Development of Resistive DCON and MATCH

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Resistive DCON

- Ideal DCON computes the MHD stability of axisymmetric toroidal plasmas. Thoroughly verified and validated, robust, reliable, easy to use, widely used.
- Integrates the Euler-Lagrange equation for Fourier components of the normal displacement from the magnetic axis to the plasma-vacuum interface. This is an initial value problem.
- Straightforward extension to compute the outer region matching data for resistive instabilities converts it to a shooting method, which is numerically unstable.
- Pletzer and Dewar introduced a singular Galerkin method, avoiding this problem.
- We improve on their implementation with a better choice of basis functions and grid packing, reusing most of our existing code.
- Solutions in the outer region are matched to the inner region resistive MHD model of Glasser, Greene & Johnson, solved by DELTAR, and a vacuum region, solved by Chance's VACUUM.
- We have obtained some excellent agreement with the straight-through linear MARS code.

Pletzer & Dewar References

- A. D. Miller & R. L. Dewar, "Galerkin method for differential equations with singular points," *J. Comp. Phys.* **66**, 356-390 (1986). Introduces Galerkin method for singular ODEs, solves test problems.

- R. L. Dewar & A. Pletzer, "Two-dimensional generalization of the Newcomb equation," *J. Plasma. Phys.* **43**, 2, 291-310 (1990). Derives 2D Newcomb equations, equivalent to DCON equation.

- A. Pletzer & R. L. Dewar, "Non-ideal Variational method for determination of the outer-region matching data," J. Plasma Phys. 45, 3, 427-451 (1991).
  Solves cylindrical problem with non-monotonic q profile.

- A. Pletzer, A. Bondeson, and R. L. Dewar, "Linear stability of resistive MHD modes: axisymmetric toroidal computation of the outer region matching data," J. Comp. Phys. 115, 530-549 (1994). Solves toroidal problem, PEST 3, verified against MARS code.

Galerkin Expansion

Euler-Lagrange Equation

$$\mathbf{L}\Xi = -(\mathbf{F}\Xi' + \mathbf{K}\Xi)' + (\mathbf{K}^\dagger\Xi' + \mathbf{G}\Xi) = 0$$

Galerkin Expansion

$$(u, v) \equiv \int_0^1 u^\dagger(\psi) v(\psi) d\psi$$

$$\Xi(\psi) = \sum_{i=0}^{N} \Xi_i \alpha_l(\psi)$$

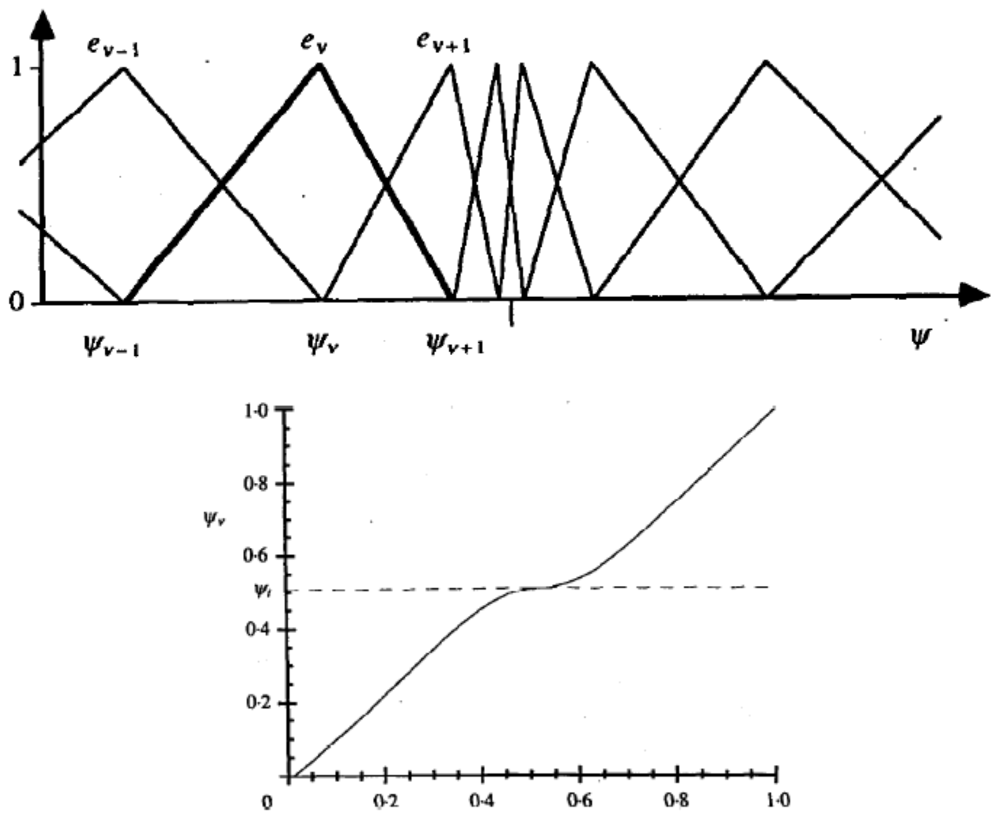
$$(\alpha_i, \mathbf{L}\Xi) = (\alpha_i, \mathbf{L}\alpha_j)\Xi_j = 0$$

$$\mathbf{L}_{ij} = (\alpha_i', \mathbf{F}\alpha_j') + (\alpha_i', \mathbf{K}\alpha_j) + (\alpha_i, \mathbf{K}^\dagger\alpha_j') + (\alpha_i, \mathbf{G}\alpha_j)$$

