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STABILITY CRITERIA FOR KINETIC MAGNETOHYDRODYNAMICS*

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OUTLINE

I. INTRODUCTORY REVIEW

II. NEW STUDY OF KMHD STABILITY CONDITIONS

- LINEAR STABILITY DEFINED ACCORDING TO TIME EVOLUTION OF INITIAL-VALUE SOLUTIONS
- DOES NOT USE THE KRUSKAL-OBERMAN, ROSENBLUTH-ROSTOKER COMPARISON THEOREMS
- DOES NOT REQUIRE SELF-ADJOINTNESS OR A COMPLETE BASIS OF NORMAL MODES

I. INTRODUCTORY REVIEW

KINETIC MAGNETOHYDRODYNAMICS (KMHD)
[Kruskal-Oberman (1958), Rosenbluth-Rostoker (1959)]:

COLLISIONLESS, FLUID-KINETIC MODEL OF A QUASINEUTRAL, MAGNETIZED PLASMA IN ITS ZERO-LARMOR-RADIUS LIMIT

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Further assumptions:

- **Single ion species of unit charge**
- **No mass ratio approximations**
- **Mean flow velocity of the order of the sound speed**
- **Kinetic pressures comparable to the magnetic pressure**

Then, $\mathbf{u}_e \to \mathbf{u}_i \to \mathbf{u}$ (common, single-fluid mean velocity) besides $n_e = n_i = n$ (fluid quasineutrality)

SINGLE-FLUID, HYDROMAGNETIC SYSTEM WITH ZERO-LARMOR-RADIUS KINETIC CLOSURE

$$\frac{\partial \mathbf{B}}{\partial t} \;=\; \nabla \times (\mathbf{u} \times \mathbf{B}) \;, \qquad \mathbf{j} \;=\; \nabla \times \mathbf{B}$$

$$\frac{\partial n}{\partial t} \;+\; \nabla \cdot (n\mathbf{u}) \;=\; 0 \;, \qquad \rho = (m_i + m_e) n$$

$$\rho \left[\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} \right] \;-\; \mathbf{j} \times \mathbf{B} \;+\; \sum_{s=i,e} \nabla \cdot \mathbf{P}_s \;=\; 0$$

$$\mathbf{P}_s \;=\; p_{s\parallel} \mathbf{bb} \;+\; p_{s\perp} (\mathbf{I} - \mathbf{bb})$$

$$p_{s\parallel} = m_s \int d^3\mathbf{v} \; (v_\parallel - u_\parallel)^2 \; f_s, \qquad p_{s\perp} = \frac{m_s}{2} \int d^3\mathbf{v} \; |\mathbf{v}_\perp - \mathbf{u}_\perp|^2 \; f_s$$

f_s are zero-Larmor-radius-limit solutions of the collisionless Vlasov kinetic equation

POTENTIAL ENERGY FUNCTIONAL FOR SMALL-AMPLITUDE PERTURBATIONS ABOUT A STATIC, ISOTROPIC EQUILIBRIUM

$$\mathbf{u}_0 = 0 \; , \qquad \mathbf{u}_1 = \frac{\partial \boldsymbol{\xi}}{\partial t} \; , \qquad K = \frac{1}{2} \int d^3\mathbf{x} \; \rho_0 \left| \frac{\partial \boldsymbol{\xi}}{\partial t} \right|^2$$

$$\delta W = - \int^t dt' \int d^3\mathbf{x} \; \rho_0 \; \frac{\partial^2 \boldsymbol{\xi}(t')}{\partial t'^2} \cdot \frac{\partial \boldsymbol{\xi}(t')}{\partial t'} = - \int^t dt' \; \frac{dK(t')}{dt'}$$

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$$\hat{f}_s = f_{s1} + \boldsymbol{\xi} \cdot \frac{\partial f_{s0}}{\partial \mathbf{x}}$$

$$\delta W[\boldsymbol{\xi}, \hat{f}_s] = \delta W_\perp^F[\boldsymbol{\xi}_\perp] + \delta W^K[\hat{f}_s] =$$

$$= -\frac{1}{2} \int d^3\mathbf{x} \; \boldsymbol{\xi}_\perp \cdot \mathbf{F}_\perp^F[\boldsymbol{\xi}_\perp] - \frac{1}{2} \sum_{s=i,e} \int d^3\mathbf{x} \int d^3\mathbf{v} \; \frac{\hat{f}_s^2}{\partial f_{s0}/\partial \varepsilon}$$

where $\mathbf{F}_\perp^F$ is the force operator in perpendicular ideal-MHD (ideal-MHD closed with $dp/dt = \partial p/\partial t + \mathbf{u} \cdot \nabla p = 0$**)**

