045013 (2009).
[5] J.D. Callen and C.C. Hegna, "RMP-Induced Plasma Transport Near X Point," poster GP8.23, New Orleans APS-DPP Meeting, October 27–31, 2014.

MODELING SUMMARY: Status, Issues And Research Topics

- ***Modeling of Tokamaks:*** **There are many effects in plasma transport equations:**

  transients, collision- & microturbulence-induced transport, sources & sinks, small 3-D fields.

- ***3-D Field Effects*****: Fundamental physics of the effects of 3-D magnetic field perturbations on toroidal plasmas has "come of age" over the past decade:**

  transport-time-scale equation for $\Omega_{\rm t}$ (E_ρ) evolution including 3-D effects is now available,

  NTV theory nearly validated — torque magnitude, offset frequency, peak at $\omega_E \to 0$,

  toroidal field ripple reduces $\Omega_{\rm t}$ — edge direct ion losses plus NTV effects,

  resonant $n=1$ field error effects, correction — mode locking criteria including RFA effects,

  resonant field amplification (RFA) — via "least stable" $n=1$ kink MHD plasma responses,

  progress on interpreting RMP effects on H-mode pedestals — flutter at top, ?near separatrix.

- **Combination of 3-D field effects with extended MHD needs to be developed:**

  NTMs and RWMs interact with low n external $\delta\vec{B}$ — sensitivity increases at low $\Omega_{\rm t}$, high β,

  resonant magnetic perturbations (RMPs) — stochasticity is limited by $\Omega_{\rm t}$ "flow screening,"

  3-D plasma transport effects — directly on $\Omega_{\rm t}$, but mostly indirectly on n_e, T transport.

- **For a comprehensive approach for "whole device" transport modeling**

  we need to develop and begin using a self-consistent theoretical framework for combining extended MHD, consistent KE, closures and plasma transport equations for tokamaks.

Characteristic Length And Time Scales In Plasmas Span Many Orders of Magnitude (ITER and ICF)

- **Projected parameters for ITER where $B_t = 5.6$ T, $T_e \sim T_i \sim 10$ keV, $n_e \sim 10^{20}$ m^{-3}, and major/mid-plane minor radius $\simeq$ 6 m/2 m are:**

Length Scales

minimum impact distance	b_{min}^{qm}	10^{-12} m
mean particle spacing	$n_e^{-1/3}$	2×10^{-7} m
Debye shielding length	λ_{De}	7×10^{-5} m
deuteron gyroradius	ϱ_D	3×10^{-3} m
average minor radius	$\bar{a}$	3 m
collision length	λ	1.2×10^{4} m

Time Scales

plasma period	$1/\omega_p$	2×10^{-12} s
deuteron gyroperiod	$1/\omega_{cD}$	3×10^{-9} s
Alfvén period	$\bar{a}/c_A$	5×10^{-7} s
electron collision time	$1/\nu_e$	2×10^{-4} s
deuteron collision time	$1/\nu_D$	3×10^{-2} s
energy confinement time	τ_E	6 s
fusion collision time	$1/\nu_{fus}$	200 s

Dimensionless Parameters	formula	ICF	ITER
number of electrons in a Debye cube	$n_e\lambda_{De}^3$	3.5×10^3	4×10^7
electron collision to plasma frequency	ν_e/ω_p	4×10^{-5}	10^{-8}
ϱ_*: deuteron gyroradius to average minor radius	$\varrho_D/\bar{a}$		10^{-3}
collision length to relevant length	λ/L	$\lambda/\bar{a} \sim 0.4$	$\lambda/2\pi R = 320$

Characteristic Length And Time Scales In Plasmas Span Many Orders of Magnitude (DIII-D and ITER)

- **Typical parameters for present magnetic fusion experiments such as DIII-D where $B_{\rm t} = 2$ T, $T_e \sim T_i \sim 3$ keV, $n_e \sim 3\times 10^{19}$ m^{-3}, $Z_{\rm eff} \simeq 2$ and major/mid-plane minor radius $\simeq 1.7$ m/0.6 m are:**

Length Scales

minimum impact distance	$b_{\rm min}^{\rm qm}$	2×10^{-12} m
mean particle spacing	$n_e^{-1/3}$	3×10^{-7} m
Debye shielding length	λ_{De}	7×10^{-5} m
deuteron gyroradius	ϱ_D	5×10^{-3} m
average minor radius	$\bar{a}$	0.8 m
collision length	λ	1.8×10^{3} m

Time Scales

plasma period	$1/\omega_p$	3×10^{-12} s
deuteron gyroperiod	$1/\omega_{cD}$	10^{-8} s
Alfvén period	$\bar{a}/c_{\rm A}$	10^{-7} s
electron collision time	$1/\nu_e$	5×10^{-5} s
deuteron collision time	$1/\nu_D$	3×10^{-3} s
energy confinement time	τ_E	10^{-1} s

Dimensionless Parameters	formula	DIII-D	ITER
number of electrons in a Debye cube	$n_e\lambda_{De}^3$	10^7	4×10^7
electron collision to plasma frequency	ν_e/ω_p	6×10^{-8}	10^{-8}
deuteron gyroradius to average minor radius	$\varrho_* \equiv \varrho_D/\bar{a}$	6×10^{-3}	10^{-3}
collision length to toroidal circumference	$\lambda/2\pi R$	170	320

Modeling Tokamaks Employs Some Key Approximations

- The fundamental approximation used in modeling tokamak plasmas is that the gyroradii ($\rho_i \equiv v_{Ti}/\omega_{ci}$) of the charged ions in the magnetic field $\vec{B}$ are small compared to macroscopic gradient scale lengths $L \sim 1/|\vec{\nabla} \ln(B, n, T)|$ (i.e., $\varrho_* \equiv \varrho_i/L \ll 1$). This facilitates analytic analyses on hierarchy of ever longer time scales:

  "collisionless" parallel (to $\vec{B}$) free streaming, sound and Alfvén waves ($\sim \mu$s),

  particle drifts across magnetic field lines, fluid descriptions, collisional effects on species flows, magnetic reconnection etc. ($\gtrsim$ ms), and finally

  transport of plasma across magnetic field lines and flux surfaces due to collisions, radially localized microturbulence and 3-D field effects ($\gtrsim$ s).

- Modeling of ITER plasmas involves a hierarchy of time scales:

  "collisionless," $t \ll 1/\nu_e \sim 0.2$ ms — Landau damping, ion gyromotion, Alfvén and sound waves, ideal MHD and kinetic drift-wave instabilities,

  collisional — Ohm's law ($> 1/\nu_e \sim 0.2$ ms), damping of "poloidal" flows ($> 1/\nu_i \sim 34$ ms), fluid equations, resistive/neoclassical MHD instabilities,

  transport $\tau_E \sim a^2/\chi_\perp \sim 6$ s — "radial" (cross-field) plasma transport due to collisions and microturbulence, 3-D field effects.

Microscopic Approach Yields Plasma Kinetic Equation

- In Klimontovich approach microscopic distribution f^μ is sum of N_s delta functions along particle trajectories in 6-D phase space:

$$f^\mu(\vec{x},\vec{v},t) = \sum_i^{N_s} \delta[\vec{x}-\vec{x}_i(t)]\,\delta[\vec{v}-\vec{v}_i(t)] \Longrightarrow \frac{df^\mu}{dt} \equiv \frac{\partial f^\mu}{\partial t} + \vec{v}\cdot\frac{\partial f^\mu}{\partial \vec{x}} + \frac{\vec{F}_s^\mu}{m_s}\cdot\frac{\partial f^\mu}{\partial \vec{v}} = 0,$$

$$\frac{d\vec{x}_i}{dt} = \vec{v}_i,\;\; m_i\frac{d\vec{v}_i}{dt} = \vec{F}_s^\mu \equiv q_i[\vec{E}^\mu + \vec{v}_i\times\vec{B}^\mu], \quad \vec{\nabla}\cdot\vec{E}^\mu = \frac{\rho_q^\mu}{\epsilon_0}, \quad \vec{\nabla}\times\vec{B}^\mu = \mu_0\vec{J}^\mu.$$

- Averaging $df^\mu/dt = 0$ over small volume in 6-D phase space yields plasma kinetic equation (PKE) for average distribution $f_s \equiv \langle f^\mu\rangle$:

$$\underbrace{\frac{\partial f_s}{\partial t} + \vec{v}\cdot\frac{\partial f_s}{\partial \vec{x}} + \frac{\vec{F}_s}{m_s}\cdot\frac{\partial f_s}{\partial \vec{v}}}_{\text{macro, } |\vec{x}|>\lambda_D} = \underbrace{-\left\langle \frac{q_s}{m_s}\vec{E}^\mu\cdot\frac{\partial f^\mu}{\partial \vec{v}}\right\rangle}_{\text{micro, } |\vec{x}|<\lambda_D} \;\; \text{in which } \vec{F}_s \equiv q_s(\vec{E}+\vec{v}\times\vec{B});$$

LHS characteristic curves are particle trajectory equations $\dfrac{d\vec{x}}{dt} = \vec{v},\ \dfrac{d\vec{v}}{dt} = \dfrac{\vec{F}_s}{m_s}$;

RHS represents Coulomb collision effects since $-\left\langle \dfrac{q_s}{m_s}\vec{E}^\mu\cdot\dfrac{\partial f^\mu}{\partial \vec{v}}\right\rangle \Longrightarrow \mathcal{C}\{f_s\}$.

