CEMM Meeting. New Orleans LA, October 2014.

LOW COLLISIONALITY, SECOND-FLR-ORDER ION NEOCLASSICAL THEORY*

J. J. Ramos

M.I.T. Plasma Science and Fusion Center

*Work supported by the U.S. Department of Energy

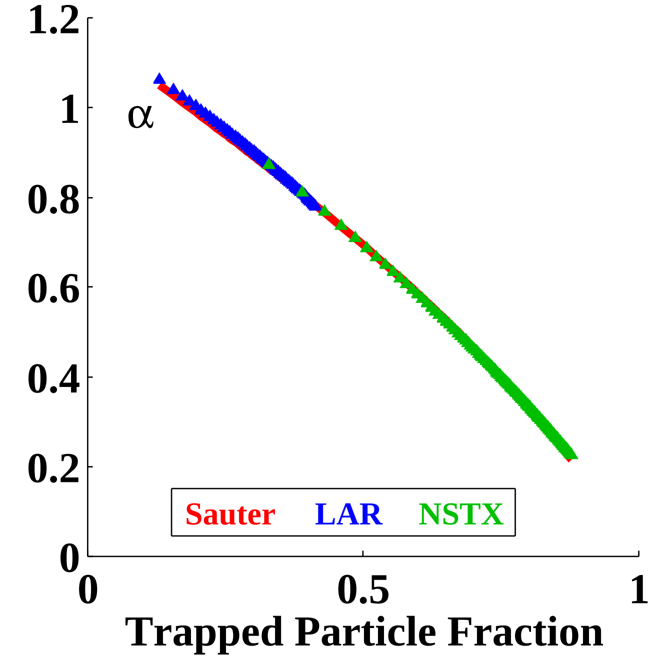
MOTIVATION

- **EXTEND THE NEOCLASSICAL SOLUTION FOR AN AXISYMMETRIC EQUILIBRIUM ION DISTRIBUTION FUNCTION TO THE REALISTICALLY LOW COLLISIONALITY REGIME** $\nu_\iota L/v_{th\iota} \sim \rho_\iota/L \equiv \delta \ll 1$**. NO GEOMETRICAL APPROXIMATIONS ARE TO BE MADE (I.E.** $\nu_* \sim \delta \ll 1$ **WITH** $\epsilon \sim B_p/B_t \sim 1$**).**

MOTIVATION

- EXTEND THE NEOCLASSICAL SOLUTION FOR AN AXISYMMETRIC EQUILIBRIUM ION DISTRIBUTION FUNCTION TO THE REALISTICALLY LOW COLLISIONALITY REGIME $\nu_\imath L/v_{th\imath} \sim \rho_\imath/L \equiv \delta \ll 1$. NO GEOMETRICAL APPROXIMATIONS ARE TO BE MADE (I.E. $\nu_* \sim \delta \ll 1$ WITH $\epsilon \sim B_p/B_t \sim 1$).

- THE STANDARD BANANA REGIME SOLUTION IS BASED ON A FIRST-FLR-ORDER ION DRIFT-KINETIC EQUATION AND APPLIES TO $\delta \ll \nu_* \ll 1$. THE "NIES" CODE [Lyons, Jardin, Ramos PoP 2012] SOLVES FOR THIS REGIME AND OBTAINS THE ION PARALLEL FLOW COEFFICIENT $\alpha = e\langle B^2\rangle U_\imath/(IdT_\imath/d\psi)$ THAT FACTORS IN THE ION BOOTSTRAP CURRENT.



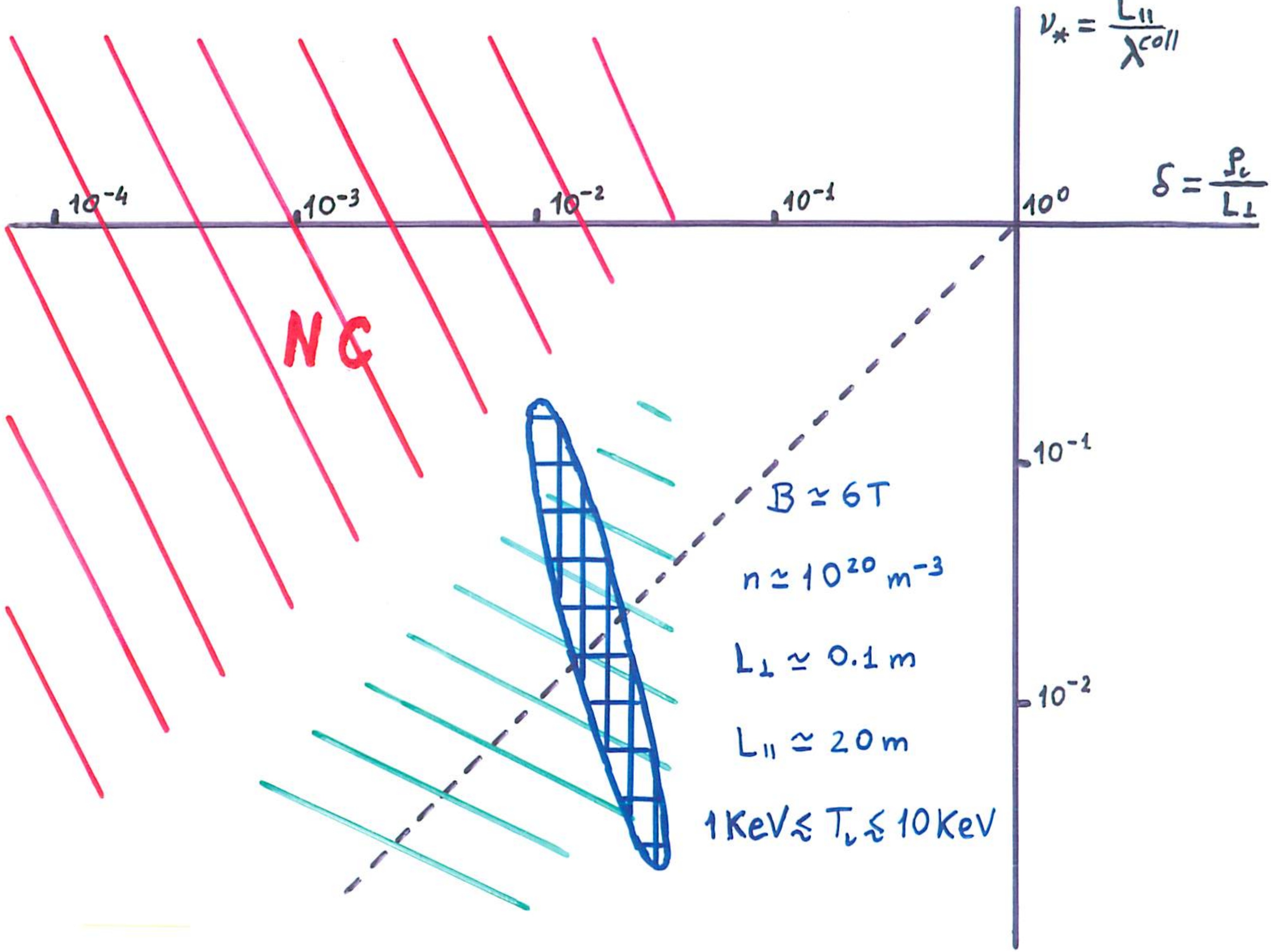
1.2
1
α
0.8
0.6
0.4
0.2
0
0
0.5
1
Sauter
LAR
NSTX
Trapped Particle Fraction