A POSITIVE DEFINITE $\delta W[\boldsymbol{\xi}, \hat{f}_s]$ IS A SUFFICIENT CONDITION FOR KMHD STABILITY

If $\delta W \geq 0$:

$$K(t) \;=\; K(0) \;+\; \delta W(0) \;-\; \delta W(t) \;\leq\; K(0) \;+\; \delta W(0)\;,$$

therefore the kinetic energy is a bounded function of time.

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For $\partial f_{s0}/\partial \varepsilon < 0$ **,** δW^K **is positive definite, hence stability in perpendicular ideal-MHD is sufficient for stability in KMHD.**

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The density continuity condition $\int d^3\mathbf{v} f_{s1} = -\boldsymbol{\xi} \cdot \nabla n_0 - n_0 \nabla \cdot \boldsymbol{\xi}$, or $\int d^3\mathbf{v} \ \hat{f}_s = -n_0 \nabla \cdot \boldsymbol{\xi}$, yields

$$\delta W[\boldsymbol{\xi}, \hat{f}_s] \geq -\frac{1}{2} \int d^3\mathbf{x} \ \boldsymbol{\xi}_\perp \cdot \mathbf{F}^F_\perp[\boldsymbol{\xi}_\perp] + \frac{1}{2} \int d^3\mathbf{x} \ (p_{i0} + p_{e0})(\nabla \cdot \boldsymbol{\xi})^2,$$

hence stability in isothermal ideal-MHD (ideal-MHD closed with $d(pn^{-1})/dt = 0$) is sufficient for stability in KMHD.

NO RIGOROUS PROOF THAT A POSITIVE $\delta W[\boldsymbol{\xi}, \hat{f}_s]$ IS NECESSARY FOR KMHD STABILITY BECAUSE KMHD IS NOT SELF-ADJOINT

The two standard methods to prove that an instability follows if a trial perturbation that makes the potential energy negative is used as initial condition, do not work in KMHD due to its lack of self-adjointness.

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The two standard methods to prove that an instability follows if a trial perturbation that makes the potential energy negative is used as initial condition, do not work in KMHD due to its lack of self-adjointness.

- **In the first method, the initial condition is expanded as a superposition of normal modes and it is argued that, in order to make δW negative, at least one of the normal modes must have a positive growth rate, which will cause an exponential growth of the perturbation.**
  This requires the existence of a complete basis of normal modes and this has not been proved because the KMHD normal modes are not eigenfunctions of a self-adjoint operator.

- **The second method is the one used by Laval et al. (1965) to prove that a negative δW results in an exponentially growing kinetic energy in ideal-MHD, without recourse to the expansion in normal modes.**

  This method requires that the "force-times-displacement" functional

$$U \ = \ -\frac{1}{2}\int d^3\mathbf{x}\ \mathbf{F}\cdot\boldsymbol{\xi} \ = \ -\frac{1}{2}\int d^3\mathbf{x}\ \rho_0\ \frac{\partial^2\boldsymbol{\xi}}{\partial t^2}\cdot\boldsymbol{\xi}$$

  be equal to the potential energy δW.

  The equality $U = \delta W$ holds when the force operator $\mathbf{F}$ is self-adjoint, so that $d(\int d^3\mathbf{x}\ \mathbf{F}\cdot\boldsymbol{\xi})/dt = 2\int d^3\mathbf{x}\ \mathbf{F}\cdot\partial\boldsymbol{\xi}/\partial t$, but it does not hold in KMHD.