- Thus, PKE is $\boxed{\dfrac{df_s(\vec{x},\vec{v},t)}{dt} \equiv \dfrac{\partial f_s}{\partial t} + \vec{v}\cdot\dfrac{\partial f_s}{\partial \vec{x}} + \dfrac{\vec{F}_s}{m_s}\cdot\dfrac{\partial f_s}{\partial \vec{v}} = \mathcal{C}\{f_s\} + \mathcal{S}\{f_s\}.}$

Fokker-Planck (F-P) Collision Operator Is Developed From Coulomb Collision F-P Coefficients

- Fokker-Planck collision operator on particle distribution f is

$$\mathcal{C}\{f\} = -\frac{\partial}{\partial \vec{v}} \cdot \vec{J}_{\vec{v}} = -\frac{\partial}{\partial \vec{v}} \cdot \left(\frac{\langle \Delta \vec{v} \rangle}{\Delta t} f \right) + \frac{1}{2} \frac{\partial^2}{\partial \vec{v}\, \partial \vec{v}} : \left(\frac{\langle \Delta \vec{v}\, \Delta \vec{v} \rangle}{\Delta t} f \right), \quad \text{in which}$$

  $\langle \Delta \vec{v} \rangle / \Delta t$ and $\langle \Delta \vec{v}\, \Delta \vec{v} \rangle / \Delta t$ are Fokker-Planck drag and diffusion coefficients

- Fokker-Planck collision operator is small momentum transfer limit ($m|\langle \Delta \vec{v} \rangle| \ll m|\vec{v}|$) of the Boltzmann collision operator.

- Simplest, Lorentz (subscript L) collision operator $\mathcal{C}_{\rm L}$ in spherical velocity-space coordinates of speed v, "pitch-angle" ϑ, phase φ is

$$\mathcal{C}_{\rm L}\{f\} = \frac{\nu(v)}{2} \left[\frac{1}{\sin \vartheta} \frac{\partial}{\partial \vartheta} \sin \vartheta \frac{\partial f}{\partial \vartheta} + \frac{1}{\sin^2 \vartheta} \frac{\partial^2 f}{\partial \varphi^2} \right], \quad \vartheta, \varphi \text{ diffusion at constant } v.$$

- Fokker-Planck Coulomb collision operator is

  "local" in $\vec{x}$ since it results from collisions within a Debye length λ_D,
  a second order differential operator in velocity space $\vec{v}$.

Properties Of Collisionless Vlasov And Full PKE With Coulomb Collision Operator Are Different

- **When collisions are neglected, PKE is a first order partial differential equation in the 6-D phase space called the Vlasov equation:**

  $\frac{\partial f_s}{\partial t} + \vec{v}\cdot\frac{\partial f_s}{\partial \vec{x}} + \frac{\vec{F}_s}{m_s}\cdot\frac{\partial f_s}{\partial \vec{v}} = 0$, **which has characteristic curves that are (without chaos) deterministic particle trajectories via** $\frac{d\vec{x}\,'}{dt'} = \vec{v}', \quad m_s\frac{d\vec{v}\,'}{dt'} = \vec{F}_s$, **whose time** t **solutions are** $\vec{x}\,'(\vec{x}_0, \vec{v}_0, t - t_0), \ \vec{v}\,'(\vec{x}_0, \vec{v}_0, t - t_0)$ **for I.C.** $\vec{x}_0, \vec{v}_0$ **at** t_0, **which yields Vlasov solution mapping** $f_s \to f_s[\vec{x}\,', \vec{v}\,', t']$ **in which** $t' \equiv t - t_0$

  that does no smoothing of f_s **and produces no entropy in stable plasmas.**

- **Adding second derivative F-P collision operator changes things:**

  PKE becomes a (diffusive) parabolic second order differential equation;

  particle trajectories become slightly probabilistic, not fully deterministic;

  PKE becomes singular differential equation where $|df_s/dt| \lesssim |\mathcal{C}\{f_s\}| \sim \nu_{\rm eff} f_s$

  at $\delta\vec{v}$ **singular layers where** $\nu_{\rm eff} \sim \nu/(|\delta\vec{v}|/v)^2 \gg \nu$ **increases plasma entropy.**

Perturbed PKE Is Often Basic Starting Point

- In a plasma with small inhomogeneities in f_s, $\vec{E}_0$, $\vec{B}_0$ on λ, ϱ scale lengths, the equilibrium PKE is $\mathcal{C}\{f_0\} = 0$, whose solution is

$$f_0 = f_{\rm iMs} = \frac{n_s\, e^{-v^2/v_{Ts}^2}}{\pi^{3/2} v_{Ts}^3}, \quad \text{the isotropic Maxwellian with } v_{Ts} \equiv \sqrt{2T_s/m_s}\,.$$

- When small electromagnetic field perturbations are introduced, $f_s \to f_{\rm iMs} + \tilde{f}_s$ and $\tilde{f}_s(\vec{x}, \vec{v}, t)$ is governed by the perturbed PKE:

$\boxed{\mathcal{L}_\nu\{\tilde{f}_s\} = S,}$ where $\mathcal{L}_\nu$ is the linear second order partial differential operator

$$\mathcal{L}_\nu\{\tilde{f}_s\} \equiv \frac{\partial \tilde{f}_s}{\partial t} + \vec{v}\cdot\frac{\partial \tilde{f}_s}{\partial \vec{x}} + \frac{q_s}{m_s}(\vec{E}_0 + \vec{v}\times\vec{B}_0)\cdot\frac{\partial \tilde{f}_s}{\partial \vec{v}} - \mathcal{C}\{\tilde{f}_s\}, \quad \text{and "source" is}$$

$$S = -\frac{q_s}{m_s}(\tilde{\vec{E}} + \vec{v}\times\tilde{\vec{B}})\cdot\frac{\partial f_{\rm iMs}}{\partial \vec{v}} - \frac{q_s}{m_s}(\tilde{\vec{E}} + \vec{v}\times\tilde{\vec{B}})\cdot\frac{\partial \tilde{f}_s}{\partial \vec{v}}, \quad \text{response to } \tilde{\vec{E}}, \tilde{\vec{B}}.$$

- Different solution procedures are currently used in limiting cases:

collisionless — integrate along trajectories $\Longrightarrow \tilde{f}_s(t) = \tilde{f}_s(t'=0) + \int_0^t dt'\, S(t')$,

collisional — solve collision-induced dissipative singular layers in steady state.

Perturbed PKE Can Be Solved With Green Function

- Green function solution of perturbed PKE $\mathcal{L}_\nu\{\tilde{f}_s\} = S$ is

  $\tilde{f}_s(\vec{x},\vec{v},t) = \int_0^t dt_0 \int d^3x_0 \int d^3v_0 \; G_\nu(\vec{x},\vec{v},t|\vec{x}_0,\vec{v}_0,t_0)\; S(\vec{x}_0,\vec{v}_0,t_0)$.

- Defining equation for Green function in the 6-D phase space is

  $\mathcal{L}_\nu\{G_\nu\} = \delta[\vec{x}-\vec{x}_0]\,\delta[\vec{v}-\vec{v}_0]\,\delta[t-t_0] \;\leftarrow$ Dirac delta functions.