Finite-Energy Response Driven by Large Solution

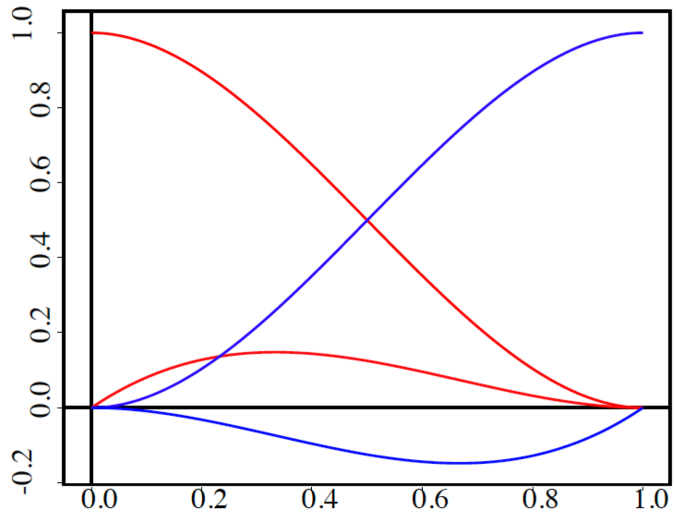
$$L_{ij}\check{\Xi}_j = -(\alpha_i, L\hat{\Xi})$$

Dewar and Pletzer:
Linear Finite Elements on a Packed Grid



The choice of basis functions determines
the rate of convergence.

Better Basis Functions: C^1 Hermite Cubics

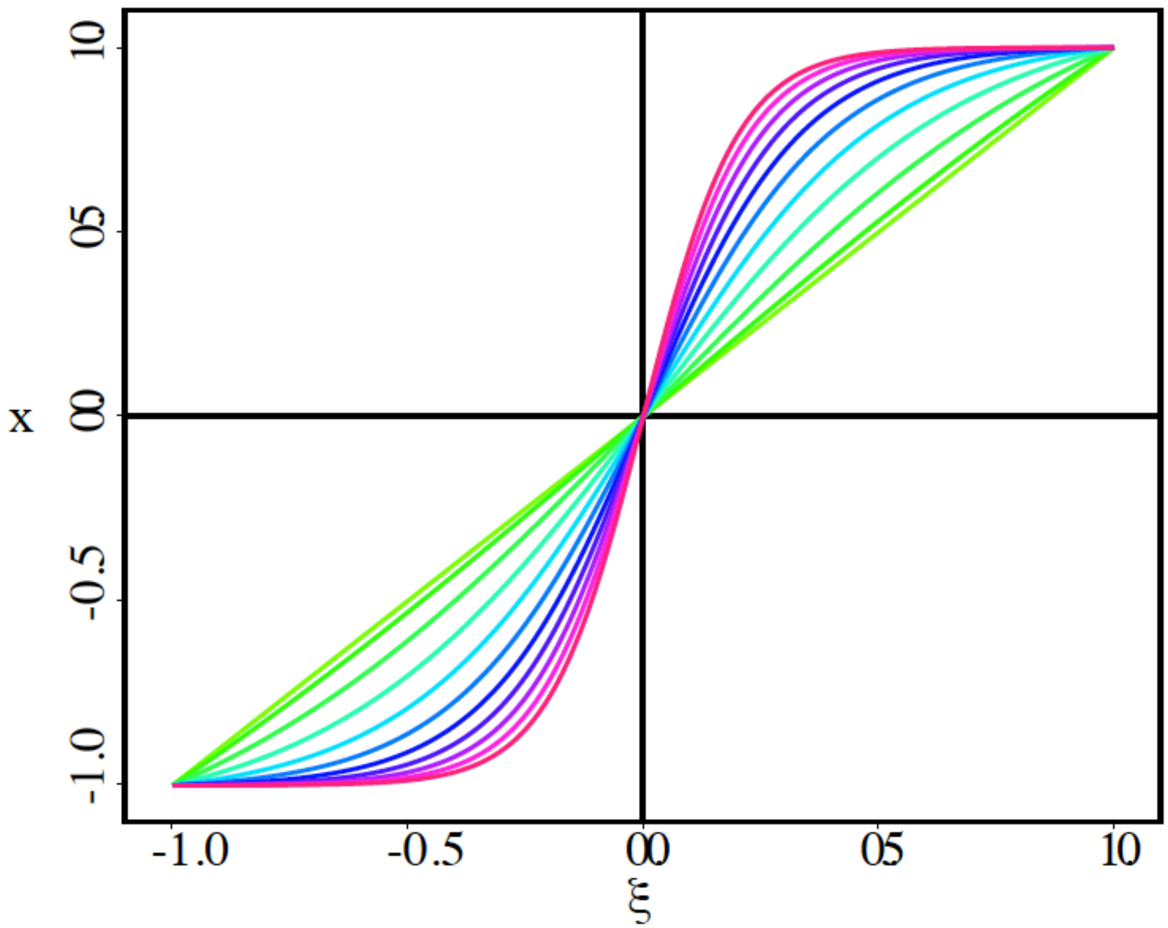


- Cubic polynomials on (0,1), within each grid cell.
- C^1 continuity of function values and first derivatives across grid cells.
- Imposes boundary conditions on nonresonant solutions across the singular surface.