THE ROSENBLUTH-ROSTOKER ENERGY PRINCIPLE

Equivalent to Kruskal-Oberman for quasineutral plasmas

Considers an auxiliary linear KMHD model (RR), obtained by specializing the pressure tensor to the distribution functions of a zero-frequency KMHD normal mode, $\hat{f}_s^{\omega=0}$:

$$\rho_0 \frac{\partial^2 \boldsymbol{\xi}_\perp}{\partial t^2} = \mathbf{F}_\perp^F[\boldsymbol{\xi}_\perp] - \sum_{s=i,e} \nabla \cdot \hat{\mathbf{P}}_s^{RR}[\boldsymbol{\xi}_\perp]$$

$$\hat{\mathbf{P}}_s^{RR}[\boldsymbol{\xi}_\perp] = \left[2 \int d^3\mathbf{v}\ (\varepsilon - \mu B_0)\hat{f}_s^{\omega=0}\right] \mathbf{bb} + \left[\int d^3\mathbf{v}\ \mu B_0 \hat{f}_s^{\omega=0}\right] (\mathbf{I} - \mathbf{bb})$$

$$\hat{f}_s^{\omega=0}[\boldsymbol{\xi}_\perp] = \frac{\oint d\tau\ [\mu B_0\ \nabla \cdot \boldsymbol{\xi}_\perp + (2\varepsilon - 3\mu B_0)\ \boldsymbol{\xi}_\perp \cdot \boldsymbol{\kappa}_0]}{\oint d\tau} \ \frac{\partial f_{s0}}{\partial \varepsilon}$$

where $\oint d\tau = \oint d\ell\ [2(\varepsilon - \mu B_0)/m_s]^{-1/2}$ along one period of the particle phase-space trajectory, assumed to be periodic.

•The RR potential energy functional

$$\delta W^{RR}[\boldsymbol{\xi}_\perp] = -\frac{1}{2}\int d^3\mathbf{x}\ \boldsymbol{\xi}_\perp \cdot \mathbf{F}_\perp^F[\boldsymbol{\xi}_\perp] - \frac{1}{2}\sum_{s=i,e}\int d^3\mathbf{x}\int d^3\mathbf{v}\ \frac{(\hat{f}_s^{\omega=0})^2}{\partial f_{s0}/\partial\varepsilon}$$

yields a variational energy principle such that a positive δW^{RR} is necessary and sufficient for RR stability.

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yields a variational energy principle such that a positive δW^{RR} is necessary and sufficient for RR stability.

• δW^{RR} **has the following bounds (comparison theorems):**

$$\delta W^{RR}[\boldsymbol{\xi}_\perp] \geq \delta W_\perp^F[\boldsymbol{\xi}_\perp] + \frac{5}{6}\int d^3\mathbf{x}\ (p_{i0}+p_{e0})\langle \nabla\cdot\boldsymbol{\xi}_\perp\rangle$$

hence stability in adiabatic ideal-MHD (ideal-MHD closed with $d(pn^{-5/3})/dt = 0$) is sufficient for RR stability

$$\delta W^{RR} \leq \delta W_\perp^F + \frac{1}{6}\int d^3\mathbf{x}(p_{i0}+p_{e0})[5(\nabla\cdot\boldsymbol{\xi}_\perp)^2 + (\nabla\cdot\boldsymbol{\xi}_\perp + 3\boldsymbol{\xi}_\perp\cdot\boldsymbol{\kappa}_0)^2]$$

hence stability against perpendicular displacements in the double-adiabatic model of Chew-Goldberger-Low is necessary for RR stability.

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•Away from the zero-frequency normal modes, the RR model is not physical:

Incorrect pressure tensor.

$\mathbf{b}_0 \cdot (\nabla \cdot \hat{\mathbf{P}}_s^{RR}) = 0$**, therefore parallel force balance would require** $\xi_\parallel = 0$**, which is incompatible with continuity because** $\int d^3\mathbf{v}\ \hat{f}_s^{\omega=0} \neq -n_0 \nabla \cdot \boldsymbol{\xi}_\perp$**.**

•No rigorous proof is known that stability in the RR model is equivalent to stability in KMHD.

- **The zero-frequency normal modes of the RR model are zero-frequency normal modes of KMHD.**

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  Incorrect pressure tensor.