- Using short time isotropic operator $\mathcal{C}_{\rm i}\{\tilde{f}_s\} = (\nu/2)\nabla_v^2 \tilde{f}_s$ or long time Krook operator $\mathcal{C}_{\rm K}\{\tilde{f}_s\} = -\,\nu \tilde{f}_s$, combined Green function is

$$G_\nu^\infty = \delta[\vec{x}-\vec{x}_0-\vec{v}_0\tau]\,H(\tau)\times\begin{cases} e^{-|\vec{v}-\vec{v}_0|^2/(2\nu\tau v_0^2)}/(2\pi\nu\tau v_0^2)^{3/2}, & \nu\tau \ll 1,\\ e^{-\nu\tau}\,\delta[\vec{v}-\vec{v}_0], & \nu\tau \gg 1,\end{cases}$$

  which includes diffusive scattering of $\vec{v}$ at rate $\nu_{\rm eff} \sim \nu/(|\vec{v}-\vec{v}_0|/v_0)^2 \gg \nu$ for short times $\tau \equiv t-t_0$, or damping at a rate ν for long times.

- Two key examples of rigorous solutions of PKE are:

  $\nu\tau \ll 1$ — Coulomb collisional scattering effects on linear Landau damping,[6]

  $\nu\tau \gg 1$ — fluid moment approach for collisional magnetized plasmas.

[6] J.D. Callen, "Coulomb collision effects on linear Landau damping," Phys. Plasmas **21**, 052106 (2014)

Solution Procedure Is Different When Collisions Dominate

- **Plasma kinetic equations are usually solved in separate limits:**
  low collisionality — kinetic solutions with test particle collision operator,
  collisional — fluid moment closures obtained with full F-P collision operator.

- **Fluid moment descriptions for a given plasma species s are:**
  viable for $t \gg 1/\nu_s$ where lowest order distribution becomes a Maxwellian,
  feasible, relevant and very useful because they
  reduce the plasma description from 6-D $(\vec{x}, \vec{v}, t)$ to 3-D $(\vec{x}, t)$, *but they*
  require solution of a relevant kinetic equation to obtain needed closures; and
  are critical for long time scale of fusion plasmas where $1/\nu_{\rm fus} \gg \tau_E \gg 1/\nu_D$.

- **Obtaining fluid moment equations requires a number of steps:**
  1) take $\int d^3v\, (1, m\vec{v}, mv^2)$ moments of the PKE to obtain equations for density $n(\vec{x}, t)$, flow velocity $\vec{V}(\vec{x}, t)$ and pressure $p(\vec{x}, t) \equiv nT$ of a given species;
  2) develop a Chapman-Enskog-type kinetic equation (CEKE) for δf;
  3) expand perturbed distribution δf in a complete set of fluid moments;
  4) obtain collisional limit by taking $\int d^3v\, (\cdots)$ moments of CEKE and invert the resultant matrix operator to obtain needed collisional & fluid moments.

Species s Fluid Moment Equations Are Fundamental

- The $\int d^3v\,(1, m_s\vec{v}, m_s v^2/2)$ moments of the plasma kinetic equation (PKE) yield the species s ***fundamental fluid moment equations***:

density $$(\partial/\partial t + \vec{V}_s \cdot \vec{\nabla})\, n_s \;=\; -\, n_s \vec{\nabla}\cdot\vec{V}_s + S_{ns},$$

mom. $$m_s n_s(\partial/\partial t + \vec{V}_s \cdot \vec{\nabla})\, \vec{V}_s \;=\; n_s q_s(\vec{E} + \vec{V}_s \times \vec{B}) - \vec{\nabla} p_s - \vec{\nabla}\cdot \overleftrightarrow{\pi}_s + \vec{R}_s + \vec{S}_{\mathrm{p}s},$$

energy $$\frac{3}{2}(\partial/\partial t + \vec{V}_s \cdot \vec{\nabla})\, p_s \;=\; -\,\frac{5}{2} p_s \vec{\nabla}\cdot\vec{V}_s + p_s\, \dot{s}_{\mathrm{M}s}, \quad \text{or,}$$

entropy $$(\partial/\partial t + \vec{V}_s \cdot \vec{\nabla})\, s_{\mathrm{M}s} \;=\; \dot{s}_{\mathrm{M}s} \equiv (-\vec{\nabla}\cdot \vec{q}_s - \overleftrightarrow{\pi}_s : \vec{\nabla}\vec{V}_s + Q_s + S_{\mathcal{E}s})/p_s,$$

in which the species s isotropic-Maxwellian-based collisional entropy is

$$s_{\mathrm{M}s}(\vec{x}, t) \equiv -\frac{1}{n}\int d^3v\, f_{\mathrm{iM}s} \ln f_{\mathrm{iM}s} = \frac{3}{2}\ln\left(\frac{p_s}{n^{5/3}}\right) + C, \quad \textit{collisional entropy}.$$

- But equations are incomplete until these "closures" are specified:

stress $\overleftrightarrow{\pi}_s$, heat flux $\vec{q}_s$;   collisional friction force $\vec{R}_s$, energy exchange Q_s.

Use Chapman-Enskog Approach For Kinetic Equation

- **The Chapman-Enskog Ansatz posits that the distribution function can be decomposed into two parts — a "dynamic" time- and spatially-dependent Maxwellian $f_{\mathrm{M}s}$ plus a kinetic distortion F_s:**

$$f_s(\vec{x},\vec{v},t) = f_{\mathrm{M}s}(\vec{x},\vec{v},t) + F_s(\vec{x},\vec{v},t), \quad \text{in which dynamic Maxwellian is}$$

$$f_{\mathrm{M}s}(\vec{x},\vec{v},t) = f_{\mathrm{M}s}[n_s(\vec{x},t),\vec{V}_s(\vec{x},t),T_s(\vec{x},t),\vec{v}] = \frac{n_s(\vec{x},t)\, e^{-m_s[\vec{v}-\vec{V}_s(\vec{x},t)]^2/2T_s(\vec{x},t)}}{[2\pi\, T_s(\vec{x},t)/m_s]^{3/2}}.$$

- **Substituting this Ansatz into the plasma kinetic equation yields**

$$\boxed{\frac{dF_s}{dt} - \mathcal{C}\{f_s\} - \mathcal{S}\{f_s\} = -\frac{df_{\mathrm{M}s}}{dt},}$$ ***Chapman-Enskog kinetic equation*** **(CEKE),**

$$\begin{aligned}
\frac{df_{\mathrm{M}}}{dt} = \quad f_{\mathrm{M}}\Bigg[& -\frac{1}{p}\,\vec{v}_r\cdot\Big(nq\,[\vec{E}+\vec{V}\times\vec{B}] - \vec{\nabla}p - mn\,(\frac{\partial}{\partial t}+\vec{V}\cdot\vec{\nabla})\vec{V}\Big) && \text{forces}\\
& + \left(\frac{mv_r^2}{2T}-\frac{5}{2}\right)\frac{1}{T}\,\vec{v}_r\cdot\vec{\nabla}T && T\ \text{gradient}\\
& + \frac{m}{T}\left(\vec{v}_r\vec{v}_r - \frac{v_r^2}{3}\mathbf{I}\right):\mathbf{W},\ \ \mathbf{W}\equiv\frac{1}{2}[\vec{\nabla}\vec{V}+(\vec{\nabla}\vec{V})^{\mathrm{T}}-\frac{2}{3}\mathbf{I}\,\vec{\nabla}\cdot\vec{V}] && \text{rate of strain}\\
& + \frac{S_n}{n} + \left(\frac{mv_r^2}{2T}-\frac{3}{2}\right)\left(\frac{2\,\dot{s}_{\mathrm{M}}}{3p}-\frac{S_n}{n}\right)\Bigg]. && \text{sources}
\end{aligned}$$

Chapman-Enskog Approach Is Useful And Important

- **Chapman-Enskog kinetic equation (CEKE) on the preceding page**

  is still exact since no approximations or truncations have been utilized;

  it is just a recast plasma kinetic equation that has used the Chapman-Enskog Ansatz to obtain a kinetic equation for the small kinetic distortion F_s.

- **Since by construction $\int d^3v\,(1, \vec{v}, v_r^2)\,F_s = 0$,**

  the kinetic distortion F_s does not produce any extraneous δn, $\delta\vec{V}$ or δp terms.

  Thus, $\vec{q}_s$, $\overleftrightarrow{\pi}_s$, $\vec{R}_s$ and Q_s closure moments obtained from velocity-space moments of accurate solutions of the CEKE for F_s will be consistent with the fluid moment equations without introducing any extraneous effects.