MOTIVATION

- EXTEND THE NEOCLASSICAL SOLUTION FOR AN AXISYMMETRIC EQUILIBRIUM ION DISTRIBUTION FUNCTION TO THE REALISTICALLY LOW COLLISIONALITY REGIME $\nu_\iota L/v_{th\iota} \sim \rho_\iota/L \equiv \delta \ll 1$. NO GEOMETRICAL APPROXIMATIONS ARE TO BE MADE (I.E. $\nu_* \sim \delta \ll 1$ WITH $\epsilon \sim B_p/B_t \sim 1$).

- THE STANDARD BANANA REGIME SOLUTION IS BASED ON A FIRST-FLR-ORDER ION DRIFT-KINETIC EQUATION AND APPLIES TO $\delta \ll \nu_* \ll 1$. THE "NIES" CODE [Lyons, Jardin, Ramos PoP 2012] SOLVES FOR THIS REGIME AND OBTAINS THE ION PARALLEL FLOW COEFFICIENT $\alpha = e\langle B^2\rangle U_\iota/(IdT_\iota/d\psi)$ THAT FACTORS IN THE ION BOOTSTRAP CURRENT.

- THE ORDERING $\nu_* \sim \delta \ll 1$ IS MORE APPROPRIATE FOR FUSION-GRADE PLASMAS. A NEOCLASSICAL SOLUTION IN THIS REGIME REQUIRES A SECOND-FLR-ORDER ION DRIFT-KINETIC EQUATION.



$\nu_* = \frac{L_{\parallel}}{\lambda^{coll}}$
$\delta = \frac{\rho_i}{L_\perp}$
10^{-4}
10^{-3}
10^{-2}
10^{-1}
10^{0}
NC
10^{-1}
10^{-2}
$B \simeq 6T$
$n \simeq 10^{20}\ m^{-3}$
$L_\perp \simeq 0.1\ m$
$L_{\parallel} \simeq 20\ m$
$1\ KeV \lesssim T_i \lesssim 10\ KeV$

SECOND-FLR-ORDER ION DRIFT-KINETIC EQUATION

- **DERIVED IN [Ramos PoP 2011] FOR GENERAL GEOMETRY, WITH THE ORDERINGS**

$$\nu_\iota L / v_{th\iota} \sim (m_e/m_\iota)^{1/2} \sim u_\iota / v_{th\iota} \sim \rho_\iota / L \equiv \delta \ll 1$$

- **NEAR-MAXWELLIAN (delta-f) FORMULATION IN THE REFERENCE FRAME OF THE MEAN FLOW VELOCITY**

$$f_\iota(\mathbf{w}, \mathbf{x}, t) \ = f_{M\iota}(w, \mathbf{x}, t) \ + \ f_{NM\iota}(\mathbf{w}, \mathbf{x}, t) \ = \ \left(\frac{m_\iota}{2\pi}\right)^{3/2} \frac{n(\mathbf{x}, t)}{T(\mathbf{x}, t)^{3/2}} \ \exp\left[-\frac{m_\iota w^2}{2T(\mathbf{x}, t)}\right] \ + \ f_{NM\iota}(\mathbf{w}, \mathbf{x}, t)$$

$$\mathbf{w} \ = \ \mathbf{v} \ - \ \mathbf{u}_\iota(\mathbf{x}, t) \ , \qquad\qquad f_{NM\iota} \sim \delta f_{M\iota}$$

- **CHAPMAN-ENSKOG-LIKE DESCRIPTION WITH THE DENSITY, MEAN FLOW VELOCITY AND TEMPERATURE CARRIED ENTIRELY BY THE MAXWELLIAN. THE DRIFT-KINETIC EQUATION FOR THE GYROPHASE-AVERAGED** $\bar{f}_{NM\iota}(w_\parallel, w_\perp, \mathbf{x}, t)$ **PRESERVES EXACTLY THE CONSTRAINTS** $\int d^3\mathbf{w}(1, w_\parallel, w^2)\bar{f}_{NM\iota} = 0$.

