Chapman-Enskog-like drift kinetic equations in NIMROD
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Existing drift kinetic equations (DKEs) in NIMROD

- NIMROD can solve

$$\frac{\partial f}{\partial t} + (\mathbf{v}_{||} + \mathbf{v}_D) \cdot \left[\nabla f - \frac{1-\xi^2}{2\xi} \nabla \ln B \frac{\partial f}{\partial \xi} - \frac{s}{2} \nabla \ln T_0 \frac{\partial f}{\partial s} \right] - C(f) +$$
$$\frac{1-\xi^2}{2\xi} \left[-\xi^2 \frac{\mathbf{b}}{B} \cdot \frac{\partial \mathbf{B}}{\partial t} + \frac{q}{s^2 T_0} (\mathbf{v}_{||} + \mathbf{v}_D) \cdot \mathbf{E} + \xi^2 \frac{\mathbf{E} \times \mathbf{B}}{B^2} \cdot \nabla \ln B \right] \frac{\partial f}{\partial \xi} +$$
$$\frac{s}{2} \left[-(1-\xi^2) \frac{\mathbf{b}}{B} \cdot \frac{\partial \mathbf{B}}{\partial t} + \frac{q}{s^2 T_0} (\mathbf{v}_{||} + \mathbf{v}_D) \cdot \mathbf{E} + (1+\xi^2) \frac{\mathbf{E} \times \mathbf{B}}{B^2} \cdot \nabla \ln B \right] \frac{\partial f}{\partial s} = 0$$

for electrons, ions and energetic ions.

- Typically solve for δf beyond a fixed f_0:

  Neoclassical transport: $f_0 = f_M(\psi(R,Z), s) = n e^{-s^2} / (\pi^{3/2} v_0^3)$

  Energetic ions: $f_0 = f_{\mathrm{slow}}(R, Z, s) + f_{\mathrm{tail}} = A/(1+s^3) + f_{\mathrm{tail}}$

Aspects of existing DKE implementation

- Relatively easy applications include:
  - solving for electron and ion δf's to predict neoclassical transport in axisymmetric toroidal geometry
  - advancing energetic particle δf and coupling to MHD through closure for anisotropic pressure tensor.

- Numerical formulation relatively easy since thermodynamic drives have a simple form.

- Allowed for easy testing of important DKE terms like collision operators, parallel free-streaming and particle trapping.

- But, for incorporating kinetic effects of bulk species in long time scale fluid simulations, it is likely best if DKE solutions only contain kinetic information for closing the fluid equations.

Chapman-Enskog-like (CEL) DKEs in NIMROD

- Assume $f = f_{\mathrm{M}} + f_{\mathrm{NM}}$ with $\bar{f}_{\mathrm{NMe}} = O(\delta^2 f_{\mathrm{Me}})$ and $\bar{f}_{\mathrm{NMi}} = O(\delta f_{\mathrm{Mi}})$.
- Write CEL-DKE in the fluid frame (Ramos, *Phys Plasmas* **17**, 082502 (2010)):

$$\frac{\partial \bar{f}_{\mathrm{NM}}}{\partial t} + v_\parallel' \mathbf{b}\cdot\nabla \bar{f}_{\mathrm{NM}} - \frac{1-\xi^2}{2\xi} v_\parallel' \mathbf{b}\cdot\nabla \ln B \frac{\partial \bar{f}_{\mathrm{NM}}}{\partial \xi}$$
$$+\frac{v_0}{2}(\mathbf{b}\cdot\nabla \ln n)[\xi \frac{\partial \bar{f}_{\mathrm{NM}}}{\partial s} + \frac{1-\xi^2}{s}\frac{\partial \bar{f}_{\mathrm{NM}}}{\partial \xi}] - s[\xi \mathbf{b}\cdot\nabla + \frac{\partial}{\partial t}]\ln v_0 \frac{\partial \bar{f}_{\mathrm{NM}}}{\partial s} = \langle C(f)\rangle$$
$$+\Bigg[(\frac{5}{2}-s^2)v_\parallel' \mathbf{b}\cdot\nabla \ln T + \frac{v_\parallel'}{nT}\mathbf{b}\cdot[\frac{2}{3}\nabla \pi_\parallel - \pi_\parallel \nabla \ln B - \mathbf{F}^{\mathrm{coll}}]$$
$$+2s^2(\frac{3}{2}\xi^2 - \frac{1}{2})[\frac{1}{3}\nabla\cdot\mathbf{u} - \mathbf{bb}\cdot\nabla\mathbf{u}] + \frac{2}{3nT}(s^2 - \frac{5}{2})[\mathbf{b}\cdot\nabla q_\parallel - q_\parallel \mathbf{b}\cdot\nabla \ln B - G^{\mathrm{coll}}]$$
$$+\frac{2}{3eB}s^2(\frac{3}{2}\xi^2 - \frac{1}{2})[(\frac{5}{2}-s^2)(\nabla \ln B - 2\kappa) + \nabla \ln n]\cdot\nabla T \times \mathbf{b}$$
$$+\frac{4}{3eB}(\frac{s^4}{2} - \frac{5}{2}s^2 + \frac{15}{8})(\nabla \ln B + \kappa)\cdot\nabla T \times \mathbf{b}\Bigg] f_M$$

