Implementation of Linear Neoclassical Inner Region Model in the DCON Package

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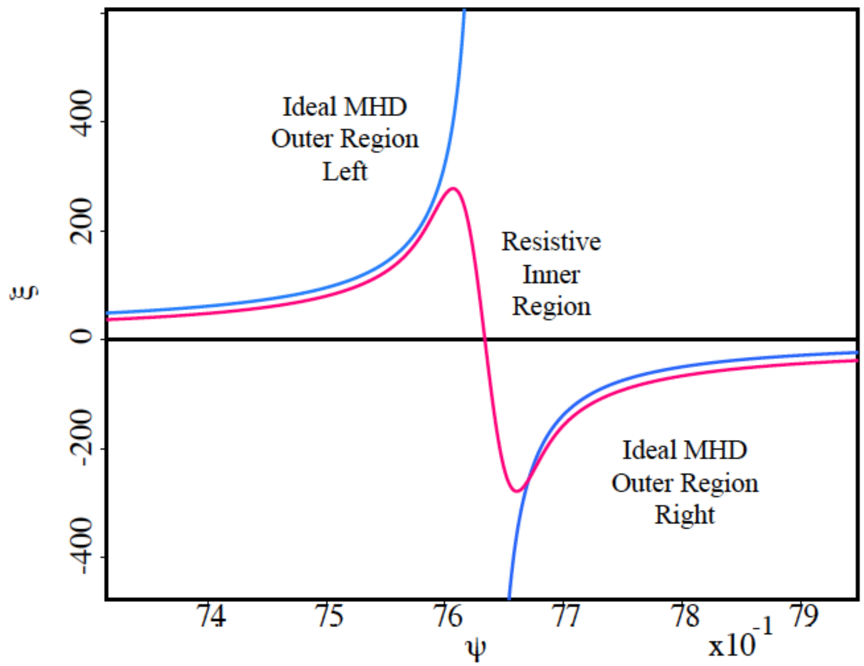
Matched Asymptotic Expansions



- Lundquist number $S = \tau_R/\tau_A >> 1$.
- Singular surface at $m = n\, q(\psi)$.
- Partition domain into multiple inner and outer regions, $x/a \sim \gamma\tau_A \sim S^{-1/3}$.
- Ideal MHD dominates outer regions.
- Thin inner regions can include resistivity, inertia, viscosity, finite Larmor radius, kinetic effects, nonlinearity, etc.
- Inner and outer regions are matched through asymptotic coefficients, matching data.
- Benchmarks with straight-through MARS code: excellent agreement, ~100x faster.

Previous Work

- Ideal DCON integrates the 2D Newcomb equation to determine the ideal MHD stability of axisymmetric toroidal plasmas.
- Resistive DCON solves the same equation in the outer region, using a different algorithm, a singular Galerkin method, à la Pletzer and Dewar. Determines outer region matching data. Re-uses most of the DCON infrastructure.
- Greatly improved convergence and speed compared to PEST III, due to advanced basis functions and grid packing scheme.
- Coupled to Morrell Chance's VACUUM code for free-boundary modes.
- Inner region equations of Glasser, Greene, and Johnson solved with new DELTAC code, more robust and reliable than older DELTAR an INNER codes.
- MATCH code couples outer region DCON and innner region DELTAC, finds multiple roots of the dispersion relation, computes and draws global eigenfunctions.
- Compared to straight-through MARS code. Excellent agreement with multiple singular surfaces, linear combination of even and odd modes. About 100x faster.

Inner Region: Beyond Resistive DCON

Fields

$$\mathbf{E} = -\partial_t \mathbf{A} - \nabla \varphi, \quad \mathbf{b} = \nabla \times \mathbf{A}$$
$$\nabla \cdot \mathbf{j} = \nabla \cdot \mathbf{A} = 0, \quad \mathbf{j} = -\nabla^2 \mathbf{A}$$

Equilibrium Toroidal Rotation

$$\mathbf{V} = \Omega(\psi) R^2 \nabla \phi, \quad \mathbf{V} \cdot \nabla = \Omega(\psi) \partial_\phi$$

Density and Pressure

$$(\partial_t + \Omega \partial_\phi)\rho + \nabla \cdot (\rho \mathbf{v}) = 0$$
$$(\partial_t + \Omega \partial_\phi) p + \mathbf{v} \cdot \nabla P + \gamma P \nabla \cdot \mathbf{v} + \tfrac{2}{3}\left(\nabla \cdot \mathbf{q} + \mathbf{R} \cdot \mathbf{u} + \pi : \nabla \mathbf{v}\right) = 0$$