$$\frac{\partial \bar{f}_{NM\iota}}{\partial t} + \dot{\mathbf{x}}_\iota \cdot \frac{\partial \bar{f}_{NM\iota}}{\partial \mathbf{x}} + \dot{w}_{\|\iota} \frac{\partial \bar{f}_{NM\iota}}{\partial w_\|} + \dot{w}_{\perp\iota} \frac{\partial \bar{f}_{NM\iota}}{\partial w_\perp} = D_\iota \; f_{M\iota} + \mathcal{C}_\iota[\bar{f}_{NM\iota}]$$

where

$$\mathcal{C}_\iota[\bar{f}_{NM\iota}] = C_{\iota\iota}[f_{M\iota}, \bar{f}_{NM\iota}] + C_{\iota\iota}[\bar{f}_{NM\iota}, f_{M\iota}] = O\left(\delta^2 \frac{v_{th\iota}}{L} f_{M\iota}\right)$$

$$D_\iota = O\left(\delta \frac{v_{th\iota}}{L}\right) + O\left(\delta^2 \frac{v_{th\iota}}{L}\right)$$

$$\dot{\mathbf{x}}_\iota = w_\| \mathbf{b} + \mathbf{u}_\iota - \mathbf{u}_{D\iota} + \frac{w_\perp^2}{2} \nabla \times \left(\frac{\mathbf{b}}{\Omega_{c\iota}}\right) + \left(w_\|^2 - \frac{v_\perp'^2}{2}\right) \frac{\mathbf{b} \times \boldsymbol{\kappa}}{\Omega_{c\iota}} = O\big(v_{th\iota}\big) + O\big(\delta v_{th\iota}\big)$$

$$\dot{w}_{\|\iota} = \frac{\mathbf{b} \cdot (\nabla \cdot \mathsf{P}_\iota^{CGL})}{m_\iota n} - \frac{w_\perp^2}{2} \mathbf{b} \cdot \nabla \ln B - w_\| (\mathbf{bb}) : \nabla(\mathbf{u}_\iota - \mathbf{u}_{D\iota}) + \frac{w_\| w_\perp^2}{2} \nabla \cdot \left(\frac{\mathbf{b} \times \boldsymbol{\kappa}}{\Omega_{c\iota}}\right) = O\left(\frac{v_{th\iota}^2}{L}\right) + O\left(\delta \frac{v_{th\iota}^2}{L}\right)$$

$$\dot{w}_{\perp\iota} = \frac{w_\perp}{2} \left[w_\| \mathbf{b} \cdot \nabla \ln B + (\mathbf{bb} - \mathsf{I}) : \nabla(\mathbf{u}_\iota - \mathbf{u}_{D\iota}) - w_\|^2 \nabla \cdot \left(\frac{\mathbf{b} \times \boldsymbol{\kappa}}{\Omega_{c\iota}}\right)\right] = O\left(\frac{v_{th\iota}^2}{L}\right) + O\left(\delta \frac{v_{th\iota}^2}{L}\right)$$

with $\mathbf{u}_{D\iota} = \dfrac{\mathbf{b} \times \nabla(n T_\iota)}{m_\iota n \Omega_{c\iota}}$ **and** $\mathsf{P}_\iota^{CGL} = n T_\iota \mathsf{I} + (p_{\iota\|} - p_{\iota\perp})(\mathbf{bb} - \mathsf{I}/3)$

AXISYMMETRIC EQUILIBRIUM

THE EQUATIONS FOR THE ELECTROMAGNETIC FIELDS AND THE FLUID VARIABLES IN AN AXISYMMETRIC EQUILIBRIUM [with cylindrical coordinates (R, ζ, Z)] YIELD

$$\mathbf{B} \;=\; \nabla\psi \times \nabla\zeta \;+\; RB_\zeta \nabla\zeta \;, \qquad RB_\zeta \;=\; I(\psi)[1 + O(\delta^2)]$$

$$\mathbf{E} \;=-\; \nabla\phi \;-\; V_0 \nabla\zeta \;, \qquad e\phi \;=\; e\Phi(\psi) \;+\; O(\delta^2 n m_\iota v_{th\iota}^2) \;, \qquad eV_0 \;=\; O(\delta^3 n m_\iota v_{th\iota}^2)$$

$$n \;=\; N(\psi)[1 + O(\delta^2)]$$

$$T_\iota \;=\; T_\iota^{(0)}(\psi)[1 + O(\delta^2)]$$

$$\mathbf{u}_\iota \;=\; U_\iota(\psi)\mathbf{B} \;+\; R^2\Omega_\iota(\psi)\nabla\zeta \;+\; O(\delta^3 v_{th\iota}) \;, \qquad \Omega_\iota(\psi) \;\equiv\; \frac{d\Phi}{d\psi} + \frac{1}{eN}\frac{d(NT_\iota^{(0)})}{d\psi}$$