Definition of closure moments

- Desired closure moments computed using random velocity:

$$
\begin{aligned}
\pi_{\|} = p_{\|} - p_{\perp} &= \frac{m}{2}\int d\mathbf{v}\big(3[\mathbf{b}\cdot(\mathbf{v}-\mathbf{u})]^2 - |\mathbf{v}-\mathbf{u}|^2\big)\bar{f}_{\mathrm{NM}} \\
&= 2\pi m(\frac{2T}{m})^{5/2}\int_0^\infty ds s^4 \int_{-1}^{1} d\xi(\frac{3}{2}\xi^2 - \frac{1}{2})\bar{f}_{\mathrm{NM}} \\
q_{\|} &= \frac{m}{2}\int d\mathbf{v}\big([\mathbf{b}\cdot(\mathbf{v}-\mathbf{u})]|\mathbf{v}-\mathbf{u}|^2\big)\bar{f}_{\mathrm{NM}} \\
&= \frac{8\pi T^3}{m^2}\int_0^\infty ds s^5 \int_{-1}^{1} d\xi\,\xi\,\bar{f}_{\mathrm{NM}} \\
\mathbf{F}_e^{\mathrm{coll}}/(n_e e) &= (m_e/n_e e)\int d\mathbf{v}(\mathbf{v}-\mathbf{u})C_{\mathrm{ei}}[f_e, f_i] \\
&= \eta_\perp \mathbf{J} - \frac{3}{2}\frac{n_e}{B}\eta_\perp \mathbf{b}\times\nabla T_e + \int d\mathbf{v}\, m_e v_{\|}' \langle C(f_{\mathrm{NMe}}, f_{\mathrm{Mi}})\rangle \mathbf{b} \\
G_e^{\mathrm{coll}} &= \frac{m_e}{2}\int d\mathbf{v}|\mathbf{v}-\mathbf{u}|^2 C_{\mathrm{ei}}[f_e, f_i] \approx \frac{3e^2 n_e^2 \eta_\perp}{m_i}(T_i - T_e)
\end{aligned}
$$

Fluid moments equations - electrons

- Low-order electron fluid moments evolve according to

$$\frac{\partial n}{\partial t} + \nabla \cdot n\mathbf{u}_e = 0$$

$$m_e n(\frac{\partial \mathbf{u}_e}{\partial t} + \mathbf{u}_e \cdot \nabla \mathbf{u}_e) + en(\mathbf{E} + \mathbf{u}_e \times \mathbf{B}) + \nabla(nT_e) + \nabla \cdot [\pi_{\parallel e}(\mathbf{bb} - \mathbf{I}/3)] - \mathbf{F}_e^{\mathrm{coll}} = 0$$

$$\frac{3n}{2}\frac{dT_e}{dt} + nT_e \nabla \cdot \mathbf{u}_e + \nabla \cdot (q_{\parallel e}\mathbf{b} - \frac{5nT_e}{2eB}\mathbf{b} \times \nabla T_e) - G_e^{\mathrm{coll}} - \pi_{\parallel e}[\frac{1}{3}\nabla \cdot \mathbf{u}_e - \mathbf{bb} \cdot \nabla \mathbf{u}_e] = 0$$

Aspects of CEL-DKE formulation

- Allows for a tight coupling, *i.e.*, hybrid fluid/kinetic capability that is rigorous and consistent: 1, $v'_{\|}$ and v'^2 moments of $\bar{f}_{\mathrm{NM}}$ vanish.