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- **No rigorous proof is known that stability in the RR model is equivalent to stability in KMHD.**

- **The orbit periodicity requirement necessitates the not very satisfactory argument of nearly periodic orbits for passing particles on ergodic magnetic lines.**

II. NEW STUDY OF A KMHD NECESSARY STABILITY CONDITION

- LINEAR STABILITY DEFINED ACCORDING TO TIME EVOLUTION OF INITIAL-VALUE SOLUTIONS

- DOES NOT USE THE KRUSKAL-OBERMAN, ROSENBLUTH-ROSTOKER COMPARISON THEOREMS

- DOES NOT REQUIRE PARTICLE ORBIT PERIODICITY, SELF-ADJOINTNESS OR A COMPLETE BASIS OF NORMAL MODES

(THE RESULT THAT STABILITY IN ISOTHERMAL IDEAL-MHD IS SUFFICIENT FOR STABILITY IN KMHD ALREADY FULFILLS THESE CRITERIA)

ZERO-LARMOR-RADIUS DRIFT-KINETIC EQUATION IN THE MACROSCOPIC FLOW REFERENCE FRAME

$$\mathbf{w} \;=\; \mathbf{v} \;-\; \mathbf{u}(\mathbf{x},t) \;, \qquad f_s \;=\; f_s(w_\|, w_\perp, \mathbf{x}, t)$$

$$\frac{\partial f_s}{\partial t} + \left(\mathbf{u} + w_\|\mathbf{b}\right) \cdot \left.\frac{\partial f_s}{\partial \mathbf{x}}\right|_{w_\|, w_\perp} + \frac{w_\perp}{2}\left[(\mathbf{bb} - \mathbf{I}) : (\nabla \mathbf{u}) - w_\| \nabla \cdot \mathbf{b}\right] \frac{\partial f_s}{\partial w_\perp}$$

$$+ \;\left[\frac{\mathbf{b} \cdot (\nabla \cdot \mathbf{P}_s)}{m_s n} - w_\|(\mathbf{bb}) : (\nabla \mathbf{u}) + \frac{w_\perp^2}{2} \nabla \cdot \mathbf{b}\right] \; \frac{\partial f_s}{\partial w_\|} \;=\; 0$$

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$$\frac{\partial f_s}{\partial t} + (\mathbf{u} + w_\parallel \mathbf{b}) \cdot \left.\frac{\partial f_s}{\partial \mathbf{x}}\right|_{w_\parallel, w_\perp} + \frac{w_\perp}{2} \left[(\mathbf{bb} - \mathbf{I}) : (\nabla \mathbf{u}) - w_\parallel \nabla \cdot \mathbf{b} \right] \frac{\partial f_s}{\partial w_\perp}$$

$$+ \left[\frac{\mathbf{b} \cdot (\nabla \cdot \mathbf{P}_s)}{m_s n} - w_\parallel (\mathbf{bb}) : (\nabla \mathbf{u}) + \frac{w_\perp^2}{2} \nabla \cdot \mathbf{b} \right] \frac{\partial f_s}{\partial w_\parallel} \;=\; 0$$

Calling $n_s^{kin} \equiv \int d^3\mathbf{w} \; f_s$ **and** $c_{s\parallel} \equiv \int d^3\mathbf{w} \; w_\parallel \; f_s$ **:**

$$\frac{\partial (n_s^{kin} - n)}{\partial t} + \nabla \cdot \left[(n_s^{kin} - n)\mathbf{u} + c_{s\parallel} \mathbf{b} \right] = 0$$

$$\frac{\partial c_{s\parallel}}{\partial t} \;+\; \nabla \cdot (c_{s\parallel} \mathbf{u}) + c_{s\parallel} (\mathbf{bb}) : (\nabla \mathbf{u}) - \frac{(n_s^{kin} - n)}{m_s n} \mathbf{b} \cdot (\nabla \cdot \mathbf{P}_s) = 0$$

so the KMHD system preserves the constraints $n_s^{kin} - n = 0$ **and** $c_{s\parallel} = 0$ **once they are imposed on the initial condition.**

The KMHD system has the energy conservation law

$$\frac{\partial}{\partial t}\left[\frac{\rho u^2}{2} + \frac{B^2}{2} + \sum_{s=i,e}\left(\frac{p_{s\|}}{2} + p_{s\perp}\right)\right] \; +$$

$$\nabla\cdot\left\{\frac{\rho u^2}{2}\mathbf{u} - (\mathbf{u}\times\mathbf{B})\times\mathbf{B} + \sum_{s=i,e}\left[\left(\frac{p_{s\|}}{2} + p_{s\perp}\right)\mathbf{u} + \mathbf{P}_s\cdot\mathbf{u} + \mathbf{q}_s\right]\right\} = 0$$

where $\mathbf{q}_s = \frac{1}{2}m_s(\int d^3\mathbf{w}\;\, w_{\|}w^2 f_s)\mathbf{b}$ **is the parallel heat flux.**