Momentum Conservation and Ohm's Law

$$\rho(\partial_t + \Omega \partial_\phi)\mathbf{v} = \mathbf{j} \times \mathbf{B} + \mathbf{J} \times \mathbf{b} - \nabla p - \nabla \cdot \boldsymbol{\pi}$$
$$\mathbf{E} + \mathbf{v} \times \mathbf{B} + \mathbf{V} \times \mathbf{b} = \mathbf{R} + \frac{1}{N_e e}\left(\mathbf{j} \times \mathbf{B} + \mathbf{J} \times \mathbf{b} - \nabla p_e - \nabla \cdot \boldsymbol{\pi}_e\right)$$

Short mean free path: Braginskii.
Long mean free path: neoclassical; Connor, Hastie & Helander.

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Neoclassical Inner Region

Reference

"Linear tearing mode stability equations for a low collisionality toroidal plasma,"
J. W. Connor, R. J. Hastie, and P. Helander, Plasma Phys. Control. Fusion 51, 015009 (2009).

Linearized Gyrokinetic Equation

$$\left(v_\parallel \mathbf{b} + \mathbf{v}_{dj}\right) \cdot \nabla g_j - i\omega g_j - C_j(g_j) = -i\frac{e_j}{T_j} f_{0j}\left(\omega - \omega_{*j}^T\right)\left[J_0(z_j)\left(\Phi - v_\parallel A_\parallel\right) + \frac{v_\perp}{k_\perp}\tilde{B}_\parallel J_1(z_j)\right]$$

Ordering Assumptions

$$\varepsilon \equiv \left(\frac{\rho_e}{\delta}\right)^\lambda \ll 1, \quad \rho_e \equiv \frac{v_{\mathrm{the}}}{\Omega_e}, \quad x \sim \delta, \quad \left(\frac{m_e}{m_i}\right)^{1/2} \sim \varepsilon^\mu$$

species	ω_{bj}	$k_\parallel v_{\mathrm{th}j}$	$\omega_{\mathrm{dr}j}$	ω	ν_j	$k_r^2 D_{\perp j}^{\mathrm{neo}}$
$j =$ electrons	1	$\varepsilon^{\lambda+1}$	ε^λ	$\varepsilon^{\lambda+2}$	ε^λ	$\varepsilon^{3\lambda}$
$j =$ ions	1	$\varepsilon^{\lambda+1}$	$\varepsilon^{\lambda-\mu}$	$\varepsilon^{\lambda+2-\mu}$	ε^λ	$\varepsilon^{3\lambda-2\mu}$

$$\lambda = 7/4, \quad \mu = 3/2, \quad \omega\nu_e \sim k_\parallel^2 v_{\mathrm{the}}^2$$

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Collision Terms: Pitch Angle Scattering, Momentum Conservation

Electron Collision Operator

$$C_e(h) = \nu_{ei}(v)\left[Lh + \frac{m_e}{T_e}v_\parallel u_{\parallel i} f_{0e}\right] + \nu_{ee}(v)\left[Lh + \frac{m_e}{T_e}v_\parallel u^*_{\parallel e} f_{0e}\right]$$

$$u_{\parallel i} = \frac{1}{n_i}\int d\mathbf{v} v_\parallel h_i, \quad u^*_{\parallel e} = \frac{\int d\mathbf{v}\nu_{ee}(v)v_\parallel h_{1e}}{\int d\mathbf{v}(m_e v_\parallel^2/T_e)\nu_{ee} f_{0e}}$$

$$\nu_{ei} = \frac{\nu_0}{(m_e v^2/2T_e)^{3/2}}, \quad \nu_{ee} = \frac{\nu_0\phi(m_e v^2/2T_e)}{(m_e v^2/2T_e)^{3/2}}$$

$$\nu_0 = \frac{\sqrt{2}\pi n_e e^4 \ln\Lambda}{m_e^{1/2}T_e^{3/2}}, \quad L = \frac{2v_\parallel}{v^2 B}\frac{\partial}{\partial\lambda}\lambda v_\parallel \frac{\partial}{\partial\lambda}$$

$$\phi(x) = \left(1 - \frac{1}{2x}\right)\eta(x) + \eta'(x), \quad \eta(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t}t^{1/2}dt$$