Better Basis Functions: Singular Elements

- Weierstrass Convergence Theorem:
  Polynomial approximation uniformly convergent for analytic functions.
- Large and small resonant solutions are non-analytic near the singular surface.
- Supplement Hermite basis with power series for resonant solution near singular surface.
- Evaluation of singular element quadratures with LSODE.
- DCON fits equilibrium data to Fourier series and cubic splines, computes resonant power series to arbitrarily high order. Recent work extends this to the degenerate zero-β limit.
- Convergence requires that the large solution be computed to at least $n = 2\sqrt{(-D_I)}$ terms. PEST 3 is limited to n = 1. Higher n required for small shear and high β.

Better Basis Functions: Adjustable Grid Packing Between Singular Surfaces



Layout of Basis Functions

$$\mathbf{L}\bar{\mathbf{u}} = -\left(\mathbf{F}\bar{\mathbf{u}}' + \mathbf{K}\bar{\mathbf{u}}\right)' + \left(\mathbf{K}^{\dagger}\bar{\mathbf{u}}' + \mathbf{G}\bar{\mathbf{u}}\right) = \mathbf{r}$$

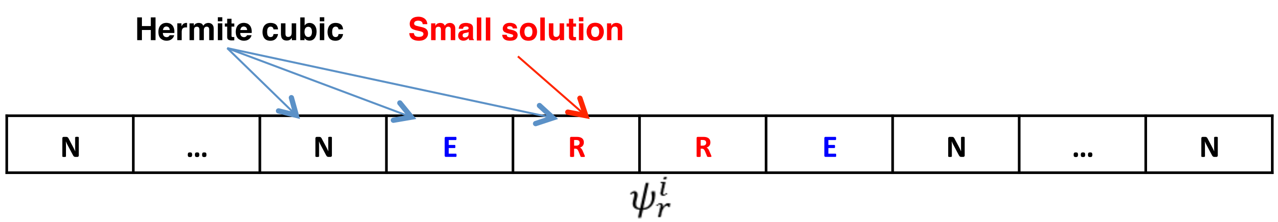
Variational Principle

$$W = \frac{1}{2}\left(\bar{\mathbf{u}}, \mathbf{L}\bar{\mathbf{u}}\right) - \left(\bar{\mathbf{u}}, \mathbf{r}\right)$$

$$\delta W = \left(\delta\bar{\mathbf{u}}, \mathbf{L}\bar{\mathbf{u}}\right) - \left(\delta\bar{\mathbf{u}}, \mathbf{r}\right) = 0$$

Resonant-Galerkin Expansion

$$\mathbf{u}(\psi) = \sum_{k=0}^{p} \bar{\mathbf{u}}_k \alpha_k(\psi) + \sum_{j,m=L,R} \mathbf{u}_m^{j(s)}(\psi)\Delta_{lm}^{ij} \qquad \delta\mathbf{u}(\psi) = \sum_{k=0}^{p} \alpha_k(\psi)\delta\bar{\mathbf{u}}_k + \sum_{j,m=L,R} \mathbf{u}_m^{j(s)}(\psi)\delta\Delta_{lm}^{ij}$$



Extension element (E) connecting Resonant element (R) and Normal element (N) allows the resonant small solution smoothly vanishes.

Adjustable grid packing is applied to the interval between each two adjacent resonant surfaces.

Inner Region: Coordinates

Straight Fieldline Coordinates

$$\mathbf{B} = \chi'[\nabla\zeta - q(\psi)\nabla\theta] \times \nabla\psi, \quad \mathcal{J} \equiv (\nabla\psi \times \nabla\theta \cdot \nabla\zeta)^{-1}$$

$$\mathcal{J}\mathbf{B} \cdot \nabla\psi = 0, \quad \mathcal{J}\mathbf{B} \cdot \nabla\theta = \chi', \quad \mathcal{J}\mathbf{B} \cdot \nabla\zeta = q\chi'$$

$$\mathcal{J}\mathbf{B} \cdot \nabla f(\psi, \theta, \zeta) = \chi'(\partial_\theta f + q\partial_\zeta f)$$

Singular Surface Taylor Expansion

$$q(\psi) = q_0 + q_0' x + \cdots, \quad x \equiv \psi - \psi_0$$

$$q_0 \equiv q(\psi_0) = \frac{m}{n}, \quad q_0' \equiv q'(\psi_0) \neq 0$$

Inner Region Coordinates

$$x \equiv \psi - \psi_0, \quad y \equiv \theta, \quad z \equiv \zeta - q_0\theta$$

$$(\psi, \theta, \zeta) \to (x, y, z), \quad (\nabla x \times \nabla y \cdot \nabla z)^{-1} = \mathcal{J}$$

$$\mathcal{J}\mathbf{B} \cdot \nabla x = 0, \quad \mathcal{J}\mathbf{B} \cdot \nabla y = \chi', \quad \mathcal{J}\mathbf{B} \cdot \nabla z = \chi' q_0' x$$

$$\mathcal{J}\mathbf{B} \cdot \nabla f(x, y, z) = \chi'(\partial_y f + q_0' x \partial_z f)$$

Inner Region: Equations and Ordering

Fields

$$\mathbf{E} = -\partial_t \mathbf{A} - \nabla\varphi, \quad \mathbf{b} = \nabla \times \mathbf{A}$$
$$\nabla \cdot \mathbf{j} = \nabla \cdot \mathbf{A} = 0, \quad \mathbf{j} = -\nabla^2 \mathbf{A}$$

