Computational Modeling of Fully Ionized Magnetized Plasmas Using the Fluid Approximation

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- Pioneer in computational physics
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- Leader and mentor

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The Problem

- Compute the *low frequency* dynamics of hot magnetized plasmas in *realistic geometry* in the presence of *high frequency* oscillations
- Incorporate the effects of *lowest order kinetic corrections* to the usual MHD equations
- Develop *accurate and efficient algorithms* that enable these goals

Modeling Magnetized Plasmas

- Plasma kinetic equation

$$\frac{d}{dt} f_\alpha(\mathbf{x},\mathbf{v},t) = \frac{\partial f_\alpha}{\partial t} + \mathbf{v}\cdot\nabla f_\alpha + \frac{q_\alpha}{m_\alpha}\left(\mathbf{E}+\mathbf{v}\times\mathbf{B}\right)\cdot\nabla_{\mathbf{v}} f_\alpha = \sum_\beta C_{\alpha,\beta}(f_\alpha, f_\beta)$$

- Maxwell's equations

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla\times\mathbf{E} \qquad \nabla\cdot\mathbf{E} = \rho_q$$

$$\rho_q = \sum_\alpha q_\alpha \int f_\alpha d^3\mathbf{v} \qquad \nabla\times\mathbf{B} - \frac{\partial \mathbf{E}}{\partial t} = \mu_0 \mathbf{J} \qquad \mathbf{J} = \sum_\alpha q_\alpha \int f_\alpha \mathbf{v} d^3\mathbf{v}$$

- Contains all information about plasma dynamics
- Impossible to solve analytically except in special cases
- Impractical for low frequencies, global geometry

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Fluid equations defined by taking moments of distribution function

- Define moments of distribution function

$$M_n(x,t) = \int_{-\infty}^{\infty} f(x,v,t) v^n dv$$

- Knowledge of N moments allows (in principle) reconstruction of f at N points in velocity space
- N moments of plasma kinetic equation => N fluid equations satisfied by M_{N+1}
  - Each additional moment equation yields more information about velocity distribution
- Must truncate moment equation hierarchy
  - Approximate solution of kinetic equation

Closure of Moment Equations

- Use low-order truncation and closures
- Need to express high-order moments in terms of low-order moments

$$\mathbf{q} = \mathbf{q}[n, T, \ldots], \qquad \Pi = \Pi(p, \mathbf{V}, \ldots.)$$

- Must be obtained from approximate solution of kinetic equation
  - Analytical
  - Numerical
- *There is no general agreement on the closure of the moment equations for hot, magnetized plasmas!*

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Two-fluid Equations ($m_e \sim 0$, $n_e \sim n_i$)

Lowest order moments for ions and electrons:

$$\frac{\partial n}{\partial t} = -\nabla \cdot n\mathbf{V}_e = -\nabla \cdot n\mathbf{V}_i$$

$$mn\frac{d\mathbf{V}_i}{dt} = -\nabla p_i + ne\left(\mathbf{E} + \mathbf{V}_i \times \mathbf{B}\right) - \nabla \cdot \Pi_i + \mathbf{R}$$

$$0 = -\nabla p_e - ne\left(\mathbf{E} + \mathbf{V}_e \times \mathbf{B}\right) - \nabla \cdot \Pi_e - \mathbf{R}$$

$$\mathbf{J} = ne\left(\mathbf{V}_i - \mathbf{V}_e\right)$$

+ Energy equation (+??)

+ Maxwell's equations ($V^2/c^2 << 1$)

+ Closure expressions

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Two-Fluid Equations Present Challenges for Computation

- Extreme separation of time scales

$$\underbrace{\tau_A}_{\textit{Alfvén transit time}} < \underbrace{\tau_S}_{\textit{Sound transit time}} \ll \underbrace{\tau_{evol}}_{\textit{MHD evolution time}} \ll \underbrace{\tau_R}_{\textit{Resistive diffusion time}}$$

- Extreme separation of spatial scales
  - Internal boundary layers, localized and extended along magnetic field lines

$$\delta / L \sim S^{-\alpha} \ll 1 \text{ for } S \gg 1 \qquad S = \tau_R / \tau_A$$