  When a CEKE is not used (e.g., as in present gyrokinetics studies), it should be shown that $\int d^3v\,(1, \vec{v}, v_r^2)\,F_s = 0$ so there are no δn, $\delta\vec{V}$ or δp terms, which would be inconsistent with using kinetic-based results in the fluid equations.

- **In the original Chapman-Enskog approaches (with, without $\vec{B}$)**

  collisions were assumed to be dominant to solve for the kinetic distortion F_s from the CEKE to obtain closures for the Braginskii equations.

  However, in low collisionality magnetized plasmas this assumption must be modified for collisional effects on particle motion along magnetic field lines.

Collisional & Closure Moments Different $\parallel$, $\wedge$ and $\perp$ to $\vec{B}$

- **Braginskii moments and closures for an electron-ion plasma are**

$$\vec{R}_e \equiv n_e e \left(\frac{\vec{J}_\parallel}{\sigma^{\rm Sp}} + \frac{\vec{J}_\perp}{\sigma_\perp} \right) - C_{\nabla T} n_e \hat{\rm b}\,(\hat{\rm b}\cdot\vec{\nabla} T_e) - \frac{3}{2}\frac{\nu_e}{\omega_{ce}}\,\hat{\rm b}\times\vec{\nabla} T_e, \quad \vec{R}_i = -\vec{R}_e,$$

$$Q_e \equiv -Q_\Delta - \vec{R}_e\cdot\vec{J}/n_e e, \quad Q_i = Q_\Delta,$$

$$\vec{q}_e \equiv -n_e\chi_{e\parallel}\,\hat{\rm b}(\hat{\rm b}\cdot\vec{\nabla} T_e) - n_e\chi_{e\wedge}\hat{\rm b}\times\vec{\nabla} T_e - n_e\chi_{e\perp}[-\hat{\rm b}\times(\hat{\rm b}\times\vec{\nabla} T_e)]$$
$$+ C_{\nabla T}\, n_e T_e J_\parallel/n_e e + (3/2)\,(\nu_e/\omega_{ce})\,\hat{\rm b}\times\vec{J}_\perp/n_e e,$$

$$\vec{q}_i \equiv -n_i\chi_{i\parallel}\,\hat{\rm b}(\hat{\rm b}\cdot\vec{\nabla} T_i) - n_i\chi_{i\wedge}\,\hat{\rm b}\times\vec{\nabla} T_i - n_i\chi_{i\perp}\,[-\hat{\rm b}\times(\hat{\rm b}\times\vec{\nabla} T_i)],$$

plus similar electron and ion parallel, cross, perpendicular stress tensors $\overset{\leftrightarrow}{\pi}_s$.

- **Here, new coefficients are $\sigma_\perp \equiv n_e e/m_e\nu_e$ and the diffusivities are**

$$\chi_{e\parallel} \equiv C_{\chi_e}\frac{v_{Te}^2}{\nu_e}, \quad \chi_{e\wedge} \equiv \frac{5}{4}\frac{v_{Te}^2}{\omega_{ce}}, \quad \chi_{e\perp} \equiv C_{\chi_{e\perp}}\,\nu_e\,\frac{v_{Te}^2}{\omega_{ce}^2}, \qquad C_{\chi_{e\perp}} = 2.33\,(Z=1),$$

$$\chi_{i\parallel} \equiv C_{\chi_i}\frac{v_{Ti}^2}{\nu_i}, \quad \chi_{i\wedge} \equiv \frac{5}{4}\frac{v_{Ti}^2}{\omega_{ci}}, \quad \chi_{i\perp} \equiv C_{\chi_{i\perp}}\,\nu_i\,\frac{v_{Ti}^2}{\omega_{ci}^2}, \qquad C_{\chi_{i\perp}} = 1,$$

which scale as $\chi_\perp/\chi_\wedge \sim \chi_\wedge/\chi_\parallel \sim \nu/\omega_c \ll 1$ and thus $\chi_\perp/\chi_\parallel \sim (\nu/\omega_c)^2 \lll 1$.

Braginskii Collisional Plasma Transport Equations Are Not Directly Applicable To Tokamak Plasmas

- The small gyroradius ($\varrho_* \ll 1$) and strongly magnetized plasma ($\nu \ll \omega_c$) criteria are well satisfied in magnetic fusion experiments.

- However, collision lengths are longer than gradient scale lengths along $\vec{B}$, e.g., $\lambda_e/L_\| \sim 12\,000\,\mathrm{m}/12\,\mathrm{m} = 1\,000 \gg 1$, which violates the high collisionality assumption for the Braginskii equations.

- Nonetheless, the Braginskii collisional equations are often adapted and used for modeling tokamak plasmas because the

  parallel transport diffusivities $D_\| \sim \nu\lambda^2$ are so large ($\sim 10^{12}$ m^2/s in ITER) that the basic thermodynamic variables (n, T) are equilibrated along $\vec{B}$,

  and then the Braginskii equations are applied to the remaining 2-D geometry.

- But dissipative collisional effects along $\vec{B}$ are important:

  parallel viscous forces increase the electrical resistivity, produce a "bootstrap current" driven by $\vec{\nabla} P$ and damp poloidal ion flows, and

  the parallel dissipative effects are important in numerical simulations.

Extended MHD Model Includes Ideal MHD And The Dissipative Effects Of Collisional And Closure Moments

- **The extended MHD equations for a magnetized plasma are obtained by summing the fluid moment equations over species. Together with equations for electric and magnetic fields they are**

Extended MHD plasma description ($\vec{R}_e$, $\overleftrightarrow{\Pi}$, $\overleftrightarrow{\pi}_e$, $\sum_s \dot{s}_{\mathrm{M}s} \to 0$ for ideal MHD):

mass density $\quad (\partial/\partial t + \vec{V}\cdot\vec{\nabla})\,\rho_m \;=\; -\,\rho_m \vec{\nabla}\cdot\vec{V},$

charge continuity $\quad \vec{\nabla}\cdot\vec{J} \;=\; 0,$

momentum $\quad \rho_m(\partial/\partial t + \vec{V}\cdot\vec{\nabla})\vec{V} \;=\; \vec{J}\times\vec{B} - \vec{\nabla}P - \vec{\nabla}\cdot\overleftrightarrow{\Pi},$

Ohm's law $\quad \vec{E} \;=\; -\,\vec{V}\times\vec{B} + \vec{R}_e/n_e e + (\vec{J}\times\vec{B} - \vec{\nabla}p_e - \vec{\nabla}\cdot\overleftrightarrow{\pi}_e)/n_e e,$

equation of state $\quad (\partial/\partial t + \vec{V}\cdot\vec{\nabla})\ln(P/\rho_m^{5/3}) \;=\; \sum_s \dot{s}_{\mathrm{M}s}.$

Maxwell Equations for extended MHD (no Gauss' law, $\vec{E}$ from Ohm's law):

Faraday's law $\quad \partial\vec{B}/\partial t \;=\; -\,\vec{\nabla}\times\vec{E},$

no magnetic monopoles $\quad \vec{\nabla}\cdot\vec{B} \;=\; 0,$

nonrelativistic Ampere's law $\quad \vec{J} \;=\; \vec{\nabla}\times\vec{B}/\mu_0.$

Ideal MHD Provides Tokamak Plasma Constraints

- **Stable compressional Alfvén waves enforce equilibrium radial force balance on very short time scales ($\bar{a}/c_{\rm A} \sim 10^{-7} - 10^{-6}$ s) and yield**

  ideal MHD equilibrium equations: $\vec{J}\times\vec{B} = \vec{\nabla}P, \quad \vec{J} = \vec{\nabla}\times\vec{B}/\mu_0, \quad \vec{\nabla}\cdot\vec{B} = 0.$

- **If the shear Alfvén or sound waves become unstable,**

  they grow on very fast time scales ($R/c_{\rm A} \sim 10^{-5} - 10^{-6}$ s), and

  usually lead to virulent global instabilities and hence plasma "disruptions."

- **Stability criteria for avoiding these ideal MHD instabilities provide limits on parameter regimes in which tokamaks operate:**

  sound wave stability, $\beta \equiv \dfrac{P}{B^2/2\mu_0} \lesssim \dfrac{a}{Rq} \sim 0.1$ (analogous to Rayleigh-Taylor),

  shear Alfvén stability, $q \simeq \dfrac{aB_{\rm t}}{RB_{\rm p}} \geq 1$ (Kruskal-Shafranov criterion, kink modes).