AXISYMMETRIC EQUILIBRIUM

THE EQUATIONS FOR THE ELECTROMAGNETIC FIELDS AND THE FLUID VARIABLES IN AN AXISYMMETRIC EQUILIBRIUM [with cylindrical coordinates (R, ζ, Z)] YIELD

$$\mathbf{B} = \nabla\psi \times \nabla\zeta + RB_\zeta \nabla\zeta \, , \qquad RB_\zeta = I(\psi)[1 + O(\delta^2)]$$

$$\mathbf{E} = -\nabla\phi - V_0 \nabla\zeta \, , \qquad e\phi = e\Phi(\psi) + O(\delta^2 n m_\iota v_{th\iota}^2) \, , \qquad eV_0 = O(\delta^3 n m_\iota v_{th\iota}^2)$$

$$n = N(\psi)[1 + O(\delta^2)]$$

$$T_\iota = T_\iota^{(0)}(\psi)[1 + O(\delta^2)]$$

$$\mathbf{u}_\iota = U_\iota(\psi)\mathbf{B} + R^2 \Omega_\iota(\psi) \nabla\zeta + O(\delta^3 v_{th\iota}) \, , \qquad \Omega_\iota(\psi) \equiv \frac{d\Phi}{d\psi} + \frac{1}{eN} \frac{d(NT_\iota^{(0)})}{d\psi}$$

THE PARALLEL COMPONENT OF THE AXISYMMETRIC EQUILIBRIUM ION MOMENTUM EQUATION BECOMES

$$\mathbf{b} \cdot \left\{ \nabla \left[m_\iota n \left(\frac{u_\iota^2}{2} - \Omega_\iota R^2 \right) - en\phi + nT_\iota + \frac{2}{3}(p_{\iota\parallel} - p_{\iota\perp}) \right] - (p_{\iota\parallel} - p_{\iota\perp}) \nabla \ln B + \nabla \cdot \mathbf{P}_\iota^{GV} \right\} = O\left(\delta^3 \frac{n m_\iota v_{th\iota}^2}{L} \right)$$

IN THE AXISYMMETRIC EQUILIBRIUM, THE NON-MAXWELLIAN PART OF THE ION DISTRIBUTION FUNCTION HAS THE FORM

$$\bar{f}_{NM\iota} = \bar{f}_{NM\iota}^{odd} + \bar{f}_{NM\iota}^{even} = O(\delta f_{M\iota}) + O(\delta^2 f_{M\iota})$$

IN THE AXISYMMETRIC EQUILIBRIUM, THE NON-MAXWELLIAN PART OF THE ION DISTRIBUTION FUNCTION HAS THE FORM

$$\bar{f}_{NM\iota} = \bar{f}^{odd}_{NM\iota} + \bar{f}^{even}_{NM\iota} = O(\delta f_{M\iota}) + O(\delta^2 f_{M\iota})$$

AND, SEPARATING ITS EVEN AND ODD PARTS WITH RESPECT TO $w_\parallel$, THE DRIFT-KINETIC EQUATION BECOMES

$$\left[w_\parallel \mathbf{b}\cdot\frac{\partial}{\partial \mathbf{x}} + \frac{w_\perp}{2}\mathbf{b}\cdot\nabla\ln B\left(-w_\perp\frac{\partial}{\partial w_\parallel} + w_\parallel\frac{\partial}{\partial w_\perp}\right)\right]\bar{f}^{odd}_{NM\iota} = D^{even}_\iota f^{(0)}_{M\iota}$$

$$\left[w_\parallel \mathbf{b}\cdot\frac{\partial}{\partial \mathbf{x}} + \frac{w_\perp}{2}\mathbf{b}\cdot\nabla\ln B\left(-w_\perp\frac{\partial}{\partial w_\parallel} + w_\parallel\frac{\partial}{\partial w_\perp}\right)\right]\bar{f}^{even}_{NM\iota} +$$

$$+ \left(\dot{\mathbf{x}}^{(1)}_\iota\cdot\frac{\partial}{\partial \mathbf{x}} + \dot{w}^{(1)}_{\parallel\iota}\frac{\partial}{\partial w_\parallel} + \dot{w}^{(1)}_{\perp\iota}\frac{\partial}{\partial w_\perp}\right)\bar{f}^{odd}_{NM\iota} = D^{odd}_\iota f^{(0)}_{M\iota} + \mathcal{C}_\iota[\bar{f}^{odd}_{NM\iota}]$$

IN THE AXISYMMETRIC EQUILIBRIUM, THE NON-MAXWELLIAN PART OF THE ION DISTRIBUTION FUNCTION HAS THE FORM

$$\bar{f}_{NM\iota} \;=\; \bar{f}^{odd}_{NM\iota} \;+\; \bar{f}^{even}_{NM\iota} \;=\; O(\delta f_{M\iota}) + O(\delta^2 f_{M\iota})$$

AND, SEPARATING ITS EVEN AND ODD PARTS WITH RESPECT TO $w_\parallel$, THE DRIFT-KINETIC EQUATION BECOMES

$$\left[w_\parallel \mathbf{b}\cdot\frac{\partial}{\partial\mathbf{x}} \;+\; \frac{w_\perp}{2}\mathbf{b}\cdot\nabla\ln B\left(-w_\perp\frac{\partial}{\partial w_\parallel} \;+\; w_\parallel\frac{\partial}{\partial w_\perp}\right)\right]\bar{f}^{odd}_{NM\iota} \;=\; D^{even}_\iota f^{(0)}_{M\iota}$$

$$\left[w_\parallel \mathbf{b}\cdot\frac{\partial}{\partial\mathbf{x}} \;+\; \frac{w_\perp}{2}\mathbf{b}\cdot\nabla\ln B\left(-w_\perp\frac{\partial}{\partial w_\parallel} \;+\; w_\parallel\frac{\partial}{\partial w_\perp}\right)\right]\bar{f}^{even}_{NM\iota} \;+$$

$$+\;\left(\dot{\mathbf{x}}^{(1)}_\iota\cdot\frac{\partial}{\partial\mathbf{x}} \;+\; \dot{w}^{(1)}_{\parallel\iota}\,\frac{\partial}{\partial w_\parallel} \;+\; \dot{w}^{(1)}_{\perp\iota}\,\frac{\partial}{\partial w_\perp}\right)\bar{f}^{odd}_{NM\iota} \;=\; D^{odd}_\iota f^{(0)}_{M\iota} \;+\; \mathcal{C}_\iota[\bar{f}^{odd}_{NM\iota}]$$