Density and Pressure

$$\partial_t \rho + \nabla \cdot (\rho \mathbf{v}) = 0$$
$$\partial_t p + \mathbf{v} \cdot \nabla p + \gamma p \nabla \cdot \mathbf{v} = 0$$

Momentum Conservation and Ohm's Law

$$\partial_t(\rho \mathbf{v}) = \mathbf{j} \times \mathbf{B} + \mathbf{J} \times \mathbf{b} - \nabla p$$
$$\mathbf{E} + \mathbf{v} \times \mathbf{B} = \eta \mathbf{j}$$

Ordering Assumptions

$$x \sim \epsilon \ll 1, \quad \partial_x \sim \epsilon^{-1}, \quad \partial_y \sim \partial_z \sim 1, \quad \partial_t \sim \epsilon, \quad \eta \sim \epsilon^3$$
$$\mathbf{v} \sim p \sim \mathbf{A} \sim 1, \quad \mathbf{v} \cdot \nabla x \sim \nabla \cdot \mathbf{v} \sim \mathbf{A} \cdot \nabla z \times \nabla x \sim \mathbf{A} \cdot \nabla x \times \nabla y \sim \varphi \sim \epsilon$$

Inner and Outer Region Solutions

Outer Region Basis Functions and Linear Combination

$$\mathbf{u}_{i,k}(\psi) \equiv \sum_{j=1}^{n} \sum_{l=L}^{R} \left[\delta_{i,j}\delta_{k,l}\mathbf{u}_{j,l}^{b}(\psi) + \Delta'_{i,k;j,l}\mathbf{u}_{j,l}^{s}(\psi) \right]$$

$$\mathbf{u}(\psi) = \sum_{i=1}^{n} \sum_{k=L}^{R} c_{i,k}\mathbf{u}_{i,k}(\psi)$$

Inner Region Basis Functions and Linear Combination

$$\mathbf{v}_{i,\pm}(x) \equiv \mathbf{v}_{i,\pm}^{b}(x) + \Delta_{i,\pm}(Q)\mathbf{v}_{i,\pm}^{s}(x) = \pm\mathbf{v}_{i,\pm}(-x)$$

$$\mathbf{v}_{i}(x) = d_{i,+}\mathbf{v}_{i,+}(x) + d_{i,-}\mathbf{v}_{i,-}(x)$$

Inner region solutions computed with DELTAR.
Glasser, Jardin & Tesauro, Phys. Fluids **27**, 1225 (1984).

Matching Conditions

Matching Conditions

$$c_{j,L} = d_{j,+} - d_{j,-}, \quad c_{j,R} = d_{j,+} + d_{j,-}$$

$$\sum_{i=1}^{n} \sum_{k=L}^{R} c_{i,k} \Delta'_{i,k;j,L} = d_{j,+} \Delta_{j,+}(Q) - d_{j,-} \Delta_{j,-}(Q)$$

$$\sum_{i=1}^{n} \sum_{k=L}^{R} c_{i,k} \Delta'_{i,k;j,R} = c_{j,+} \Delta_{j,+}(Q) + c_{j,-} \Delta_{j,-}(Q)$$

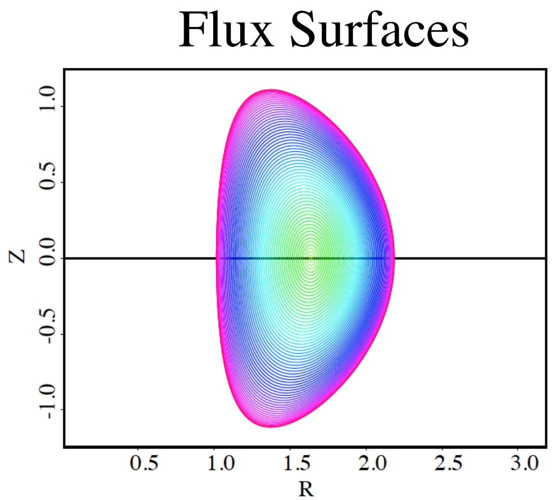
Matrix Form and Dispersion Relation

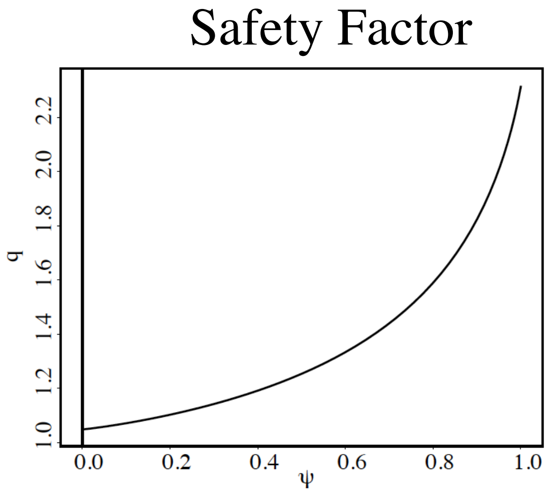
$$\mathbf{c} \equiv (c_{1L}, d_{1+}, d_{1-}, c_{1R}, c_{2L}, d_{2+}, d_{2-}, c_{2R}, \cdots)^T$$