- **Further tokamak analyses assume these ideal MHD stability criteria are satisfied so these virulent instabilities are circumvented.**

Collisional Dissipative Effects Are Important For $t > 1/\nu$

- **Resistive effects reconnect (or tear) magnetic field lines in thin singular layers around low order rational surfaces where $q(\psi_{\rm p}) = m/n$ on which the helical magnetic field lines close on themselves.**

- **This reconnection process violates the frozen flux theorem of ideal MHD and can allow slowly growing, radially isolated tearing-type (classical $\vec{\nabla} J_{\rm t}$-driven and neoclassical $\vec{\nabla} P$-driven) instabilities.**

- **These modes can cause magnetic island topologies to develop in the plasma which sometimes continue to grow and violently disrupt plasma confinement, i.e., lead to a plasma "disruption."**

- **When such deleterious modes are controlled, the equilibrium extended MHD equations yield prescriptions for the first order (in the small gyroradius expansion) equilibrium and perturbed flows and currents (and hence Ohm's law) on magnetic flux surfaces.**

- **Key closure for low collisionality tokamaks is viscous stress $\overleftrightarrow{\pi}_s$.**

Consider Collisional Stresses In A Magnetized Plasma

- Collisional Braginskii viscous stresses are defined relative to $\vec{B}$ direction:

$$\overleftrightarrow{\pi} = \overleftrightarrow{\pi}_\parallel + \overleftrightarrow{\pi}_\wedge + \overleftrightarrow{\pi}_\perp, \quad \text{parallel, cross (gyroviscous) and perpendicular stresses.}$$

- For strongly magnetized ($\omega_c \gg 1/\nu$) toroidal plasmas of fusion interest a small gyroradius expansion is usually appropriate: $\varrho_* \equiv \varrho/a \ll 1$.

- For arbitrary $\vec{V}$, the characteristic scalings of the parallel, cross and perpendicular stresses can be written schematically for $Rq \gtrsim \lambda \gtrsim a$ as

$$\overleftrightarrow{\pi}_\parallel \sim \nu\lambda^2\, \vec{\nabla}_\parallel \vec{V}, \quad \overleftrightarrow{\pi}_\wedge \sim \nu\varrho\,\lambda\, \vec{B}\times\vec{\nabla}\vec{V}/B \sim \varrho_*\, \overleftrightarrow{\pi}_\parallel, \quad \overleftrightarrow{\pi}_\perp \sim \nu\varrho^2\, \vec{\nabla}_\perp \vec{V} \sim \varrho_*^2\, \overleftrightarrow{\pi}_\parallel.$$

- Thus, the parallel viscous stress $\overleftrightarrow{\pi}_\parallel$ is dominant in small gyroradius, magnetized toroidal plasmas. We concentrate on it. The $\overleftrightarrow{\pi}_\wedge$ and $\overleftrightarrow{\pi}_\perp$ are changed less.

- The parallel viscous stresses for electrons and ions were originally written by Braginskii for each species in the form (z here is coordinate along $\vec{B}$, $Z_i = 1$)

$$\overleftrightarrow{\pi}_\parallel = -\,\eta_0\, W_{zz}\, \hat{\vec{e}}_z \hat{\vec{e}}_z, \quad W_{zz} \equiv 2\,\frac{\partial V_z}{\partial z} - \frac{2}{3}\,(\vec{\nabla}\cdot\vec{V}), \quad \eta_0^i = 0.48\, n_i m_i \frac{v_{Ti}^2}{\nu_i}, \quad \eta_0^e = 0.37\, n_e m_e \frac{v_{Te}^2}{\nu_e}.$$

- But this is not valid for low collisionality tokamak plasmas where $\lambda_e \equiv v_{Te}/\nu_e \gg L_\parallel$.

Low ν Flow Damping Can Be Included In $\|$ Viscous Stress

- A multi-collisionality parallel stress that yields the Braginskii and flux-surface-averaged (FSA) neoclassical closures has been proposed[4]

$$\pi_\| = \pi_\|^{\rm f} + \pi_\|^{\rm r},$$

$$\text{fast,} \quad \pi_\|^{\rm f} \equiv -\,3\,\eta_{00}\left(\frac{\vec{B}\cdot\vec{\nabla}\times(\vec{V}\times\vec{B})}{B^2} + \frac{2}{3}\vec{\nabla}\cdot\vec{V} - \frac{(\vec{B}\cdot\vec{V})(\vec{\nabla}\cdot\vec{B})}{B^2}\right),$$

$$\text{residual,} \quad \pi_\|^{\rm r} \equiv -\,mn\,\mu\,\langle B_0^2\rangle \frac{\hat{\rm b}\cdot\vec{\nabla}B_0}{\langle(\hat{\rm b}\cdot\vec{\nabla}B_0)^2\rangle}\,(U_\theta - U_\theta^0)\,, \quad \hat{\rm b} \equiv \vec{B}_0/B_0.$$

- Neoclassical poloidal flow damping frequency μ is of the form

$$\mu \simeq \frac{1.46\sqrt{\epsilon}\,\nu}{(1+\nu_*^{1/2}+\nu_*)(1+\epsilon^{3/2}\nu_*)}, \quad \text{for collisionality parameter } \nu_* \equiv \frac{\nu}{\epsilon^{3/2}\omega_t} = \frac{R_0 q}{\epsilon^{3/2}\lambda}.$$

$\Longrightarrow$ banana regime for $\nu_* \ll 1$, plateau for $1 \ll \nu_* \ll \epsilon^{-3/2}$, Braginskii for $\nu_* \gg \epsilon^{-3/2}$.

- The "offset" poloidal flow velocity for ions is given by

$$U_{i\theta}^0(\psi_p) \simeq k_i\,\frac{I(\psi_p)}{q_i\langle B_0^2\rangle}\,\frac{dT_0}{d\psi_p}, \quad \text{in which } k_i = \frac{\mu_{i01}/\mu_{i00}}{1+(\mu_{i11}-\mu_{i01}^2/\mu_{i00})/\nu_{i11}} \sim \frac{1.17}{1+0.67\sqrt{\epsilon}}.$$

Tokamak Extended MHD Model Is Obtained From Fluid Equations

- Plasma density and charge continuity equations result from sums over species:

$$\sum_s n_s m_s \quad \Longrightarrow \quad \boxed{\left.\frac{\partial \rho_m}{\partial t}\right|_{\vec{x}} + \vec{\nabla}\cdot\rho_m\vec{V} = \sum_s m_s S_{ns},} \qquad \textstyle\sum_s n_s q_s \quad \Longrightarrow \quad \boxed{\vec{\nabla}\cdot\vec{J} = 0.}$$

- Total plasma equation of state (entropy eqn.) is unchanged from usual form (p 3).

- Plasma force balance is obtained by summing momentum equations over species:[4]

$$\boxed{\left.\frac{\partial(\rho_m\vec{V})}{\partial t}\right|_{\vec{x}} + \vec{\nabla}\cdot(\rho_m\vec{V}\vec{V}) = \vec{J}\times\vec{B} - \vec{\nabla}P - \sum_s(\vec{\nabla}\cdot\overset{\leftrightarrow}{\pi}{}^{\rm f}_{s\|} + \vec{\nabla}\cdot\overset{\leftrightarrow}{\pi}{}^{\rm r}_{s\|} + \vec{\nabla}\cdot\overset{\leftrightarrow}{\pi}_{s\wedge}) + \sum_s\vec{S}_{\rm ps}.}$$

- General Ohm's law is obtained from electron force balance equation ($\hat{\rm b} \equiv \vec{B}/B$):

$$\boxed{\begin{aligned}\vec{E} \;=\; &-\vec{V}\times\vec{B} + \frac{\vec{J}\times\vec{B} - \vec{\nabla}p_e - \vec{\nabla}\cdot\overset{\leftrightarrow}{\pi}{}^{\rm f}_{e\|} - \vec{\nabla}\cdot\overset{\leftrightarrow}{\pi}_{e\wedge} - C_{\nabla T}n_e\,\hat{\rm b}\,(\hat{\rm b}\cdot\vec{\nabla}T_e) + \vec{S}_{\rm pe}}{n_e e}\\ &+\frac{1}{\sigma_\perp}\left(\vec{J}_\perp - \frac{3}{2}\frac{n_e\vec{B}\times\vec{\nabla}T_e}{B^2}\right) + \eta^{\rm nc}_\|\left(\vec{J}_\| - \vec{J}_{\|\rm drives}\right) - \frac{m_e}{e}\left(\frac{\partial}{\partial t} + \vec{V}_e\cdot\vec{\nabla}\right)\vec{V}_e,\end{aligned}}$$

$$\vec{J}_\perp \equiv -\hat{\rm b}\times(\hat{\rm b}\times\vec{J}), \qquad \vec{J}_\| \equiv \hat{\rm b}\,(\hat{\rm b}\cdot\vec{J}).$$

- Use $\vec{J}_{\|\rm drives} \equiv \dfrac{\vec{B}}{B}\langle\vec{B}_0\cdot\vec{J}_{\rm drives}\rangle$ fom p 32, but $I\,\dfrac{dP}{d\psi_p} \to B^2\dfrac{\vec{J}_\perp\cdot\vec{\nabla}\theta}{\vec{B}\cdot\vec{\nabla}\theta}$ in $\langle\vec{B}_0\cdot\vec{J}_{\rm bs}\rangle$.