Ion Collision Operator

$$C_i(h) = \frac{2\nu_{ii}(v)v_\parallel}{Bv^2}\frac{\partial}{\partial\lambda}\lambda v_\parallel\frac{\partial}{\partial\lambda}h + \nu_{ii}v_\parallel f_{0i}\frac{m_i}{T_I}u^*_{\parallel i}$$

$$u^*_{\parallel i} = \frac{T_i}{m_i}\frac{\int d^3v\nu_{ii}(v)v_\parallel h}{\int d^3v\nu_{ii}(v)v_\parallel^2 f_{0i}}$$

$$\nu_{ii}(v) = \nu_i\phi(u)/u^3, \quad \nu_i = \frac{\sqrt{2}\pi n_e e^4\ln\Lambda}{m_i^{1/2}T_i^{3/2}}, \quad u = m_i v^2/2T_i$$

Density and Temperature Equations

Electron Density

$$-D\frac{\hat{n}_e}{n_e} = \frac{e}{T_e}\left\langle \Phi - \Psi \right\rangle \left[[1 + (\lambda_4 - \lambda_2)s^2]\left(1 - \frac{\omega_{*e}}{\omega}\right) + \lambda_2 s^2 \frac{\omega_{*e}}{\omega}\eta_e \right]$$
$$- \left\langle \frac{\tilde{B}_\parallel}{B} \right\rangle \left[\left(1 - \frac{\omega_{*e}}{\omega}\right)\left(1 + \lambda_4 s^2 - \frac{5}{3}\lambda_2 s^2\right) - \frac{\omega_{*e}}{\omega}\eta_e \left(1 + \lambda_4 s^2 - \frac{10}{3}\lambda_2 s^2\right) \right]$$

Electron Temperature

$$-D\frac{\hat{T}_e}{T_e} = \frac{e}{T_e}\left\langle \Phi - \Psi \right\rangle \left[(\lambda_1 s^2 - \lambda_2 s^2)\left(1 - \frac{\omega_{*e}}{\omega}\right) - (1 + \lambda_1 s^2)\frac{\omega_{*e}}{\omega}\eta_e \right]$$
$$- \left\langle \frac{\tilde{B}_\parallel}{B} \right\rangle \left[\left(1 - \frac{\omega_{*e}}{\omega}\right)\left(\frac{2}{3} + \frac{5}{3}\lambda_1 s^2 - \lambda_3 s^2\right) - \frac{\omega_{*e}}{\omega}\eta_e \left(\frac{7}{3} + \frac{10}{3}\lambda_1 s^2 - \lambda_3 s^2\right) \right]$$

Ion Density

$$\omega\frac{\hat{n}_i}{n_i} = \frac{e(\Phi - \Psi)}{T_i}(\omega - \omega_{*i}) + \left\langle \frac{\tilde{B}_\parallel}{B} \right\rangle [\omega - \omega_{*i}(1 + \eta_i)]$$

Ion Temperature

$$\hat{T}_i - \frac{2i}{3}\frac{T_i I^2}{m_i \Omega_{i0}^2 \tau_{ii}\omega}\left\langle \frac{\langle B^2 \rangle}{B^2} - f_c \right\rangle \hat{T}_i'' = -e\left\langle \Phi - \Psi \right\rangle \frac{\eta_i \omega_{*i}}{\omega}$$
$$- \frac{2i}{3}\frac{T_i I^2}{m_i \Omega_{i0}^2 \tau_{ii}\omega}\left\langle \frac{\langle B^2 \rangle}{B^2} - f_c \right\rangle \frac{\omega_{*i}}{\omega}\eta_i e\Phi'' + \frac{2}{3}T_i \left\langle \frac{\tilde{B}_\parallel}{B} \right\rangle \left[\left(1 - \frac{\omega_{*i}}{\omega}\right) - \frac{7}{2}\frac{\eta_i \omega_{*i}}{\omega} \right]$$

Parallel Magnetic Field and Ohm's Law

Pressure and Parallel Magnetic Field

$$\tilde{n}_j = \hat{n}_j - \frac{n_j e_j}{T_j}\Phi + \frac{n_j e_j}{T_j}\left(1 - \frac{\omega_{*j}}{\omega}\right)\Psi, \quad \tilde{T}_j = \hat{T}_j + \frac{n}{\omega}T'_{0j}\Psi$$

$$\hat{p}_j = T_j \hat{n}_j + n_j \hat{T}_j, \quad \hat{p} = \sum_j \hat{p}_j, \quad \tilde{p} = \sum_j \tilde{p}_j = \hat{p} + \frac{n}{\omega}p'_0\Psi, \quad \frac{\tilde{B}_\parallel}{B} = -\frac{\tilde{p}}{B^2}$$