- Extreme anisotropy
  - E.g., accurate treatment of $\mathbf{B} \cdot \nabla$, $\chi_{\|} / \chi_{\perp} \sim 10^{10}$, etc.
- "Parasitic" modes
  - High frequency modes inherent in the formulation that may affect the low frequency dynamics

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Dealing with Parasitic Modes

- The fundamental problem of computational MHD: *Compute low frequency dynamics in presence of high frequency parasitic modes*
  - "Reduction" of mathematical model
    - Eliminate parasitic modes analytically
      - Example: $\nabla \cdot \mathbf{V} = 0$ eliminates sound waves
      - Strong toroidal field allows elimination of fast waves from MHD model
  - "Primitive" equations and "strongly" implicit methods
    - No analytic reduction of equations
    - Use algorithms that allow very large time steps (CFL ~ $10^{4\text{-}5}$)
- Will concentrate on the second approach

There are Different Fluid Models

- Within fluid formulation, different terms are important in different parameter regimes
- Leads to different fluid models of plasmas
  - MHD
  - Hall MHD
  - Drift MHD
  - Transport
- Models distinguished by degree of force balance
- Obtained by "non-dimensionalizing" equations and systematically ordering small parameters:

$$\delta = \rho_i / L << 1, \quad \varepsilon = \omega / \Omega_i, \quad \xi = V / V_{thi}$$

Non-dimensional Equations

Continuity:

$$\varepsilon \frac{\partial n}{\partial t} = -\xi\delta\nabla \cdot n\mathbf{V}_i = -\xi\delta\nabla \cdot n\mathbf{V}_e$$

Ion momentum:

$$\varepsilon\xi \frac{\partial \mathbf{V}_i}{\partial t} + \xi^2\delta\mathbf{V}_i \cdot \nabla\mathbf{V}_i = -\frac{1}{n}\delta\left(\nabla p_i + \frac{\Pi_{i0}}{p_0}\nabla \cdot \Pi_i\right) + \xi\left(\mathbf{E} + \mathbf{V}_i \times \mathbf{B}\right) \quad ,$$

Electron momentum:

$$\xi\mathbf{E} = -\xi\mathbf{V}_e \times \mathbf{B} - \frac{1}{n}\delta\left(\nabla p_e + \frac{\Pi_{e0}}{p_0}\nabla \cdot \Pi_e\right)$$

Pre-Maxwell:

$$\varepsilon \frac{\partial \mathbf{B}}{\partial t} = -\xi\delta\nabla \times \mathbf{E} \quad , \quad \mathbf{J} = \xi\nabla \times \mathbf{B} \quad , \quad \mathbf{J} = n\left(\mathbf{V}_i - \mathbf{V}_e\right)$$

Orderings:

$$\text{time } \varepsilon = \frac{\omega}{\Omega} \quad , \quad \text{flow } \xi = \frac{V_0}{V_{thi}} \quad , \quad \text{length } \delta = \frac{\rho_i}{L} << 1$$

Normalizations:

$$E_0 = V_0 B_0 \quad , \quad J_0 = n_0 e V_0 \quad , \quad p_0 = m n_0 V_{thi}^2$$

Equation of Motion and Generalized Ohm's Law

- Adding and subtracting ion and electron equations:

$$\underbrace{\xi \mathbf{J} \times \mathbf{B} - \frac{1}{n} \delta \nabla p}_{\text{"Equilibrium" forces}} = \underbrace{n\left(\varepsilon \xi \frac{\partial \mathbf{V}_i}{\partial t} + \xi^2 \delta \mathbf{V}_i \cdot \nabla \mathbf{V}_i \right) - \frac{1}{n} \delta \frac{\Pi_{i0}}{p_0} \nabla \cdot \Pi_i}_{\text{Dynamical response}}$$

$$\underbrace{\xi \left(\mathbf{E} + \mathbf{V}_i \times \mathbf{B} \right)}_{\text{Ideal MHD}} = \underbrace{\xi \frac{1}{n} \mathbf{J} \times \mathbf{B} - \delta \frac{1}{n} \left(\nabla p_e + \frac{\Pi_{e0}}{p_0} \nabla \cdot \Pi_e \right)}_{\text{2- fluid and FLR effects}}$$