Key Properties Of Tokamak Extended MHD Model

- **Tokamak extended MHD model adds collisional effects for $t > 1/\nu_s$**

  primarily via the viscous forces due to the parallel viscous stresses $\overleftrightarrow{\pi}_{s\|}$, which

  for $t > 1/\nu_e > 0.2$ ms modifies parallel Ohm's law by increasing $\|$ resistivity and adds bootstrap current driven by the radial plasma pressure gradient,

  and adds poloidal ion flow damping at a rate $\sim \nu_i \sim 1/(34\,\text{ms})$ to the overall plasma force balance.

- **It is important to recall that the extended MHD model "owns" the current density $\vec{J}$ because:**

  in MHD models $\vec{J} = \vec{\nabla}\times\vec{B}/\mu_0$ with the magnetic field being determined from Faraday's law $\partial\vec{B}/\partial t = -\vec{\nabla}\times\vec{E}$ in which the electric field is determined from the extended MHD Ohm's law, and

  proper solutions of the Chapman-Enskog kinetic equation yield kinetic distortions F_s that have no momentum moments (i.e., $\int d^3v\, m_s\vec{v}\, F_s = \vec{0}$) and hence produce no contributions to $\vec{J}$.

- **Next (final) step will be to obtain net radial transport equations for a tokamak plasma on the long transport time scale $t \gg 1/\nu_s$.**

Next Step: Develop Modern Transport Equations For Tokamaks

- **Tokamak plasma transport equations for modeling codes (e.g., ONETWO, TRANSP) were developed in late 70's from n, T fluid moment equations with collisional Braginskii closures; and then *ad hoc* terms are added for**

  neoclassical effects on ∥ Ohm's law (trapped particle effects on $\eta_\parallel$ and bootstrap current),

  fluctuation-induced transport induced by micro-turbulence,

  heating & current-drive and flow sources & sinks,

  effects of small 3-D magnetic field asymmetries, etc.

- **But tokamak plasmas are not in a collisional regime! ($\lambda \gg Rq$) — and we should develop transport equations that naturally include all these effects.**

- **Here, we will develop[7,8] self-consistent fluid-moment-based radial transport equations that include all these effects for nearly axisymmetric single-ion-species tokamak plasmas using neoclassical-based closures.**

- **The procedures used (solve for flows in flux surfaces first) and net plasma transport equations are analogous to those developed for stellarator transport.[9]**

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[7] J.D. Callen, A.J. Cole and C.C. Hegna, "Toroidal rotation in tokamak plasmas," Nucl. Fusion **49**, 085021 (2009).

[8] J.D. Callen, A.J. Cole, and C.C. Hegna, "Toroidal flow and particle flux in tokamak plasmas," Phys. Plasmas **16**, 082504 (2009); Erratum Phys. Plasmas **20**, 069901 (2013).

[9] See for example K.C. Shaing and J.D. Callen, Phys. Fluids **26**, 3315 (1983) and references cited therein.

Multi-Stage Strategy Is Used To Develop Transport Equations[10]

- **I. Average the density, momentum and energy equations over fluctuations (i.e., average over toroidal angle ζ) and then flux-surface-average (FSA) them.**

- **II.** *Key Elements Of New Approach:* **Consider sequentially specific components of the equilibrium force balance equations and their consequences:**

  IIA. *Radial:* **Use zeroth order radial force balance enforced by compressional Alfvén waves to obtain relation between toroidal & poloidal flows and radial electric field E_ρ & $dp_i/d\rho$.**

  IIB. *Parallel:* **Determine the parallel neoclassical Ohm's law and first order poloidal flows & heat flows within a flux surface from equilibrium momentum & heat flux equations.**

  IIC. *Toroidal:* **Require net radial current from all particle fluxes to vanish and thereby determine FSA toroidal momentum equation, and hence toroidal rotation Ω_t (and thus E_ρ).**

- **III. Substitute net second order ambipolar fluxes into FSA transport equations to obtain final comprehensive "radial" transport equations — for ambipolar n, p_e, p_i, and $\Omega_t \equiv \vec{V}\cdot\vec{\nabla}\zeta \simeq V_t/R$ (toroidal plasma rotation frequency).**

[10] J.D. Callen, C.C. Hegna, and A.J. Cole, "Transport equations in tokamak plasmas," Phys. Plasmas **17**, 056113 (2010).

This Approach Is New And Has Some Consequences

- Key differences from usual approaches for plasma transport equations are:

  first solve for electrons & ion flows within flux surfaces $\rightarrow \parallel$ Ohm's law & poloidal ion flow;

  derivation of non-ambipolar density fluxes and toroidal rotation ($\rightarrow E_\rho$) are naturally joined;

  comprehensive transport equations are obtained for $\Omega_{\rm t}$ ($\rightarrow E_\rho$) and $\psi_{\rm p}$, as well as usual n_e, p_s;

  effects of micro-turbulence on $\parallel$ Ohm's law, poloidal ion flow, particle fluxes, momentum transport and E_ρ are all included self-consistently;

  fluctuation-induced density flux is obtained from electron $\langle \overline{\tilde{n}_e \vec{\tilde{V}}_{\tilde{E}} \cdot \vec{\nabla} \rho} \rangle$ plus Rey., Max. stresses;

  source effects (e.g., NBI momentum input and $\vec{J}_{\rm CD}$) are included self-consistently;

  poloidal field transients ($\dot{\psi}_p \neq 0$) and current diffusion time scale effects are included; and

  net transport equations follow naturally from extended two-fluid moment equations and hence are consistent with M3D, NIMROD, JOREK etc. extended MHD code frameworks.

- Some new consequences that result from this approach are:

  radial electric field is determined self-consistently and enforces ambipolar density transport;

  micro-turbulence should be determined from Chapman-Enskog kinetic equation (CEKE) — so closures and transport they induce are consistent with these FSA transport equations,

  paleoclassical n, $\Omega_{\rm t}$ ($\rightarrow E_\rho$), p_s diffusion and pinch effects are included naturally; and

  poloidal flux transients ($\dot{\psi}_{\rm p} \neq 0$) induce radial motion of n, $\Omega_{\rm t}$ ($\rightarrow E_\rho$), p_s.