- DKEs written in moving frame of fluid makes taking moments easy.

- Numerical considerations:
  - time centering of fluid and kinetic variables,
  - enforcing the requirement that fluid moments of $\bar{f}_{\mathrm{NM}}$ vanish,
  - ability to evolve linearized system that expands about an axisymmetric $\bar{f}_{\mathrm{NM}}$ and its closure moments.

Spitzer problem verification

- Advance coupled system for flows and electron CEL-DKE.

$$m_e n_e \frac{\partial u_e}{\partial t} = -e n_e E + \underbrace{e n_e \eta_\perp J_\parallel + \int d\mathbf{v} m_e v_\parallel' \langle C(\bar{f}_{\mathrm{NMe}}, f_{\mathrm{Mi}})\rangle}$$

$$m_i n_i \frac{\partial u_i}{\partial t} = +e n_e E - \qquad F_{ei}^{\mathrm{coll}}$$

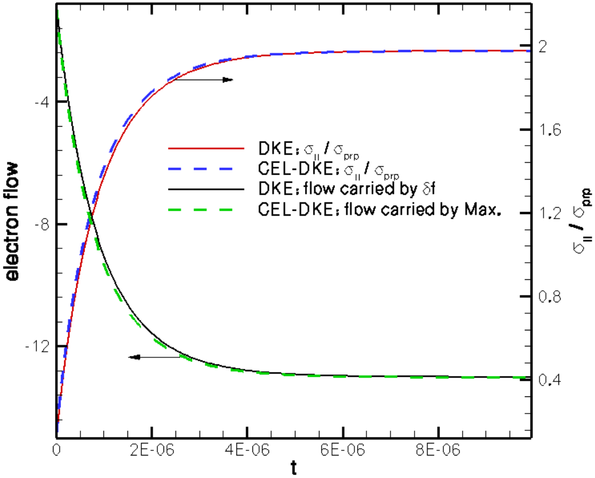
$$\frac{\partial \bar{f}_{\mathrm{NMe}}}{\partial t} = \langle C(\bar{f}_{\mathrm{NMe}}, f_{\mathrm{Me}}) + C(f_{\mathrm{Me}}, \bar{f}_{\mathrm{NMe}})\rangle + \langle C(\bar{f}_{\mathrm{NMe}}, f_{\mathrm{Mi}})\rangle$$

$$+ \underbrace{\frac{e\eta_\perp}{m_e v_{\mathrm{Te}}} \frac{3\sqrt{\pi}}{2s^2} \left[E(s_i) - s_i E'(s_i)\right] \xi J_\parallel f_{\mathrm{Me}}}_{\text{from } \langle C(f_{\mathrm{Me}}, f_{\mathrm{i}})\rangle} - \frac{v_\parallel' f_{\mathrm{Me}}}{n T_e} \left[n_e e \eta_\perp J_\parallel + \int d\mathbf{v} m_e v_\parallel' \langle C(\bar{f}_{\mathrm{NMe}}, f_{\mathrm{Mi}})\rangle\right]$$

- In δf approach flow is carried by δf.

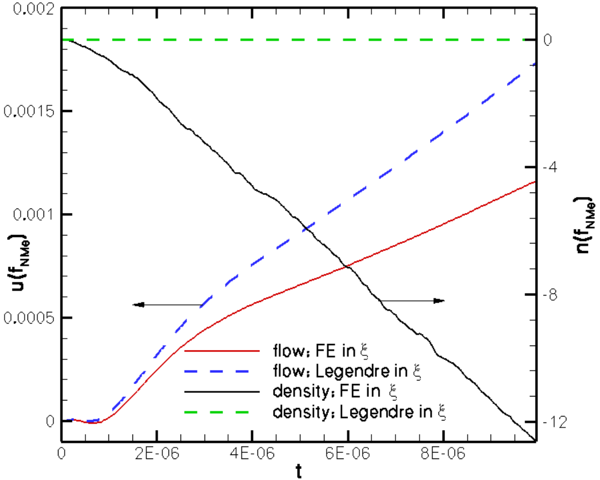
Spitzer conductivity recovered

- CEL-DKE and δf DKE approaches agree.