With ideal-wall boundary conditions:

$$\frac{d}{dt}\int d^3\mathbf{x}\left[\frac{\rho u^2}{2} + \frac{B^2}{2} + \sum_{s=i,e}\left(\frac{p_{s\|}}{2} + p_{s\perp}\right)\right] \;=\; 0$$

STATIC MAXWELLIAN EQUILIBRIUM

$$\partial/\partial t \;=\; 0\;, \qquad\qquad \mathbf{u}_0 \;=\; 0$$

$$f_{Ms0} \;=\; \left(\frac{m_s}{2\pi}\right)^{3/2} \frac{n_0}{T_{s0}^{3/2}} \;\exp\left(-\frac{m_s w^2}{2\,T_{s0}}\right)$$

$$\mathbf{B}_0 \cdot \nabla n_0 \;=\; 0\;, \qquad\qquad \mathbf{B}_0 \cdot \nabla\, T_{s0} \;=\; 0$$

$$\mathbf{j}_0 \times \mathbf{B}_0 \;=\; (\nabla \times \mathbf{B}_0) \times \mathbf{B}_0 \;=\; \nabla[n_0(T_{i0} + T_{e0})]$$

LINEARIZED SYSTEM FOR KMHD PERTURBATION

$$\mathbf{B}_1 = \nabla \times (\boldsymbol{\xi}_\perp \times \mathbf{B}_0), \qquad \mathbf{j}_1 = \nabla \times \mathbf{B}_1$$

$$n_1 = -\boldsymbol{\xi}_\perp \cdot \nabla n_0 - n_0 \nabla \cdot \boldsymbol{\xi}$$

$$\rho_0 \frac{\partial \mathbf{u}_1}{\partial t} \;=\; \rho_0 \frac{\partial^2 \boldsymbol{\xi}}{\partial t^2} \;=\; \mathbf{F}_\perp^F[\boldsymbol{\xi}_\perp] \;+\; \sum_{s=i,e} \left(\hat{F}_{s\|}[\hat{f}_s]\ \mathbf{b}_0 \;+\; \hat{\mathbf{F}}_{s\perp}[\hat{f}_s] \right)$$

$$\mathbf{F}_\perp^F[\boldsymbol{\xi}_\perp] \;=\; (\mathbf{j}_0 \times \mathbf{B}_1)_\perp \;+\; \mathbf{j}_1 \times \mathbf{B}_0 \;+\; \nabla_\perp \left[\boldsymbol{\xi}_\perp \cdot \nabla \left(n_0 T_{i0} + n_0 T_{e0} \right) \right]$$

$$\hat{F}_{s\|}[\hat{f}_s] \;=\; -\; m_s \int d^3\mathbf{w}\ w_\|^2\ \mathbf{b}_0 \cdot \left.\frac{\partial \hat{f}_s}{\partial \mathbf{x}}\right|_{w,\lambda}$$

$$\hat{\mathbf{F}}_{s\perp}[\hat{f}_s] = -\nabla_\perp \left[\frac{m_s}{2} \int d^3\mathbf{w}\ w_\perp^2\ \hat{f}_s \right] - \left[m_s \int d^3\mathbf{w}\ \left(w_\|^2 - \frac{w_\perp^2}{2} \right) \hat{f}_s \right] \boldsymbol{\kappa}_0$$

$$\hat{f}_s \;=\; f_{s1} \;+\; \boldsymbol{\xi} \cdot \frac{\partial f_{Ms0}}{\partial \mathbf{x}} \;=\; \hat{f}_s^{even} \;+\; \hat{f}_s^{odd}$$

$$\frac{\partial \hat{f}_s^{even}}{\partial t} \;+\; w_{\|} \; \mathbf{b}_0 \cdot \frac{\partial \hat{f}_s^{odd}}{\partial \mathbf{x}}\bigg|_{w,\lambda} \;+\; Q[\mathbf{u}_1] \; \frac{m_s w^2}{T_{s0}} \; f_{Ms0} \;=\; 0$$

$$\frac{\partial \hat{f}_s^{odd}}{\partial t} \;+\; w_{\|} \; \mathbf{b}_0 \cdot \frac{\partial \hat{f}_s^{even}}{\partial \mathbf{x}}\bigg|_{w,\lambda} \;+\; \frac{w_{\|}}{n_0 T_{s0}} \hat{F}_{s\|} \; f_{Ms0} \;=\; 0$$