V × **B** and **J** × **B** enter at same order in ξ

Stress Tensor Scaling

$$\Pi = \Pi_{||} + \Pi_{\wedge} + \Pi_{\perp}$$

$$\Pi_{||} = \mathbf{bb} \cdot \Pi \qquad \Pi_{\wedge} = (\mathbf{I} \times \mathbf{b}) \cdot \Pi \qquad \Pi_{\perp} = (\mathbf{I} - \mathbf{bb}) \cdot \Pi$$

Component	Scaling	Remarks
$\Pi_{\|\|} / p_0$ (Braginskii)	$\xi\delta / (\nu / \Omega)$	• Diverges for low collisionality ($\nu / \Omega \sim \delta^2$)
$\Pi_{\|\|} / p_0$ (Neo-classical)	$(\xi / \delta)(\nu / \Omega)$	• $O(\xi\delta)$ at low collisionality • Remains Ņn scaleÓ
$\Pi_{\perp} / p_0$	$\xi\delta(\nu / \Omega)$	• Vanishingly small at low collisionality • Ignore
$\Pi_{\wedge} / p_0$ (Gyro-viscosity)	$\xi\delta$	• Independent of collisionality • Not dissipative • Important FLR effects

Different Orderings Yield Different Fluid Models

$$\underbrace{\xi \mathbf{J}\times\mathbf{B} - \frac{1}{n}\delta\nabla p}_{\text{"Equilibrium" forces}} = \underbrace{n\left(\varepsilon\xi\frac{\partial \mathbf{V}_i}{\partial t} + \xi^2\delta\mathbf{V}_i\cdot\nabla\mathbf{V}_i\right) - \frac{1}{n}\delta\frac{\Pi_{i0}}{p_0}\nabla\cdot\Pi_i}_{\text{Dynamical response}}$$

$$\underbrace{\xi(\mathbf{E}+\mathbf{V}_i\times\mathbf{B})}_{\text{Ideal MHD}} = \underbrace{\xi\frac{1}{n}\mathbf{J}\times\mathbf{B} - \delta\frac{1}{n}\left(\nabla p_e + \frac{\Pi_{e0}}{p_0}\nabla\cdot\Pi_e\right)}_{\text{2- fluid and FLR effects}}$$

Model	V	ω	Force Balance	OhmÕs Law
Hall MHD	V_{th}/δ	Ω_{ci}	$\mathbf{J}\times\mathbf{B} = n\frac{d\mathbf{V}_i}{dt} + \frac{1}{n}\delta^2\left(\nabla p + \nabla\cdot\Pi_{gv}\right) + O(\delta^3)$	$\mathbf{E}+\mathbf{V}_i\times\mathbf{B} = \frac{1}{n}\mathbf{J}\times\mathbf{B} + O(\delta^2)$
MHD	V_{th}	$\delta\Omega_{ci}$	$\mathbf{J}\times\mathbf{B} = \delta\left(n\frac{d\mathbf{V}_i}{dt} + \nabla p\right) + \delta^2\nabla\cdot\Pi_i^{gv} + O(\delta^4)$.	$\mathbf{E}+\mathbf{V}_i\times\mathbf{B} = \frac{1}{n}\underbrace{\mathbf{J}\times\mathbf{B}}_{O(\delta)} - \delta\frac{1}{n}\nabla p_e = O(\delta)$
Drift MHD	δV_{th}	$\delta^2\Omega_{ci}$	$-\nabla p + \mathbf{J}\times\mathbf{B} = \delta^2\left(n\frac{d\mathbf{V}_i}{dt} + \nabla\cdot\Pi_i^{gv}\right) + O(\delta^4)$	$\mathbf{E}+\mathbf{V}_i\times\mathbf{B} = \frac{1}{n}\left(\mathbf{J}\times\mathbf{B} - \nabla p_e\right)$

The "Standard" Drift Model

- In MHD, $\mathbf{V}_{\perp i} = \mathbf{E} \times \mathbf{B} / B^2 = \mathbf{V}_E$
- In drift ordering, $\mathbf{V}_{\perp i} = \mathbf{V}_E + \mathbf{V}_* + O(\delta^2)$
- Write drift equations in terms of $\mathbf{V}_E$:

Ohm's Law

$$\mathbf{E} = -\left(\mathbf{V}_E + \mathbf{V}_{*i} - \frac{1}{n}\mathbf{J}_\perp\right) \times \mathbf{B} - \frac{1}{n}\nabla p_e + O(\delta^2) \quad ,$$

$$= -\mathbf{V}_E \times \mathbf{B} - \frac{1}{n}\nabla_{||} p_e + \frac{1}{n}\underbrace{\left(-\nabla_\perp p + \mathbf{J} \times \mathbf{B}\right)}_{O(\delta^2)} + O(\delta^2) \quad ,$$

$$= -\mathbf{V}_E \times \mathbf{B} - \frac{1}{n}\nabla_{||} p_e$$

•*Valid only for slight deviations from equilibrium*

Equation of Motion

$$\delta^2 \left(n\frac{d}{dt}\left(\mathbf{V}_{||i} + \mathbf{V}_E\right) + \underbrace{n\frac{d\mathbf{V}_{*i}}{dt} + \nabla \cdot \Pi_i^{gv}\left(\mathbf{V}_i\right)}_{GVC} \right) =$$

$$-\nabla p + \mathbf{J} \times \mathbf{B}$$

$$+ O(\delta^4)$$

•*Gyro-viscous cancellation gives simplified equations*
•*Exact form uncertain*
•*Only applicable to slight deviations from equilibrium*
•*We ignore for general application*

Extended MHD Model

$$Mn\frac{d\mathbf{V}}{dt} = -\nabla p + \mathbf{J}\times\mathbf{B} - \nabla\cdot\Pi_{\|i} - \nabla\cdot\Pi_{gvi}$$

$$\mathbf{E} = -\mathbf{V}\times\mathbf{B} + \frac{1}{ne}\left(\mathbf{J}\times\mathbf{B} - \nabla p_e - \nabla\cdot\Pi_{\|e}\right) + \eta\mathbf{J}$$

$$\frac{\partial\mathbf{B}}{\partial t} = -\nabla\times\mathbf{E} \quad , \qquad \mu_0\mathbf{J} = \nabla\times\mathbf{B}$$

- + Continuity and Energy equations
- + Closure expressions
- Encompasses Hall, MHD, and Drift models
- Terms can be selected by the “user”
  - GV cancellation not explicitly implemented

Extended MHD Properties

- Dispersion
  - Contains all MHD modes ($\omega^2 \sim k^2$)
  - Introduces dispersive modes ($\omega^2 \sim k^4$)
    - Electrons (whistlers)
    - Ions + electrons (kinetic Alfvén wave)
    - Ions only ("gyro-viscous" waves)
  - If extended MHD just produced more troublesome parasitic modes, who cares? However…..
- Stability
  - Drift stabilization at moderate to high k
  - Neo-classical de-stabilization of magnetic islands
  - ++++?