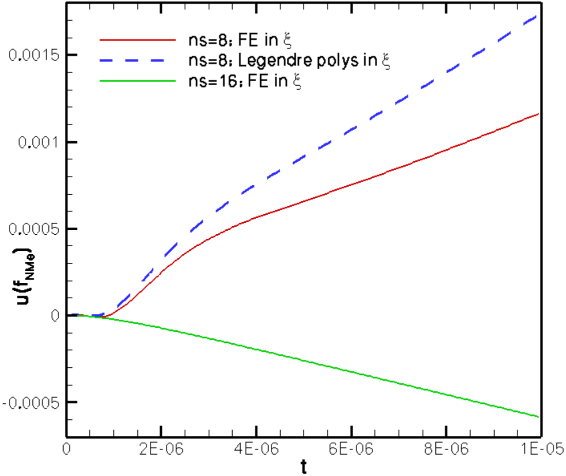
Conservation properties of ξ bases

- Flow moments of $\bar{f}_{\rm NMe}$ arise for both Legendre and FE ξ bases.
- Density moment of $\bar{f}_{\rm NMe}$ remains zero for Legendre basis.



Flow conservation depends on speed grid.

- Refining speed grid improves flow conservation but should consider projecting out Maxwellian moments.



CEL-DKE/NIMROD coupling

- Treating all $\bar{f}_{\mathrm{NM}}$ terms implicitly is desirable but couples entire velocity grid.
- Assume electrons present biggest timestep limitation: center n, $\mathbf{B}$ and T_e in $\bar{f}_{\mathrm{NMe}}$ advance.
- Most basic advance:

$$(\mathbf{V}, \bar{f}_{\mathrm{NMe}}, \bar{f}_{\mathrm{NMi}})^k \qquad\qquad (n, \mathbf{B}, T_e, T_i)^{k+1/2}$$

with $\mathbf{u}_e = \mathbf{V} - \frac{1}{1+m_e/m_i}\mathbf{J}/ne$ and $\mathbf{u}_i = \mathbf{V} + \frac{m_e/m_i}{1+m_e/m_i}\mathbf{J}/ne$.

Linearization example

- Consider linearizing stress drive in CEL-DKE:

$$\frac{v_{\|}'}{nT_e}\mathbf{b}\cdot[\frac{2}{3}\nabla\pi_{\|}-\pi_{\|}\nabla\ln B]f_M$$

- Cancel n and use s normalization to write

$$\frac{m}{2\pi^{3/2}}s\xi e^{-s^2}\delta(\frac{\mathbf{b}}{T^2}\cdot[\frac{2}{3}\nabla\pi_{\|}-\pi_{\|}\nabla\ln B])$$

$$=\frac{m}{2T_0^2\pi^{3/2}}s\xi e^{-s^2}\Bigg[-2\frac{\delta T}{T_0}\mathbf{b}_0\cdot[\frac{2}{3}\nabla\pi_{\|}-\pi_{\|}\nabla\ln B]_0$$

$$+\delta\mathbf{b}\cdot\frac{2}{3}\nabla\pi_{\|0}+\mathbf{b}_0\cdot\frac{2}{3}\nabla\delta\pi_{\|}-\delta\pi_{\|}\mathbf{b}_0\cdot\nabla\ln B_0-\pi_{\|0}\delta(\mathbf{b}\cdot\nabla\ln B)\Bigg]$$

- Linearization aided by computing structures like $\delta\mathbf{b}$, $\delta(\mathbf{b}\cdot\nabla\ln B)$, and $\delta\kappa=\delta(\mathbf{b}\cdot\nabla\mathbf{b})$.

Conclusions

- Further testing underway: neoclassical transport, tearing modes, parallel electron heat transport ....
- Apply first-order, electron CEL-DKE to NTM problem.
- Apply first-order, ion CEL-DKE to study flow damping or thermal ion effect in GS problem.
- Implement second-order terms in ion CEL-DKE?