where $$w = \left(w_{\|}^2 + w_{\perp}^2\right)^{1/2}, \qquad \lambda = \frac{w_{\perp}^2}{w^2 B_0(\mathbf{x})} = \frac{\mu}{\varepsilon}$$

and $$Q[\boldsymbol{\eta}] \;\equiv\; \frac{1}{w^2}\left[\frac{w_{\perp}^2}{2} \nabla \cdot \boldsymbol{\eta} \;+\; \left(w_{\|}^2 - \frac{w_{\perp}^2}{2}\right)(\mathbf{b}_0 \mathbf{b}_0) : (\nabla \boldsymbol{\eta})\right]$$

$$\hat{f}_s \; = \; f_{s1} \; + \; \boldsymbol{\xi} \cdot \frac{\partial f_{Ms0}}{\partial \mathbf{x}} \; = \; \hat{f}_s^{even} \; + \; \hat{f}_s^{odd}$$

$$\frac{\partial \hat{f}_s^{even}}{\partial t} \; + \; w_{\|} \; \mathbf{b}_0 \cdot \left. \frac{\partial \hat{f}_s^{odd}}{\partial \mathbf{x}} \right|_{w,\lambda} \; + \; Q[\mathbf{u}_1] \; \frac{m_s w^2}{T_{s0}} \; f_{Ms0} \; = \; 0$$

$$\frac{\partial \hat{f}_s^{odd}}{\partial t} \; + \; w_{\|} \; \mathbf{b}_0 \cdot \left. \frac{\partial \hat{f}_s^{even}}{\partial \mathbf{x}} \right|_{w,\lambda} \; + \; \frac{w_{\|}}{n_0 T_{s0}} \hat{F}_{s\|} \; f_{Ms0} \; = \; 0$$

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This system preserves the constraints $\int d^3\mathbf{w} \; \hat{f}_s^{even} = -n_0 \nabla \cdot \boldsymbol{\xi}$ and $\int d^3\mathbf{w} \; w_{\|} \hat{f}_s^{odd} = 0$ at all times, once they are imposed on the initial condition.

POTENTIAL ENERGY FUNCTIONAL

$$\frac{d\ \delta W}{dt} = -\frac{dK}{dt} = -\int d^3\mathbf{x}\ \rho_0\ \mathbf{u}_1 \cdot \frac{\partial \mathbf{u}_1}{\partial t} =$$

$$-\int d^3\mathbf{x}\ \mathbf{u}_1 \cdot \left\{ \mathbf{F}_\perp^F[\boldsymbol{\xi}_\perp] + \sum_{s=i,e} \left(\hat{F}_{s\|}[\hat{f}_s]\mathbf{b}_0 + \hat{\mathbf{F}}_{s\perp}[\hat{f}_s] \right) \right\}$$

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Substituting the expressions of $\hat{F}_{s\|}[\hat{f}_s]$ and $\hat{\mathbf{F}}_{s\perp}[\hat{f}_s]$ and integrating by parts:

$$\frac{d\ \delta W}{dt} = -\int d^3\mathbf{x}\ \left\{ \mathbf{u}_1 \cdot \mathbf{F}_\perp^F[\boldsymbol{\xi}_\perp] + \sum_{s=i,e} m_s \int d^3\mathbf{w}\ \hat{f}_s w^2 Q[\mathbf{u}_1] \right\}$$

Using the self-adjointness of $\mathbf{F}_\perp^F[\xi_\perp]$ and the DKE's for $\hat{f}_s^{even}$ and $\hat{f}_s^{odd}$:

$$\frac{d\ \delta W}{dt} = \frac{d}{dt}\left\{ -\frac{1}{2}\int d^3\mathbf{x}\ \boldsymbol{\xi}_\perp \cdot \mathbf{F}_\perp^F[\boldsymbol{\xi}_\perp] + \frac{1}{2}\sum_{s=i,e} \int d^3\mathbf{x} \int d^3\mathbf{w} \frac{T_{s0}}{f_{Ms0}} \hat{f}_s^2 \right\} =$$

$$= d(\delta W_\perp^F[\boldsymbol{\xi}_\perp] + \delta W^K[\hat{f}_s])/dt$$

"FORCE-TIMES-DISPLACEMENT" FUNCTIONAL

$$U = -\frac{1}{2}\int d^3\mathbf{x}\ \boldsymbol{\xi}\cdot\left\{\mathbf{F}_{\perp}^{F}[\boldsymbol{\xi}_{\perp}] + \sum_{s=i,e}\left(\hat{F}_{s\|}[\hat{f}_s]\mathbf{b}_0 + \hat{\mathbf{F}}_{s\perp}[\hat{f}_s]\right)\right\} = \delta W_{\perp}^{F} + U^{K}$$