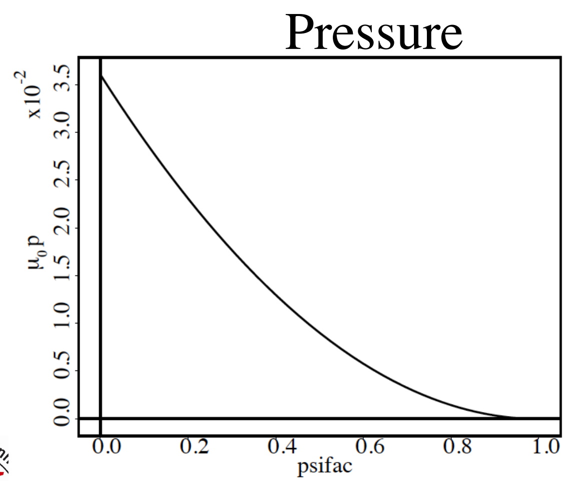
$$\mathsf{M}(Q) \cdot \mathbf{c} = 0, \quad \det \mathsf{M}(Q) = 0$$

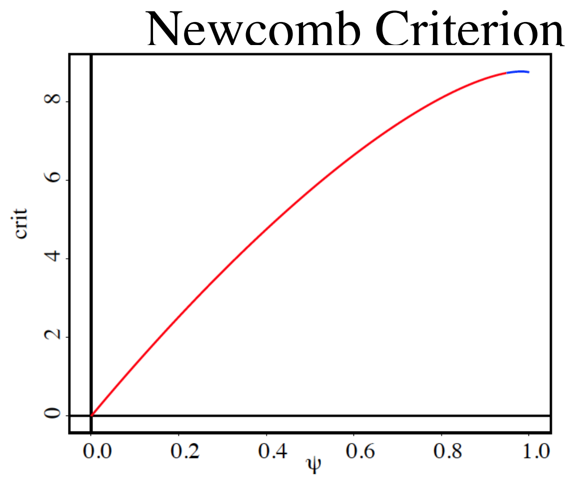
Outer region solved once in < 10 seconds.
Inner region solved many times, 20,000 per second.