Tokamak Plasma Transport Equations Include Many Effects

- **With sources of n, $L_{\rm t} \equiv \rho_m \langle R^2 \rangle \Omega_{\rm t}$ and p_s, transport equations are[10]**

$$\text{density} \qquad \frac{1}{V'}\frac{\partial}{\partial t}\bigg|_{\psi_{\rm p}} n_e V' + \dot{\rho}_{\psi_{\rm p}}\frac{\partial n_e}{\partial \rho} + \frac{1}{V'}\frac{\partial}{\partial \rho}(V'\Gamma) = \langle \overline{S}_n \rangle,$$

$$\text{tor. mom.} \qquad \frac{1}{V'}\frac{\partial}{\partial t}\bigg|_{\psi_{\rm p}} L_{\rm t} V' + \dot{\rho}_{\psi_{\rm p}}\frac{\partial L_{\rm t}}{\partial \rho} + \frac{1}{V'}\frac{\partial}{\partial \rho}(V'\,\overline{\Pi}_{\rho\zeta}) = \langle \vec{e}_\zeta \cdot \Big(\overline{\vec{J}\times\vec{B}} - \vec{\nabla}\cdot \overleftrightarrow{\Pi} + \vec{\overline{S}}_{\rm p} \Big) \rangle,$$

$$\text{energy} \qquad \frac{3}{2} p_s \frac{\partial}{\partial t}\bigg|_{\psi_{\rm p}} \ln p_s V'^{5/3} + \frac{3}{2}\,\dot{\rho}_{\psi_{\rm p}}\frac{\partial p_s}{\partial \rho} + \frac{1}{V'}\frac{\partial}{\partial \rho}(V'\Upsilon_s) + \langle \vec{\nabla}\cdot \vec{q}^{\,\rm pc}_{s*} \rangle = \overline{Q}_{s\rm net}.$$

- **There are many classes of effects in plasma transport equations:**

  transients in the poloidal flux $\psi_{\rm p}$ via $\partial/\partial t|_{\psi_{\rm t}}$ and advection of $\psi_{\rm p}$ surfaces relative to the toroidal-flux-based radial coordinate ρ via $\dot{\rho}_{\psi_{\rm p}} \equiv \dot{\psi}_{\rm p}/\psi'_{\rm p}$,

  transport fluxes of ambipolar density Γ, total momentum $\overline{\Pi}_{\rho\zeta}$ and heat Υ_s, $\vec{q}^{\,\rm pc}_{s*}$ "radially" across $\psi_{\rm p}$ axisymmetric poloidal flux surfaces with many terms induced by collision and micro-turbulent processes in each species s,

  ambipolar density $\langle \overline{S}_n \rangle$, toroidal momentum $\langle \vec{e}_\zeta \cdot \vec{\overline{S}}_{\rm p} \rangle$ and energy $\langle S_{\mathcal{E}} \rangle$ sources with energy sources contributing to net energy heating rate $\overline{Q}_{s\rm net}$, which also includes external heating sources (NBI, ECH etc.) & radiation losses,

  and toroidal torques on the plasma caused by $\overline{\vec{J}\times\vec{B}}$ & viscous stresses $\vec{\nabla}\cdot\overleftrightarrow{\Pi}$.

- **We'll now focuses on recent developments: small 3-D field effects.**

3-D Field Effects In Tokamak Plasma Transport Equations

- **With sources of n, $L_t \equiv \rho_m \langle R^2 \rangle \Omega_t$ and p_s, transport equations are**[10]

density $$\frac{1}{V'}\frac{\partial}{\partial t}\bigg|_{\psi_p} n_e V' + \dot{\rho}_{\psi_p}\frac{\partial n_e}{\partial \rho} + \frac{1}{V'}\frac{\partial}{\partial \rho}(V'\Gamma) = \langle \overline{S}_n \rangle,$$

tor. mom. $$\frac{1}{V'}\frac{\partial}{\partial t}\bigg|_{\psi_p} L_t V' + \dot{\rho}_{\psi_p}\frac{\partial L_t}{\partial \rho} + \frac{1}{V'}\frac{\partial}{\partial \rho}(V'\,\overline{\Pi}_{\rho\zeta}) = \langle \vec{e}_\zeta \cdot \Big(\overline{\vec{J}\times\vec{B}} - \vec{\nabla}\cdot\overline{\overleftrightarrow{\Pi}} + \overline{\vec{S}}_p\Big)\rangle,$$

energy $$\frac{3}{2}p_s\frac{\partial}{\partial t}\bigg|_{\psi_p} \ln p_s V'^{5/3} + \frac{3}{2}\dot{\rho}_{\psi_p}\frac{\partial p_s}{\partial \rho} + \frac{1}{V'}\frac{\partial}{\partial \rho}(V'\Upsilon_s) + \langle \vec{\nabla}\cdot \vec{q}^{\,\rm pc}_{s*}\rangle = \overline{Q}_{s\rm net}.$$

- **Small 3-D field ($|\delta\vec{B}|/B_0 \sim \varrho_*$) effects come about in many ways:**[11]

externally applied resonant $m/n \simeq q$ and non-resonant fields cause field error (FE, $\langle \vec{e}_\zeta \cdot \overline{\vec{J}\times\vec{B}}\rangle$) and neo. toroidal viscous (NTV, $\langle \vec{e}_\zeta \cdot \vec{\nabla}\cdot\overline{\overleftrightarrow{\Pi}}\rangle$) damping of Ω_t,

toroidal magnetic field “ripple” caused by the finite number of coils that produce the toroidal magnetic field which damps Ω_t via NTV,

externally applied edge resonant magnetic perturbations (RMPs) used to modify the pressure profile there and stabilize edge MHD instabilities, and

spontaneous magnetic perturbations in the plasma which are caused by extended MHD macroscopic plasma instabilities that are under control, e.g., neoclassical tearing modes (NTMs) or resistive wall modes (RWMs).

[11] J.D. Callen, topical review on “Effects of 3D magnetic perturbations on torodal plasmas,” Nucl. Fusion **51**, 094026 (2013).

Plasma Toroidal Rotation Equation Provides 3D Context

- Magnetic field magnitude will be represented in $\psi_{\rm p}, \theta, \zeta$ coordinates by

$$|\vec{B}| = \underbrace{|\vec{B}_0(\psi_{\rm p},\theta)|}_{\text{2D, axisymm.}} + \sum_{n,m} \underbrace{\delta B_n(\psi_{\rm p}, m)\cos(m\theta - n\zeta - \varphi_{m,n})}_{\text{low } m,n \text{ resonant, non-resonant}} + \underbrace{\delta B_N(\psi_{\rm p},\theta)\cos(N\zeta)}_{\text{medium } n\text{, ripple}} + \underbrace{\cdots}_{\mu\text{turb.}}.$$

- On μs time scale compressional Alfvén waves enforce radial force balance:

$$\Omega_{\rm t} \equiv \vec{V}_i \cdot \vec{\nabla}\zeta = -\left(\frac{\partial \Phi_0}{\partial \psi_{\rm p}} + \frac{1}{n_i q_i}\frac{\partial p_i}{\partial \psi_{\rm p}}\right) + q\,\vec{V}_i \cdot \vec{\nabla}\theta \quad \Longrightarrow \quad V_{\rm t} \simeq \frac{E_\rho}{B_{\rm p}} - \frac{1}{n_i q_i B_{\rm p}}\frac{dp_i}{d\rho} + \frac{B_{\rm t}}{B_{\rm p}} V_{\rm p}.$$

- On the ms time scale poloidal flow is damped to $V_{\rm p} \simeq (c_{\rm p}/q_i)(dT_i/d\psi_{\rm p}) + \cdots$.

- Toroidal plasma torques cause radial particle fluxes: $\vec{e}_\zeta \cdot \vec{F}_{\rm orce} = -\,q_s \vec{\Gamma}_s \cdot \vec{\nabla}\psi_{\rm p}$.

- Setting the total radial plasma current induced by sum of the non-ambipolar particle fluxes to zero yields transport equation[7,8] for plasma toroidal angular momentum density $L_{\rm t} \equiv \sum_{\rm ions} m_i n_i \langle R^2 \vec{V}_i \cdot \vec{\nabla}\zeta\rangle$, $\boxed{\Omega_{\rm t}(\rho,t) \equiv L_{\rm t}/m_i n_i \langle R^2\rangle :}$

$$\boxed{\underbrace{\frac{\partial L_{\rm t}}{\partial t}}_{\text{inertia}} \simeq -\underbrace{\langle \vec{e}_\zeta \cdot \vec{\nabla}\cdot \overset{\leftrightarrow}{\pi}{}^{\rm 3D}_{i\|}\rangle}_{\text{NTV from } \delta B} + \underbrace{\langle \vec{e}_\zeta \cdot \overline{\delta\vec{J}\times\delta\vec{B}}\rangle}_{\text{resonant FEs}} - \underbrace{\langle \vec{e}_\zeta \cdot \vec{\nabla}\cdot \overset{\leftrightarrow}{\pi}_{i\perp}\rangle}_{\text{cl, neo, paleo}} - \underbrace{\frac{1}{V'}\frac{\partial}{\partial\rho}(V'\Pi_{i\rho\zeta})}_{\text{Reynolds stress}^8} + \underbrace{\langle \vec{e}_\zeta \cdot \textstyle\sum_s \vec{S}_{\rm ps}\rangle}_{\text{mom. sources}}.}$$