Neoclassical Ohm's Law

$$\frac{n}{\omega}\frac{d^2(x\Psi)}{dx^2} = -\frac{inq'\sigma_\parallel^{\mathrm{sc}}(s^2)}{p'_0 q}\left\langle\frac{1}{R^2}\right\rangle Lx(\Phi - \Psi) + \frac{H\tilde{p}'}{p'_0} - \frac{(1-f_c)}{(1-0.37f_c)}\frac{L}{p'_0}\left[\alpha_n(T_e + T_i)\tilde{n}'_e + n_e\left(\alpha_e\tilde{T}'_e - \alpha_i\tilde{T}'_i\right)\right]$$

Vorticity and Transverse Viscosity

Vorticity

$$x\frac{d^2}{dx^2}(x\Psi)+D_I\Psi-\left(\frac{\omega x\hat{p}}{n}\right)'\left(\frac{L+H}{p_0'}\right)+D_I\left(\frac{\omega\hat{p}}{np_0'}\right)=\frac{m_in_i}{n^2}\left(\frac{q}{Iq'\left\langle 1/R^2\right\rangle}\right)^2\left\langle\frac{B^2}{|\nabla\psi|^2}\right\rangle$$
$$\times\left[\omega\left(\omega-\omega_{*i}(1+\eta_i)\right)\left(\left\langle\frac{|\nabla\psi|^2}{B^2}\right\rangle\frac{d^2\Phi}{dx^2}+\left\langle\frac{I^2}{B^2}\right\rangle\frac{d^2\Psi}{dx^2}\right)-1.17n\omega T_{i0}'\frac{I^2f_c}{\langle B^2\rangle}\frac{d^2\Psi}{dx^2}\right]$$
$$+\frac{m_in_i}{n^2}\left(\frac{q}{Iq'\left\langle 1/R^2\right\rangle}\right)^2\left\langle\frac{B^2}{|\nabla\Psi|^2}\right\rangle\left[\omega^2\left\langle\frac{I^2}{B^2}\right\rangle\frac{\hat{p}_i''}{en_i}-1.17\omega^2\frac{I^2f_c}{\langle B^2\rangle}\frac{\hat{T}_i''}{e}\right]$$
$$-\frac{i\omega}{n^2}\left(\frac{q}{Iq'\left\langle 1/R^2\right\rangle}\right)^2\left\langle\frac{B^2}{|\nabla\psi|^2}\right\rangle\Pi''$$

Transverse Viscosity

$$\Pi=0.1862m_in_i\left(2\hat{\varepsilon}\right)^{3/2}\langle R\rangle^2\frac{m_iT_i}{\tau_ie^3}\tilde{T}_i'',\quad \hat{\varepsilon}=\frac{R_{\max}-R_{\min}}{2<R>}$$

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General Properties of the Equations

General Form and Galerkin Discretization

$$\mathbf{u}(x) = \sum_j \mathbf{u}_j \alpha_j(x), \quad -(\mathsf{A}\mathbf{u}' + \mathsf{B}\mathbf{u})' + (\mathsf{C}\mathbf{u}' + \mathsf{D}\mathbf{u}) = 0$$

$$\mathsf{L}_{ij} = (\alpha_i', \mathsf{A}\alpha_j') + (\alpha_i', \mathsf{B}\alpha_j) + (\alpha_i, \mathsf{C}\alpha_j') + (\alpha_i, \mathsf{D}\alpha_j), \quad \mathsf{L}_{ij}\mathbf{u}_j = 0$$

Descending Power Series Solutions, $|x| \to \infty$

$$\mathsf{A} = x^2 \sum_{j=0}^{\infty} \mathsf{A}_j x^{-j}, \quad \mathsf{B} = x \sum_{j=0}^{\infty} \mathsf{B}_j x^{-j}, \quad \mathsf{C} = x \sum_{j=0}^{\infty} \mathsf{C}_j x^{-j}, \quad \mathsf{D} = \sum_{j=0}^{\infty} \mathsf{D}_j x^{-j}$$

$$\mathbf{u} = x^{\mu} \sum_{k=0}^{\infty} x^{-k} \mathbf{u}_k, \quad \mathbf{u}' = x^{\mu-1} \sum_{k=0}^{\infty} x^{-k} (\mu - k) \mathbf{u}_k$$