Chease Equilibrium, 1 Singular Surface, $\beta_N = 0.774$

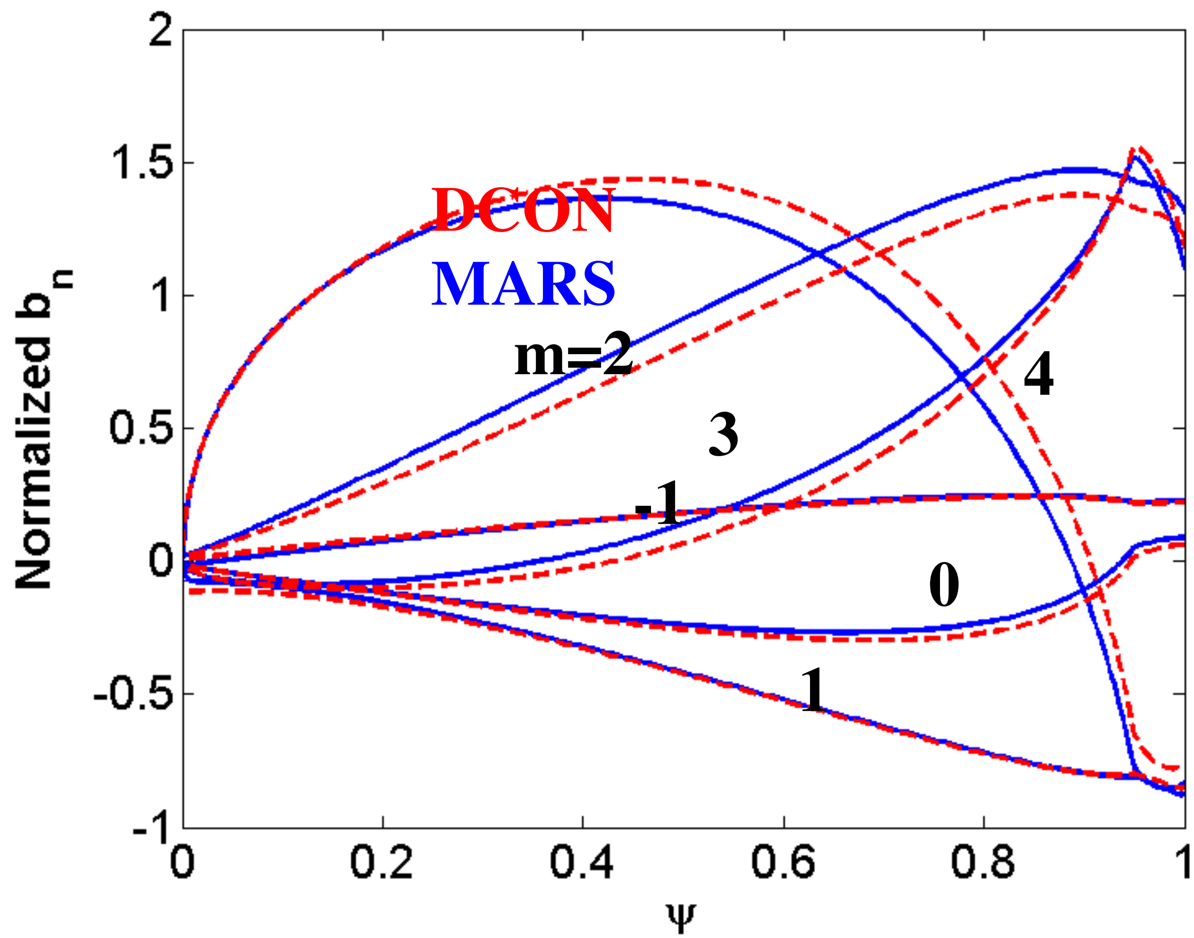




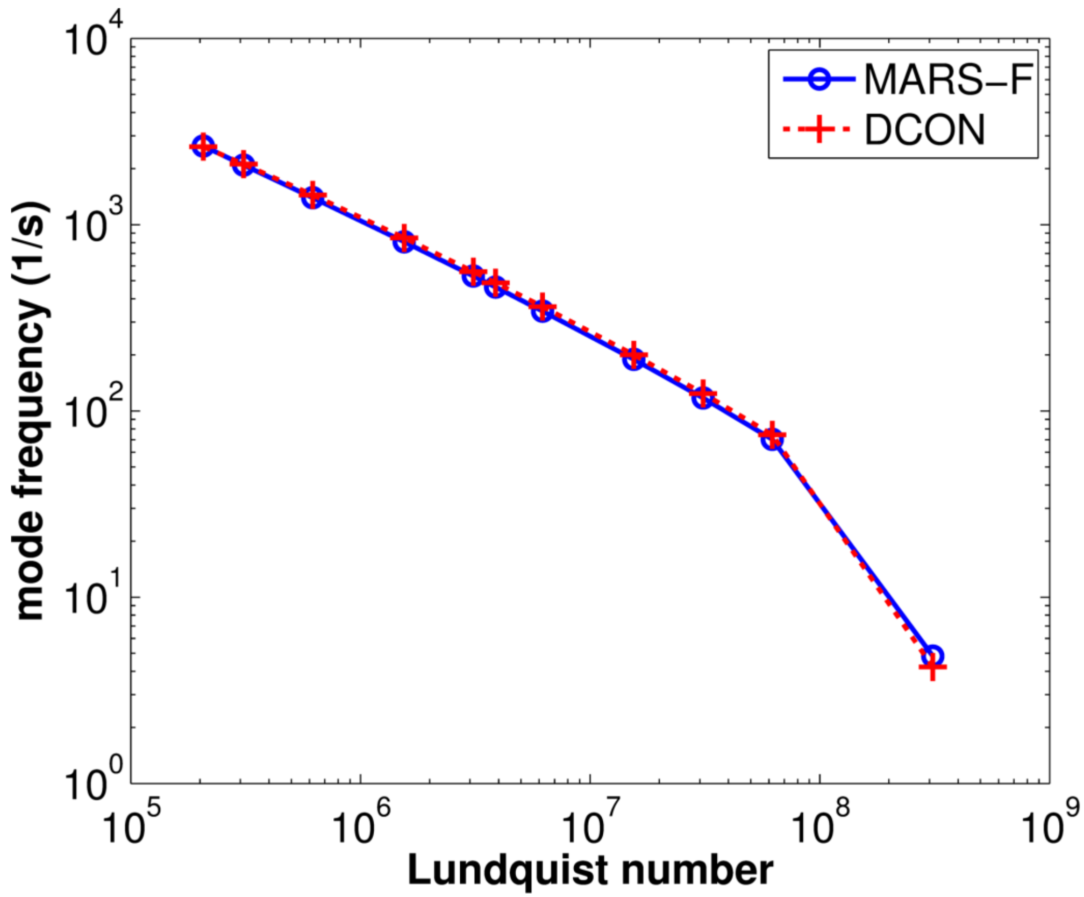




Comparison with MARS Code, 1 Singular Surface



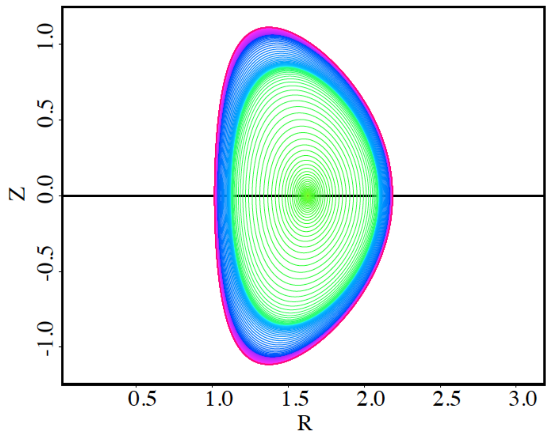
Eigenvalue Benchmark with MARS Code



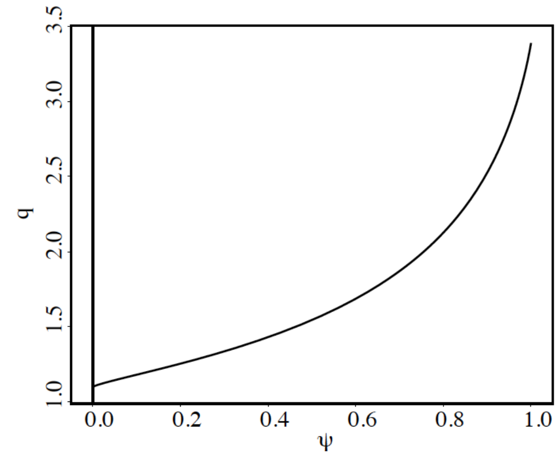
Greatly improved agreement due to bug fix:
Missing factor of $dV/d\psi$