- Radial electric field for net ambipolar transport is determined by $\Omega_{\rm t}$:

$$\boxed{E_\rho \equiv -|\vec{\nabla}\rho|\,\partial\Phi_0/\partial\rho \simeq |\vec{\nabla}\rho|\,[\,\Omega_{\rm t}\,\psi'_{\rm p} + (1/n_{i0}q_i)\,dp_i/d\rho - (c_{\rm p}/q_i)\,dT_i/d\rho],} \quad \omega_E \simeq -E_\rho/RB_{\rm p}.$$

SUMMARY

- **At present a GRAND UNIFIED TOKAMAK SIMULATION (GUTS)? — for developing "predictive capability" for ITER — seems to require us to:**

use small gyroradius expansion to order various tokamak physics effects, especially those in the radial, parallel, and toroidal components of the species force balance equation,

use extended MHD to check macrostability, obtain $\vec{B}$ field structure including plasma responses to 3-D fields, reconnecting regions and stochastic fields near separatrix X points,

determine microturbulence by solving the CEKE in this "distorted" $\vec{B}$ field geometry and produce the collision- and microturbulence-induced closures and radial transport fluxes,

solve resultant tokamak plasma transport equations simultaneously for n_e, $\Omega_{\rm t}$ (E_ρ), p_s, and

then iterate these extended MHD, CEKE, closures and transport steps for self-consistency.

What Should CEMM Do Now?

- **Kinetics: Develop general Chapman-Enskog-type kinetic equation in 6D phase space with only small gyroradius expansion, then develop drift-kinetic and perhaps gyrokinetic versions of it.**

- **Magnetic geometry: Adopt 2D $\vec{B}_0$ plus small $\delta\vec{B}$ philosophy.**

- **Viscous stresses: Develop both simple analytic-based and computational procedures for parallel viscous stresses and the forces they introduce in extended MHD.**

- **Extended MHD: Add parallel viscous stresses into extended MHD equations and then use them to begin exploring poloidal flow damping and toroidal rotation Ω_t effects on $\delta\vec{B}$ fields.**

- **Connect with gyrokinetics: Develop some benchmark "test problems" that can be used to explore to what degree gyrokinetics can be used to "do everything" or should couple with extended MHD and fluid-moment-based transport equations.**

Supplementary Material

How Do Collisions Affect Linear Landau Damping?

- The classic theoretical plasma physics issue is the damping of a linear wave for which Landau obtained a time-asymptotic solution using a Laplace transform procedure to introduce irreversibility.
- Landau damping has been observed experimentally and is often thought of as a collisionless ?, entropy-producing process because

  collisions are neglected in obtaining it (Vlasov equation is used, $\nu \lesssim 10^{-4}\,\omega_p$), damping rate $\gamma_{\rm L} \sim 10^{-2}\,\omega_p$ does not depend on the collision frequency, and wave damping would seem to imply temporal irreversibility.
- However, temporal irreversibility and entropy production in Landau damping must be due to a collisional process because

  wave transfers its phase information to the perturbed distribution, whose continued presence is demonstrated by second wave causing a nonlinear echo, but when sufficient collisions are introduced echoes are damped.
- Temporal evolution of collisional effects on linear Landau damping have been explored: J.D. Callen, Phys. Plasmas 21, 052106 (2014).

Scattering Decorrelates A Particle From A Wave

- **Dominant effect of Coulomb collisions is to scatter a particle's velocity-space pitch-angle ϑ at nearly constant speed v (energy).**
- **A particle is resonant with a wave when its speed in the wave direction $u \equiv \vec{v} \cdot \vec{k}/k$ matches the wave phase speed $V_\varphi = \omega/k$.**
- **Small pitch-angle scattering $\delta\vartheta \sim \sqrt{2\,\nu\tau}$ decorrelates u from V_φ.**

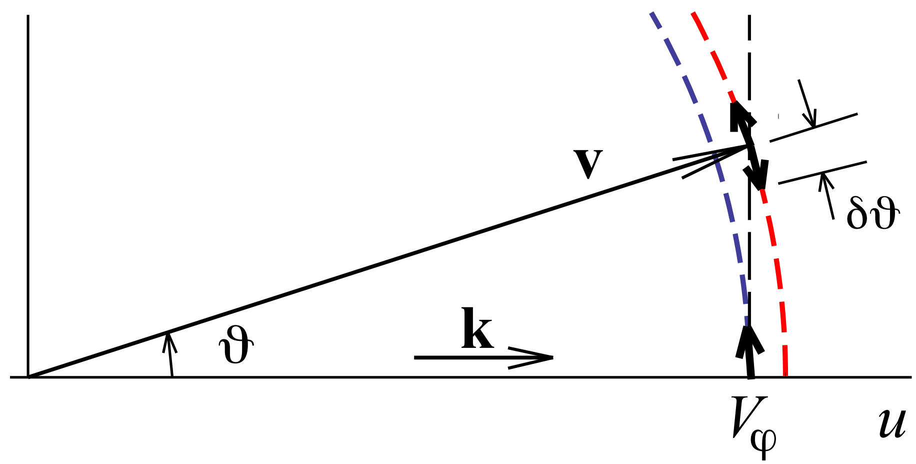

Figure 4: **Thick arrows indicate small Coulomb collisional scattering $\delta\vartheta$ of the pitch-angle ϑ_0 about ϑ for two (black, red) particle speeds v. Wave phase speed in $\vec{k}$ direction is $V_\varphi \equiv \omega/k$. Particles with $u \equiv \vec{v} \cdot \vec{k}/k = V_\varphi$ are resonant with wave.**

Discussion Of Collisional Effects On Landau Damping

- **In plasmas which are always intrinsically weakly collisional:**

  collisionless propagator $\delta[\vec{x} - \vec{x}_0 - \vec{v}_0\tau]$ is replaced by Green function G_ν,

  which causes $u/(u - V_\varphi)$ singularity to be replaced by I_ν integral

  that collisionally limits minimum resonance width with a $\nu_{\rm eff} \sim \sqrt{\omega\nu} \gg \nu$, and

  whose $t \gg 1/\nu_{\rm eff}$ limit yields Landau prescription for resolving singularity.

  Thus, Laplace transform is neither needed nor appropriate in most plasmas.

- **While collisional analysis does not change the Landau damping rate, it facilitates temporal analysis of plasma response to a wave:**

  temporally reversible response for $t \ll 1/\nu_{\rm eff}$,

  irreversible, dissipative, resonance broadening response for $t \gtrsim 1/\nu_{\rm eff}$, & finally

  collisional justification of Landau resolution of $u = V_\varphi$ singularity for $t \gg 1/\nu_{\rm eff}$.

- **Taking $t \to \infty$ limit in collisionless ($\nu \to 0$) response is incompatible with $t \ll 1/\nu_{\rm eff}$ requirement for collisionless response — Coulomb collisional scattering effects intervene at a finite time.**

Collisional Effects On Landau Damping (continued)

- ***Landau analysis*** of wave damping ***is linear theory*** that uses Vlasov equation and obtains temporal irreversibility only ***for $t \to \infty$.***
- ***However,*** it has recently been demonstrated[12] that when the collisionless Vlasov equation is used, obtaining ***temporal irreversibility*** on very long but finite time scales ***requires nonlinear*** effects.

  In particular, third order (echo-type responses) and higher order nonlinear terms are required to produce heteroclinic (temporally irreversible) solutions.

  Result is obtained in the spirit of Kolmogorov-Arnold-Moser (KAM) theorem.

- **Combination and interplay of collisional and nonlinear effects in temporal irreversibility in Landau damping are not clear.**

  Probabilistic Coulomb collision scattering effects damp echoes for $t \gtrsim 1/\nu_{\text{eff}}$.

  Unresolved question: how weak must collisional effects be (or how large must wave amplitude be) so nonlinear echo interaction effects dominate as $t \to \infty$?

- **Identifying cause of temporal irreversibility is crucial issue — for physics understanding, entropy production, numerical convergence.**

[12] C. Villani, "Landau damping," CERMAC 2010 lectures available via `http://smai.emath.fr/cemracs/cemracs10`; C. Mouhot and C. Villani, "Landau damping," J. Math. Phys. **51**, 015204 (2010); C. Villani "Particle systems and nonlinear Landau damping," Phys. Plasmas **21**, 030901 (2014).