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Substituting the expressions of $\hat{F}_{s\|}[\hat{f}_s]$ and $\hat{\mathbf{F}}_{s\perp}[\hat{f}_s]$, integrating by parts and substituting the time-integrated DKE for $\hat{f}_s^{even}$ and the the DKE for $\hat{f}_s^{odd}$:

$$U^K \ = \ -\frac{1}{2}\sum_{s=i,e} m_s\int d^3\mathbf{x}\int d^3\mathbf{w}\ \hat{f}_s w^2 Q[\boldsymbol{\xi}] \ = \ \delta W^K - \frac{R}{2} - \frac{S}{2}$$

where

$$R \ = \ \sum_{s=i,e}\int d^3\mathbf{x}\int d^3\mathbf{w}\ \frac{T_{s0}}{f_{Ms0}}\left[(\hat{f}_s^{odd})^2 \ - \ \frac{\partial \hat{f}_s^{odd}}{\partial t}\ \int_0^t dt'\ \hat{f}_s^{odd}(t')\right]$$

$$S \ = \ \sum_{s=i,e} m_s\int d^3\mathbf{x}\int d^3\mathbf{w}\ \hat{f}_s^{even}\left\{\frac{T_{s0}}{m_s f_{Ms0}}\hat{f}_s^{even}(0) + w^2 Q[\boldsymbol{\xi}(0)]\right\}$$

$U \neq \delta W$ confirms that KMHD is not self-adjoint

A special class of KMHD perturbations are the ones with

$$\hat{f}_s(0) \;=\; \hat{f}_s^{even}(0) \;=\; -\; Q[\boldsymbol{\xi}(0)]\; \frac{m_s w^2 f_{Ms0}}{T_{s0}}$$

that satisfy the constraints $\int d^3\mathbf{w}\;\hat{f}_s(0) = -n_0 \nabla \cdot \boldsymbol{\xi}(0)$ **and** $\int d^3\mathbf{w}\; w_{||} \hat{f}_s(0) = 0$ **, and make** $S = 0$ **.**

For these perturbations,

$$U = \delta W - R/2$$

and

$$\delta W^K[\hat{f}_s(0)] =$$

$$= \frac{1}{6} \sum_{s=i,e} \int d^3\mathbf{x}\; n_0 T_{s0} \Big\{ 5[\nabla \cdot \boldsymbol{\xi}(0)]^2 + \Big[\nabla \cdot \boldsymbol{\xi}(0) - 3(\mathbf{b}_0\mathbf{b}_0) : \Big(\nabla \boldsymbol{\xi}(0)\Big)\Big]^2 \Big\}$$

hence

$$\delta W[\boldsymbol{\xi}(0), \hat{f}_s(0)] \;=\; \delta W_\perp^F[\boldsymbol{\xi}_\perp(0)] \;+\; \delta W^K[\hat{f}_s(0)] \;=\; \delta W^{DA}[\boldsymbol{\xi}(0)]$$

where δW^{DA} **is the double-adiabatic potential energy.**

NECESSARY CONDITION FOR KMHD STABILITY

If an equilibrium is unstable in the double-adiabatic theory, a trial fluid displacement $\boldsymbol{\xi}^{tr}$ exists such that $\delta W^{DA}[\boldsymbol{\xi}^{tr}] < 0$.