Chease Equilibrium, 2 Singular Surfaces, $\beta_N = 0.240$

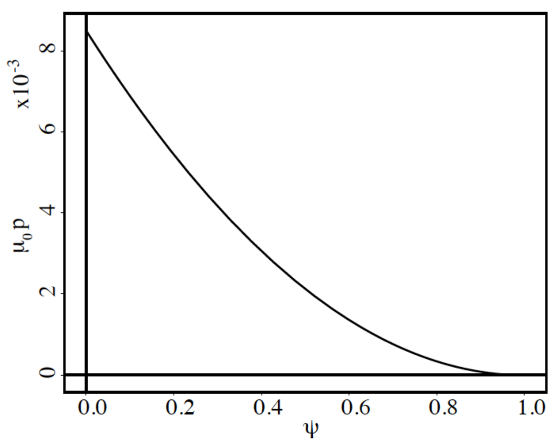
Flux Surfaces



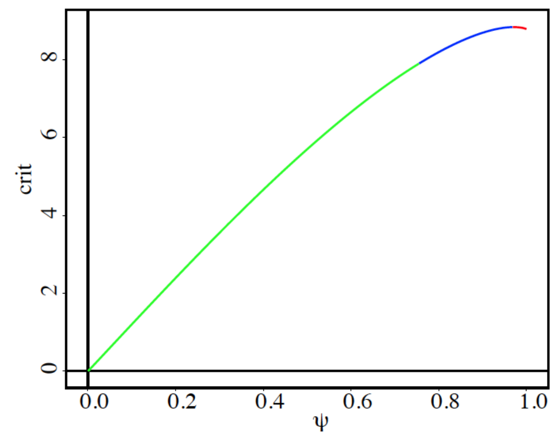
Safety Factor



Pressure

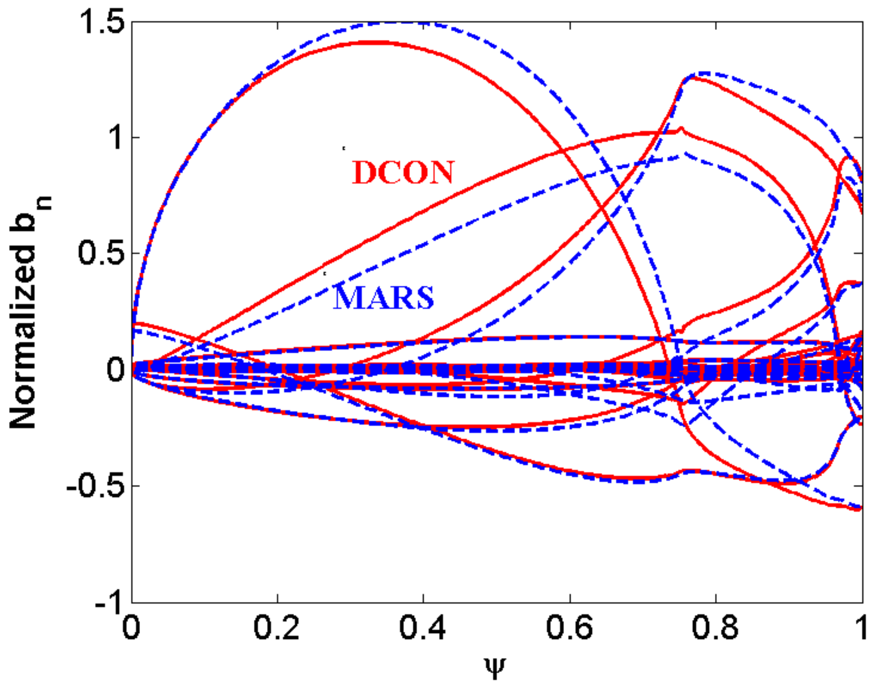


Newcomb Criterion

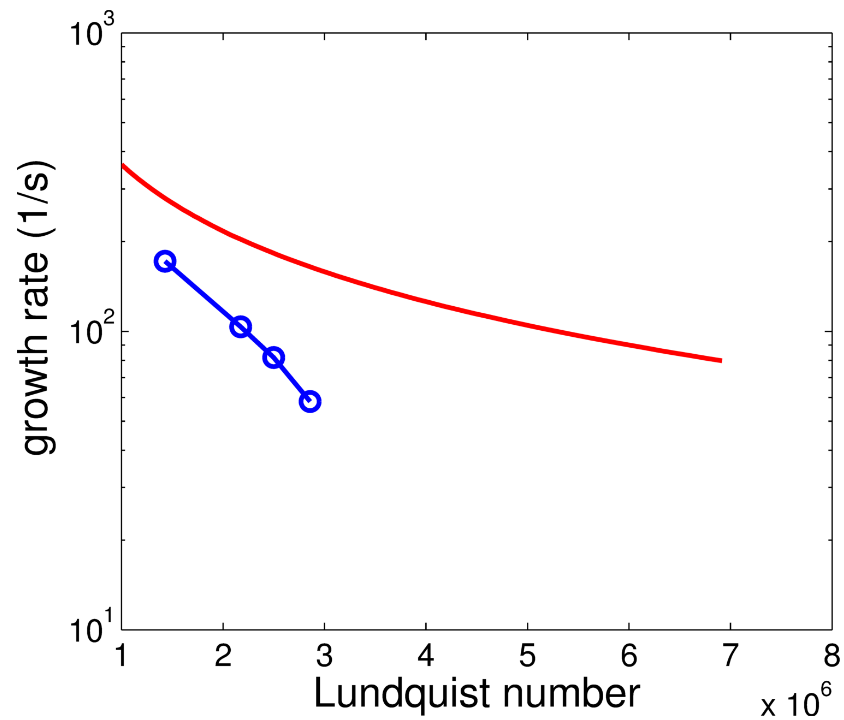


Comparison with MARS Code, 2 Singular Surfaces

Eigenfunction



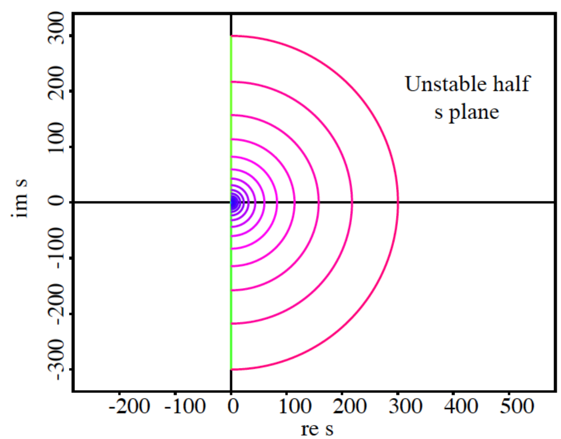
Growth Rate



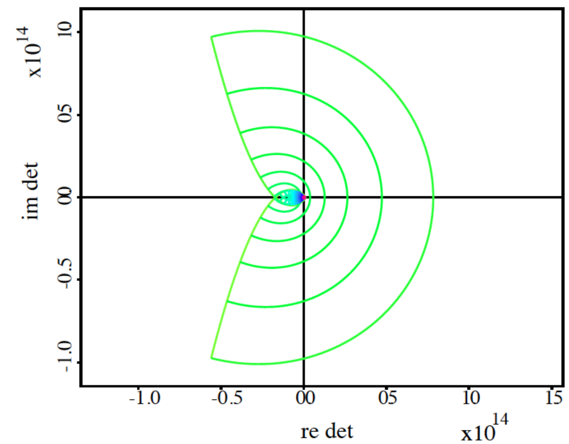
Leaves something to be desired. Another missing factor?
Careful re-derivation in progress.

Multiple Complex Roots: Generalized Nyquist Analysis and Deflation

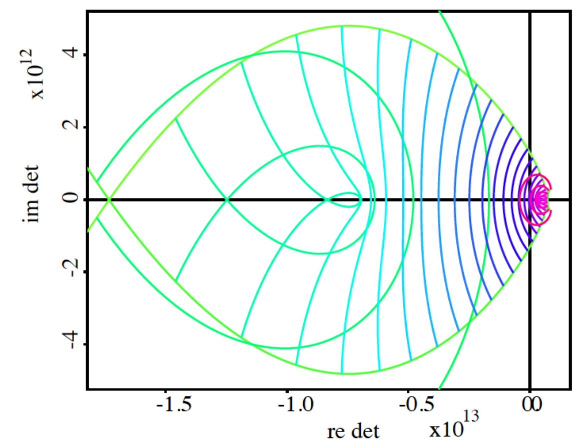
Contours in the
Complex *s* Plane



Images in the
Complex det M Plane



Zoom to the Neighborhood
of the Origin



Complex Analysis: The Principle of the Argument

$$\frac{1}{2\pi i}\oint \frac{f'(z)}{f(z)}dz = \text{Zeroes} - \text{Poles}$$

The number of times the image contour encircles the origin
is the number of unstable roots.

Newton Iteration with Deflation

$$f(z) \rightarrow \frac{f(z)}{\Pi_{i=1}^{n}(z - z_i)}, \quad z_i = \text{roots already found}$$

A Future Role for Matched Asymptotic Expansions

- The method of matched asymptotic expansions was introduced by Furth, Killeen, and Rosenbluth in order to obtain analytical results.