where

$$D^{even}_\iota \;=\; -\left[\left(\frac{2w_\parallel^2 - w_\perp^2}{2v_{th\iota}^2}\right)U_\iota B \;+\; \frac{I}{4eB}\frac{dT^{(0)}_\iota}{d\psi}\left(\frac{w^2}{v_{th\iota}^2} - 5\right)\left(\frac{2w_\parallel^2 + w_\perp^2}{v_{th\iota}^2}\right)\right]\mathbf{b}\cdot\nabla\ln B$$

with $v_{th\iota}^2 \equiv T^{(0)}_\iota/m_\iota$

THE EVEN PART OF THE DRIFT-KINETIC EQUATION

$$\left[w_{\|} \mathbf{b} \cdot \frac{\partial}{\partial \mathbf{x}} \;+\; \frac{w_{\perp}}{2} \mathbf{b} \cdot \nabla \ln B \left(-w_{\perp} \frac{\partial}{\partial w_{\|}} \;+\; w_{\|} \frac{\partial}{\partial w_{\perp}} \right) \right] \bar{f}^{odd}_{NM\iota} \;=$$

$$= \; - \left[\left(\frac{2w_{\|}^2 - w_{\perp}^2}{2v_{th\iota}^2} \right) U_{\iota} B \;+\; \frac{I}{4eB} \frac{dT_{\iota}^{(0)}}{d\psi} \left(\frac{w^2}{v_{th\iota}^2} - 5 \right) \left(\frac{2w_{\|}^2 + w_{\perp}^2}{v_{th\iota}^2} \right) \right] \mathbf{b} \cdot \nabla \ln B \;\, f^{(0)}_{M\iota}$$

HAS THE GENERAL SOLUTION

$$\bar{f}^{odd}_{NM\iota} \;=\; w_{\|} \; g_{\iota,1}(w, \psi, B) \; f^{(0)}_{M\iota}(w, \psi) \;+\; \varsigma \; H[1 - \lambda B_{max}(\psi)] \; K_{\iota}(w, \lambda, \psi)$$

where

$$g_{\iota,1}(w, \psi, B) \;=\; \frac{1}{v_{th\iota}^2} \left[-U_{\iota} B \;+\; \frac{I}{2eB} \frac{dT_{\iota}^{(0)}}{d\psi} \left(\frac{w^2}{v_{th\iota}^2} - 5 \right) \right]$$

$$\varsigma \;=\; \pm 1 \;=\; \mathrm{sign}(w_{\|}) \; , \qquad\qquad \lambda \;=\; \frac{w_{\perp}^2}{w^2 B}$$

and H is the Heaviside step function

$$\int d^3\mathbf{w} \; w_{\|} \; \bar{f}^{odd}_{NM\iota} \;=\; 0 \qquad \Rightarrow \qquad U_{\iota} \;=\; \frac{2\pi}{N} \int_0^{\infty} dw \; w^3 \int_0^{1/B_{max}} d\lambda \; K_{\iota}(w, \lambda, \psi)$$

CALL

$$\bar{f}_{NM\iota}^{even} \;=\; -\left[\frac{e(\phi-\Phi)}{T_\iota^{(0)}}+\frac{n-N}{N}+\left(\frac{w^2}{v_{th\iota}^2}-3\right)\;\frac{T_\iota-T_\iota^{(0)}}{2T_\iota^{(0)}}\right]f_{M\iota}^{(0)}\;+\;h_\iota^{even}$$

CALL

$$\bar{f}^{even}_{NM\iota} \;=\; -\left[\frac{e(\phi-\Phi)}{T_\iota^{(0)}} + \frac{n-N}{N} + \left(\frac{w^2}{v_{th\iota}^2} - 3\right)\; \frac{T_\iota - T_\iota^{(0)}}{2T_\iota^{(0)}}\right] f^{(0)}_{M\iota} \;+\; h^{even}_\iota$$

AFTER SUBSTITUTING THE PARALLEL COMPONENT EQUILIBRIUM MOMENTUM EQUATION FOR $\mathrm{b}\cdot\nabla(e\phi/T_\iota^{(0)}+n/N)$, THE ODD PART OF THE DRIFT-KINETIC EQUATION BECOMES

$$w_\parallel\;\mathbf{b}\cdot\frac{\partial h^{even}_\iota}{\partial\mathbf{x}}\Big|_{w,\lambda} \;=\; \left[\hat{D}^{odd}_\iota \;-\; \frac{m_\iota w_\parallel}{T_\iota^{(0)}}\;\mathbf{b}\cdot\nabla\left(\frac{u_\iota^2}{2} - \Omega_\iota R^2\right)\right] f^{(0)}_{M\iota} \;+$$

$$+\;\mathcal{C}_\iota[\bar{f}^{odd}_{NM\iota}] \;-\; \left(\dot{\mathbf{x}}^{(1)}_\iota\cdot\frac{\partial}{\partial\mathbf{x}} \;+\; \dot{w}^{(1)}_{\parallel\iota}\,\frac{\partial}{\partial w_\parallel} \;+\; \dot{w}^{(1)}_{\perp\iota}\,\frac{\partial}{\partial w_\perp}\right)\bar{f}^{odd}_{NM\iota}$$

where

$$\hat{D}^{odd}_\iota \;=\; D^{odd}_\iota \;+\; \frac{w_\parallel}{NT_\iota^{(0)}}\mathbf{b}\cdot\left[\frac{N}{2}\left(\frac{w^2}{v_{th\iota}^2} - 5\right)\nabla T_\iota - \frac{2}{3}\nabla(p_{\iota\parallel} - p_{\iota\perp}) + (p_{\iota\parallel} - p_{\iota\perp})\nabla\ln B - \nabla\cdot\mathbf{P}^{GV}_\iota\right]$$