Then, choose the following KMHD initial condition:

$$\boldsymbol{\xi}(0) = \boldsymbol{\xi}^{tr} \ , \qquad \mathbf{u}_1(0) = \partial\boldsymbol{\xi}(0)/\partial t = 0 \ ,$$

$$\hat{f}_s(0) \ = \ \hat{f}_s^{even}(0) \ = \ - \ Q[\boldsymbol{\xi}^{tr}] \ \frac{m_s w^2 f_{Ms0}}{T_{s0}}$$

For this perturbation,

$$\delta W(0) = \delta W^{DA}[\boldsymbol{\xi}^{tr}] < 0, \qquad K(0) = 0$$

$$\delta W(t) + K(t) = \delta W(0) + K(0) = \delta W(0)$$

$$U(t) = \delta W(t) - R(t)/2$$

Consider the fluid displacement norm:

$$N(t) \equiv \frac{1}{2} \int d^3\mathbf{x} \; \rho_0 \; |\boldsymbol{\xi}(t)|^2$$

$$\frac{dN(t)}{dt} = \int d^3\mathbf{x} \; \rho_0 \; \boldsymbol{\xi} \cdot \frac{\partial \boldsymbol{\xi}}{\partial t} \,, \qquad \frac{dN(0)}{dt} = 0$$

$$\frac{d^2N(t)}{dt^2} = \int d^3\mathbf{x} \; \rho_0 \left[\left| \frac{\partial \boldsymbol{\xi}}{\partial t} \right|^2 + \boldsymbol{\xi} \cdot \frac{\partial^2 \boldsymbol{\xi}}{\partial t^2} \right] =$$

$$= 2K(t) - 2U(t) = 4K(t) - 2\delta W(0) + R(t)$$

Therefore,

$$N(t) = N(0) - \delta W(0) \; t^2 + N_R(t) + 4 \int_0^t dt' \int_0^{t'} dt'' \; K(t'')$$

with

$$N_R(t) = \int_0^t dt' \int_0^{t'} dt'' \; R(t'')$$

It can be shown that N_R has the lower bound

$$N_R(t) \;\geq\; \frac{t^2}{2} \sum_{s=i,e} \int d^3\mathbf{x} \int d^3\mathbf{w}\; \frac{T_{s0}}{f_{Ms0}} \Phi_s(t)$$

where

$$\Phi_s(t) \;=\; \int_0^1 d\nu\;\; (3 - 4\nu)\, [\hat{f}_s^{odd}(\nu t)]^2$$

Since $K \geq 0$, this gives the lower bound for N

$$N(t) \;\geq\; N(0) \;-\; \delta W(0)\; t^2 \;+\; \frac{t^2}{2} \sum_{s=i,e} \int d^3\mathbf{x} \int d^3\mathbf{w}\; \frac{T_{s0}}{f_{Ms0}} \Phi_s(t)$$

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Since $K \geq 0$, this gives the lower bound for N

$$N(t) \;\geq\; N(0) \;-\; \delta W(0) \; t^2 \;+\; \frac{t^2}{2} \sum_{s=i,e} \int d^3\mathbf{x} \int d^3\mathbf{w} \; \frac{T_{s0}}{f_{Ms0}} \Phi_s(t)$$

Therefore, if it could be proved that, for equilibria that are stable in KMHD, $\Phi_s(t) \geq 0$ as $t \to \infty$, the equilibrium under consideration must be KMHD-unstable and double-adiabatic stability would be proven to be necessary for KMHD stability. As $t \to \infty$, $\Phi_s(t) \geq 0$ is guaranteed if $\hat{f}_s^{odd}(t)$ is bounded and

$$[\hat{f}_s^{odd}(t)]^2 \;\leq\; \frac{3}{t} \int_0^t dt' \; [\hat{f}_s^{odd}(t')]^2 \tag{1}$$

SUMMARY

- KMHD LINEAR STABILITY IS INVESTIGATED USING THE INITIAL-VALUE APPROACH

- THE ANALYSIS DOES NOT REQUIRE PARTICLE ORBIT PERIODICITY, SELF-ADJOINTNESS OF THE FORCE OPERATOR OR A COMPLETE NORMAL MODE BASIS

- RESULTS INDEPENDENT OF KRUSKAL-OBERMAN, ROSENBLUTH-ROSTOKER COMPARISON THEOREMS:
    1. STABILITY IN ISOTHERMAL IDEAL-MHD IS SUFFICIENT FOR STABILITY IN KMHD.
    2. PROVIDED THE CONDITION (1) HOLDS, STABILITY IN CGL DOUBLE-ADIABATIC MODEL (INCLUDING THE VARIATION OF $\xi_\parallel$) WOULD BE NECESSARY FOR STABILITY IN KMHD