- Most recent work uses straight-through methods, such as M3D and NIMROD, using packed grids to resolve singular layers.
- Thermonuclear plasmas are in a regime where conditions for the validity of matched asymptotic expansion are very well satisfied.
- Resistive DCON and DELTAR provide numerical methods to do the full matching problem numerically and *very* efficiently.
- Inner region dynamics can be extended to include full fluid and kinetic treatments.
- Nonlinear effects are localized to the neighborhood of the singular layers and can be solved with the 2D HiFi code, exploiting helical symmetry, matched through ideal outer regions.
- Asymptotic matching and straight-through methods can complement and verify each other.

Future Work

- Improved benchmarks vs. MARS for multiple singular surfaces. Discrepancy may be due to missing factors in the matching conditions.
- Reconstruction of inner region eigenfunction by Fourier transformation.
- More complete fluid regime model of linear inner region; Braginskii. Facilitated by new derivation of GGJ equations in terms of $\mathbf{A}$, ϕ, and p.
- Neoclassical inner region model, drift kinetic equation; Ramos.
- Nonlinear model, NTM, with nonlinear effects localized to inner regions, coupled through ideal linear outer region. 2D HiFi code, helical symmetry.
- Nonlinear verification with straight-through nonlinear codes: NIMROD, M3D-C1.