Products and Sums

$$\mathsf{D}\mathbf{u} = x^{\mu} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} x^{-(j+k)} \mathsf{D}_j \mathbf{u}_k = x^{\mu} \sum_{l=0}^{\infty} x^{-l} \sum_{j=0}^{l} \mathsf{D}_j \mathbf{u}_{l-j}$$

$$\mathsf{C}\mathbf{u}' = x^{\mu} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} x^{-(j+k)} (\mu - k) \mathsf{C}_j \mathbf{u}_k = x^{\mu} \sum_{l=0}^{\infty} x^{-l} \sum_{j=0}^{l} (\mu + j - l) \mathsf{C}_j \mathbf{u}_{l-j}$$

$$(\mathsf{B}\mathbf{u})' = \left(x^{\mu+1} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} x^{-(j+k)} \mathsf{B}_j \mathbf{u}_k \right)' = x^{\mu} \sum_{l=0}^{\infty} x^{-l} \sum_{j=0}^{l} (\mu + j - l + 1) \mathsf{B}_j \mathbf{u}_{l-j}$$

$$(\mathsf{A}\mathbf{u}')' = \left(x^{\mu+1} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} x^{-(j+k)} (\mu - k) \mathsf{A}_j \mathbf{u}_k \right)' = x^{\mu} \sum_{l=0}^{\infty} x^{-l} \sum_{j=0}^{l} (\mu + j - l)(\mu + j - l + 1) \mathsf{A}_j \mathbf{u}_{l-j}$$

Power Series Equations

Power Series Equations

$$x^{-\mu}\mathbf{L}\mathbf{u} = \sum_{l=0}^{\infty} x^{-l} \sum_{j=0}^{l} \left\{-(\mu+j-l+1)\left[(\mu+j-l)\mathbf{A}_j + \mathbf{B}_j\right] + (\mu+j-l)\mathbf{C}_j + \mathbf{D}_j\right\} \mathbf{u}_{l-j} = \mathbf{0}$$

Zeroth Order Equations

$$l = 0$$

$$\left[-(\mu+1)\left(\mu\mathbf{A}_0 + \mathbf{B}_0\right) + \mu\mathbf{C}_0 + \mathbf{D}_0\right]\mathbf{u}_0 = \mathbf{0}$$

$$\det\left[-(\mu+1)\left(\mu\mathbf{A}_0 + \mathbf{B}_0\right) + \mu\mathbf{C}_0 + \mathbf{D}_0\right] = 0$$

$$\mu = -\frac{1}{2} \pm \sqrt{-D_I}$$

Matches onto ideal outer region solutions, $\Delta' = \Delta(\omega)$.
Inner region solutions provide $\Delta(\omega)$.

Higher Order Equations

$$l > 0$$

$$\left\{-(\mu-l+1)\left[(\mu-l)\mathbf{A}_0 + \mathbf{B}_0\right] + (\mu-l)\mathbf{C}_0 + \mathbf{D}_0\right\}\mathbf{u}_l = -\sum_{j=1}^{l} \left\{-(\mu+j-l+1)\left[(\mu+j-l)\mathbf{A}_j + \mathbf{B}_j\right] + (\mu+j-l)\mathbf{C}_j + \mathbf{D}_j\right\} \mathbf{u}_{l-j}$$

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Solution Procedure

- The only velocity space integral required is f_c, the fraction of circulating particles. Computed by DCON for each singular surface.
- Advanced Galerkin basis functions: C^1 Hermite cubics on interior of asymmetric domain (-xmax,xmax); resonant power series solutions at largest $|x|$; very effective grid packing algorithm.
- Use large resonant power series solutions to drive response in small resonant power series solution.
- Solve complex banded matrix with LAPACK routines ZGBTRF and ZGBTRS.
- Coefficients of small resonant solutions provide $\Delta(\omega)$.
- Match to outer region, roots of $\det[\Delta' - \Delta(\omega)] = 0$.
- Use MATCH code to find multiple complex roots and construct global eigenfunctions.

Status and Future Work

- Equations have been derived and understood.
- Galerkin discretization with C^1 Hermite cubics is mostly complete, not yet debugged.
- Derivation of large-$|x|$ limit has been formulated but is not yet complete or coded up.
- Computation of a large number of parameters has only begin.
- Testing and debugging remain to be done.
- The linear neoclassical inner region model should be almost as fast as the GGJ model, ~50μs. The main difference is that the number of dependent variables is 8 rather than 3.
- Sheared equilibrium rotation will be added.
- Determine conditions for inner-outer overlap.
- Verification, validation, sharing.
- Nonlinear treatment, finite island widths.