CALL

$$\bar{f}^{even}_{NM\iota} \;=\; -\left[\frac{e(\phi-\Phi)}{T_\iota^{(0)}}+\frac{n-N}{N}+\left(\frac{w^2}{v_{th\iota}^2}-3\right)\;\frac{T_\iota-T_\iota^{(0)}}{2T_\iota^{(0)}}\right] f^{(0)}_{M\iota} \;+\; h^{even}_\iota$$

AFTER SUBSTITUTING THE PARALLEL COMPONENT EQUILIBRIUM MOMENTUM EQUATION FOR $\mathrm{b}\cdot\nabla(e\phi/T_\iota^{(0)}+n/N)$**, THE ODD PART OF THE DRIFT-KINETIC EQUATION BECOMES**

$$w_\parallel\;\mathbf{b}\cdot\frac{\partial h^{even}_\iota}{\partial\mathbf{x}}\Big|_{w,\lambda} \;=\; \left[\hat{D}^{odd}_\iota \;-\; \frac{m_\iota w_\parallel}{T_\iota^{(0)}}\;\mathbf{b}\cdot\nabla\left(\frac{u_\iota^2}{2}-\Omega_\iota R^2\right)\right] f^{(0)}_{M\iota} \;+$$

$$+\;\mathcal{C}_\iota[\bar{f}^{odd}_{NM\iota}] \;-\; \left(\dot{\mathbf{x}}^{(1)}_\iota\cdot\frac{\partial}{\partial\mathbf{x}} \;+\; \dot{w}^{(1)}_{\parallel\iota}\,\frac{\partial}{\partial w_\parallel} \;+\; \dot{w}^{(1)}_{\perp\iota}\,\frac{\partial}{\partial w_\perp}\right)\bar{f}^{odd}_{NM\iota}$$

where

$$\hat{D}^{odd}_\iota \;=\; D^{odd}_\iota \;+\; \frac{w_\parallel}{NT_\iota^{(0)}}\mathbf{b}\cdot\left[\frac{N}{2}\left(\frac{w^2}{v_{th\iota}^2}-5\right)\nabla T_\iota-\frac{2}{3}\nabla(p_{\iota\parallel}-p_{\iota\perp})+(p_{\iota\parallel}-p_{\iota\perp})\nabla\ln B-\nabla\cdot\mathbf{P}^{GV}_\iota\right]$$

THIS HAS THE SOLVABILITY CONDITION

$$\oint_{w,\lambda}\frac{d\ell}{w_\parallel}\;\left\{\hat{D}^{odd}_\iota f^{(0)}_{M\iota} \;+\; \mathcal{C}_\iota[\bar{f}^{odd}_{NM\iota}] \;-\; \left(\dot{\mathbf{x}}^{(1)}_\iota\cdot\frac{\partial}{\partial\mathbf{x}} \;+\; \dot{w}^{(1)}_{\parallel\iota}\,\frac{\partial}{\partial w_\parallel} \;+\; \dot{w}^{(1)}_{\perp\iota}\,\frac{\partial}{\partial w_\perp}\right)\bar{f}^{odd}_{NM\iota}\right\} \;=\; 0$$

THE EXPLICIT FORM OF THE ODD DISTRIBUTION FUNCTION SOLUTION

$$\bar{f}^{odd}_{NM\iota} \;=\; w_\| \; g_{\iota,1}(w,\psi,B) \; f^{(0)}_{M\iota}(w,\psi) \;+\; \varsigma \; H[1-\lambda B_{max}(\psi)] \; K_\iota(w,\lambda,\psi) \;=\; \bar{f}^{odd}_{NM\iota}(w_\|, w_\perp, \psi, B)$$

AND THE FIRST-ORDER DRIFT-KINETIC COEFFICIENT FUNCTIONS YIELD

$$\oint_{w,\lambda} \frac{d\ell}{w_\|} \left(\dot{\mathbf{x}}^{(1)}_\iota \cdot \frac{\partial}{\partial \mathbf{x}} \;+\; \dot{w}^{(1)}_{\|\iota} \frac{\partial}{\partial w_\|} \;+\; \dot{w}^{(1)}_{\perp\iota} \frac{\partial}{\partial w_\perp} \right) \bar{f}^{odd}_{NM\iota} \;=\; 0$$

THE EXPLICIT FORM OF THE ODD DISTRIBUTION FUNCTION SOLUTION

$$\bar{f}^{odd}_{NM\iota} \;=\; w_\parallel \; g_{\iota,1}(w,\psi,B)\; f^{(0)}_{M\iota}(w,\psi) \;+\; \varsigma\; H[1-\lambda B_{max}(\psi)]\; K_\iota(w,\lambda,\psi) \;=\; \bar{f}^{odd}_{NM\iota}(w_\parallel, w_\perp, \psi, B)$$

AND THE FIRST-ORDER DRIFT-KINETIC COEFFICIENT FUNCTIONS YIELD

$$\oint_{w,\lambda} \frac{d\ell}{w_\parallel} \left(\dot{\mathbf{x}}^{(1)}_\iota \cdot \frac{\partial}{\partial \mathbf{x}} \;+\; \dot{w}^{(1)}_{\parallel\iota} \frac{\partial}{\partial w_\parallel} \;+\; \dot{w}^{(1)}_{\perp\iota} \frac{\partial}{\partial w_\perp} \right) \bar{f}^{odd}_{NM\iota} \;=\; 0$$

SO THE SOLVABILITY CONDITION THAT DETERMINES $K_\iota(w,\lambda,\psi)$ REDUCES TO

$$\oint_{w,\lambda} \frac{d\ell}{w_\parallel}\, \mathcal{C}_\iota[\varsigma H K_\iota] \;=\; -\; S_\iota$$

where

$$S_\iota \;=\; S^{coll}_\iota \;+\; \oint_{w,\lambda} \frac{d\ell}{w_\parallel}\; \hat{D}^{odd}_\iota \; f^{(0)}_{M\iota}$$

AND S^{coll}_ι IS THE STANDARD BANANA REGIME SOURCE

$$S^{coll}_\iota \;=\; \oint_{w,\lambda} \frac{d\ell}{w_\parallel}\, \mathcal{C}_\iota\left[w_\parallel g_{\iota,1} f^{(0)}_{M\iota} \right] \;=\; -\; \frac{2\nu_\iota m_\iota I}{e T^{(0)}_\iota} \frac{dT^{(0)}_\iota}{d\psi} \frac{v_{th\iota}}{w} \left[\varphi\left(\frac{w}{v_{th\iota}}\right) - 5\xi\left(\frac{w}{v_{th\iota}}\right) \right] f^{(0)}_{M\iota} \; \oint \frac{d\ell}{B}$$

with φ and ξ the error and Chandrasekhar functions

THE ADDITIONAL SOURCE IN THE NEW REGIME IS GIVEN BY

$$
\begin{aligned}
\frac{\hat{D}_\iota^{odd}}{w_\parallel} = & - \frac{\lambda B w^2}{2NT_\iota^{(0)} v_{th\iota}^2} \nabla \cdot \left\{ \frac{NT_\iota^{(0)}}{\Omega_{c\iota}} \mathbf{b} \times [2(\mathbf{b}\cdot\nabla)\mathbf{u}_\iota + \mathbf{b} \times (\nabla \times \mathbf{u}_\iota)] \right\} - \\
& - \frac{w^2}{\Omega_{c\iota} v_{th\iota}^2} \left\{ (1 - 2\lambda B)\ (\mathbf{b} \times \boldsymbol{\kappa}) \cdot [2(\mathbf{b}\cdot\nabla)\mathbf{u}_\iota + \mathbf{b} \times (\nabla \times \mathbf{u}_\iota)] \ + \ \frac{\lambda B}{4} \mathbf{M}_\times : (\nabla \mathbf{u}_\iota) \right\} + \\
& + \left(1 - \frac{\lambda B w^2}{2 v_{th\iota}^2}\right) \left[\frac{\mathbf{b}}{\Omega_{c\iota}} \times \nabla \ln (NT_\iota^{(0)})\right] \cdot [2(\mathbf{b}\cdot\nabla)\mathbf{u}_\iota + \mathbf{b} \times (\nabla \times \mathbf{u}_\iota)] + \left(\frac{w^2}{v_{th\iota}^2} - 5\right) \left[\frac{\mathbf{b}}{\Omega_{c\iota}} \times \nabla \ln T_\iota^{(0)}\right] \cdot [(\mathbf{b}\cdot\nabla)\mathbf{u}_\iota] \\
& + \frac{\lambda B w^2}{4NT_\iota^{(0)2}} \left(\frac{w^2}{v_{th\iota}^2} - 5\right) \nabla \cdot \left(\frac{NT_\iota^{(0)} j_\parallel}{\Omega_{c\iota}^2 B} \mathbf{b} \times \nabla T_\iota^{(0)}\right) \ + \ \frac{j_\parallel w^2}{2\Omega_{c\iota}^2 B} \left(\frac{w^2}{v_{th\iota}^2} - 5\right) (1 - 2\lambda B)(\mathbf{b} \times \boldsymbol{\kappa}) \cdot \nabla \ln T_\iota^{(0)} + \\
& + \frac{\lambda B w^2}{8\Omega_{c\iota}^2} \mathbf{M} : \left[\left(\frac{w^2}{v_{th\iota}^2} - 7\right) \nabla \ln N \ \nabla \ln T_\iota^{(0)} \ - \ \left(\frac{w^4}{2v_{th\iota}^4} - \frac{8w^2}{v_{th\iota}^2} + \frac{49}{2}\right) \nabla \ln T_\iota^{(0)} \ \nabla \ln T_\iota^{(0)}\right] - \\
& - \frac{\lambda B w^2}{8\Omega_{c\iota} N T_\iota^{(0)2}} \left(\frac{w^2}{v_{th\iota}^2} - 5\right) \mathbf{M} : \left[\nabla \left(\frac{NT_\iota^{(0)}}{\Omega_{c\iota}} \nabla T_\iota^{(0)}\right)\right]
\end{aligned}
$$

where

$$\mathbf{M}_{jk} = \frac{\partial b_k}{\partial x_j} + \frac{\partial b_j}{\partial x_k} - (\nabla \cdot \mathbf{b})(\delta_{jk} - b_j b_k) - (b_j \kappa_k + b_k \kappa_j)$$

and

$$\mathbf{M}_{\times jk} = \mathbf{M}_{jl}\ \epsilon_{lkm} b_m$$

ITS MAGNETIC SURFACE AVERAGE YIELDS

$$S_\iota \; - \; S_\iota^{coll} \; = \; \oint_{w,\lambda} \frac{d\ell}{w_\|} \, \hat{D}_\iota^{odd} \, f_{M\iota}^{(0)} \; = \; \frac{m_\iota^2 \lambda w^2}{8 e^2 T_\iota^{(0)}} \, \frac{dT_\iota^{(0)}}{d\psi} \left(\frac{w^2}{v_{th\iota}^2} - 5 \right) f_{M\iota}^{(0)} \; \Gamma(\psi)$$

where

$$\Gamma(\psi) \; = \; - \; \oint d\ell \; \mathbf{M} : \left[\nabla \left(\frac{1}{B} \nabla \psi \right) \right] \; =$$

$$= \; 2 \, \frac{d}{d\psi} \oint d\ell \; \mathbf{B} \cdot \nabla R^2 \; + \; \oint \frac{d\ell}{B} \; \left[2 \mathbf{b} \cdot \nabla (\nabla \psi \cdot \nabla \ln R^2) \; + \; \nabla \psi \cdot \nabla (\mathbf{b} \cdot \nabla \ln B) \right]$$

IS A GEOMETRICAL FACTOR THAT VANISHES IF THE MAGNETIC SURFACE CONFIGURATION HAS UP-DOWN SYMMETRY

CONCLUSIONS

- **IN AN AXISYMMETRIC EQUILIBRIUM WITH FURTHER UP-DOWN SYMMETRY, THE BANANA REGIME RESULT FOR THE ION PARALLEL FLOW COEFFICIENT**

$$\alpha = \frac{e \, \langle B^2 \rangle \, U_\iota}{I \, dT_\iota^{(0)}/d\psi} \;=\; \frac{2\pi \; e \, \langle B^2 \rangle}{I \;\, N \;\, dT_\iota^{(0)}/d\psi} \; \int_0^\infty dw \;\, w^3 \int_0^{1/B_{max}} d\lambda \;\, K_\iota \; ,$$

WHERE K_ι **IS THE SOLUTION OF THE GENERALIZED SPITZER PROBLEM**

$$\oint_{w,\lambda} \frac{d\ell}{w_\parallel} \, \mathcal{C}_\iota[\varsigma H K_\iota] \;=\; \frac{2\nu_\iota m_\iota I}{eT_\iota^{(0)}} \, \frac{dT_\iota^{(0)}}{d\psi} \, \frac{v_{th\iota}}{w} \left[\varphi\left(\frac{w}{v_{th\iota}}\right) - 5\xi\left(\frac{w}{v_{th\iota}}\right)\right] f_{M\iota}^{(0)} \; \oint \frac{d\ell}{B} \; ,$$

REMAINS APPLICABLE TO THE LOWER COLLISIONALITY REGIME $\nu_* \sim \rho_\iota/L$

- **IF THE EQUILIBRIUM IS NOT UP-DOWN SYMMETRIC, THE GENERALIZED SPITZER PROBLEM HAS AN ADDITIONAL SECOND-FLR-ORDER SOURCE FOR** $\nu_* \sim \rho_\iota / L$

$$\oint_{w,\lambda} \frac{d\ell}{w_\parallel}\, \mathcal{C}_\iota[\varsigma H K_\iota] = \frac{m_\iota}{eT_\iota^{(0)}} \frac{dT_\iota^{(0)}}{d\psi}\, f_{M\iota}^{(0)} \left\{ 2\nu_\iota \frac{v_{th\iota}}{w} \left[\varphi\left(\frac{w}{v_{th\iota}}\right) - 5\xi\left(\frac{w}{v_{th\iota}}\right)\right] I \oint \frac{d\ell}{B} - \frac{m_\iota \lambda w^2}{8e}\left(\frac{w^2}{v_{th\iota}^2} - 5\right)\Gamma \right\}$$

where

$$\Gamma = 2\frac{d}{d\psi}\oint d\ell\ \mathbf{B}\cdot\nabla R^2 + \oint \frac{d\ell}{B}\left[2\mathbf{b}\cdot\nabla(\nabla\psi\cdot\nabla\ln R^2) + \nabla\psi\cdot\nabla(\mathbf{b}\cdot\nabla\ln B)\right]$$

- **IF THE EQUILIBRIUM IS NOT UP-DOWN SYMMETRIC, THE GENERALIZED SPITZER PROBLEM HAS AN ADDITIONAL SECOND-FLR-ORDER SOURCE FOR** $\nu_* \sim \rho_\iota / L$

$$\oint_{w,\lambda} \frac{d\ell}{w_\|}\, \mathcal{C}_\iota[\varsigma H K_\iota] \;=\; \frac{m_\iota}{eT_\iota^{(0)}} \frac{dT_\iota^{(0)}}{d\psi}\, f_{M\iota}^{(0)} \left\{ 2\nu_\iota \frac{v_{th\iota}}{w} \left[\varphi\left(\frac{w}{v_{th\iota}}\right) - 5\xi\left(\frac{w}{v_{th\iota}}\right) \right] I \oint \frac{d\ell}{B} - \frac{m_\iota \lambda w^2}{8e} \left(\frac{w^2}{v_{th\iota}^2} - 5 \right) \Gamma \right\}$$

where

$$\Gamma \;=\; 2\,\frac{d}{d\psi} \oint d\ell\; \mathbf{B}\cdot\nabla R^2 \;+\; \oint \frac{d\ell}{B}\; \left[2\mathbf{b}\cdot\nabla(\nabla\psi\cdot\nabla \ln R^2) \;+\; \nabla\psi\cdot\nabla(\mathbf{b}\cdot\nabla \ln B) \right]$$

- **TERMS ASSOCIATED WITH THE EQUILIBRIUM FLOW AND WITH FIRST-ORDER DRIFTS ACTING ON THE NON-MAXWELLIAN PART OF THE DISTRIBUTION FUNCTION ARE COMPARABLE TO THE RETAINED SECOND-ORDER TERMS, BUT THEIR CONTRIBUTION AVERAGES OUT AND HAVE ZERO NET EFFECT**