Dispersion in Extended MHD

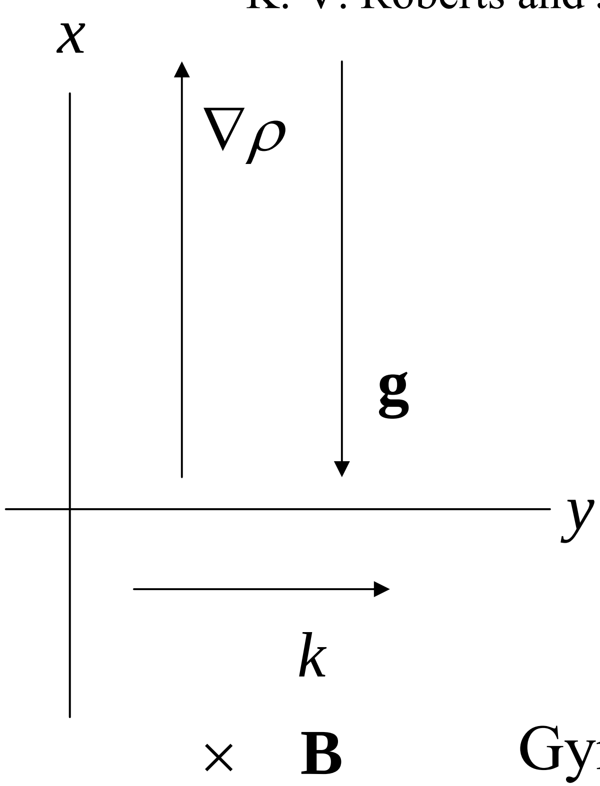
Mode	Origin	Wave Equation	Dispersion	Comments
Whistler	$\mathbf{J}\times\mathbf{B}$ in Ohm	$\frac{\partial^2 \mathbf{B}}{\partial t^2} = -\left(\frac{V_A^2}{\Omega}\right)^2 (\mathbf{b}\cdot\nabla)^2 \nabla^2 \mathbf{B}$	$\omega^2 = V_A^2 k^2 \left[1 + \frac{1}{\beta}\left(\rho_i k_\parallel\right)^2\right]$	• finite $k_\parallel$ • electron response
KAW	$\nabla_\parallel p_e$ in Ohm	$\frac{\partial^2 \mathbf{B}}{\partial t^2} = \left(\frac{V_A V_{th*}}{\Omega}\right)^2 (\mathbf{b}\cdot\nabla)^2$ $\nabla\times\left[\mathbf{bb}\cdot\nabla\times\mathbf{B}\right]$	$\omega^2 = V_A^2 k_\parallel^2 \left[1 + \left(\rho_s k_\perp\right)^2\right]$	• finite $k_\parallel$, $k_\perp$ • ion and electron response
Parallel ion GV	η_4 term in $\nabla\cdot\Pi^{GV}$	$\rho\frac{\partial^2 \mathbf{V}_\perp}{\partial t^2} = -\eta_4^2 \nabla_\parallel^4 \mathbf{V}_\perp$	$\omega_{L\pm} = V_A k_\parallel \left[\pm 1 + \frac{1+\beta}{2\sqrt{\beta}}\left(\rho_i k_\parallel\right)\right]$ $\omega_{R\pm} = V_A k_\parallel \left[\pm 1 - \frac{1+\beta}{2\sqrt{\beta}}\left(\rho_i k_\parallel\right)\right]$	• finite $k_\parallel$ • ion response
Perp. ion GV	η_3 term in $\nabla\cdot\Pi^{GV}$	$\rho\frac{\partial \mathbf{V}_\perp}{\partial t} = -\eta_3^2 \nabla_\perp^4 \mathbf{V}_\perp$	$\omega^2 = V_A^2 k_\perp^2 \left[1 + \frac{\gamma\beta}{2} + \frac{\beta}{16}\left(\rho_i k_\perp\right)^2\right]$	• finite $k_\perp$ • ion response

Notation: $\rho_i = V_{th_i}/\Omega$ is the ion gyro-radius; $V_{th*} = \sqrt{T_e/m_i}$; $\rho_s = V_{th*}/\Omega$; $\eta_4 = nT_i/2\Omega$; $\eta_3 = 2\eta_4$

$$\omega^2 \sim k^4 \rightarrow \Delta t \sim \Delta x^2$$ Requires implicit methods

Stability: Gravitational Interchange

K. V. Roberts and J. B. Taylor, Phys. Rev. Letters **8**, 197 (1962).



$$\frac{\partial\rho}{\partial t}+\nabla\cdot\rho\mathbf{V}=0$$

$$\rho\frac{d\mathbf{V}}{dt}=-\nabla\left(p+\frac{B^2}{2\mu_0}\right)+\rho\mathbf{g}-\nabla\cdot\Pi$$

$$\mathbf{E}=-\mathbf{V}\times\mathbf{B}+\frac{M}{\rho e}\left[\rho\frac{d\mathbf{V}}{dt}+\nabla p_i-\rho\mathbf{g}+\nabla\cdot\Pi\right]$$

Assume electrostatic:

$$\nabla\times\mathbf{E}=0 \quad \Rightarrow$$

$$\nabla\cdot\mathbf{V}+\underbrace{\boxed{\frac{1}{\Omega}\nabla\times\frac{d\mathbf{V}}{dt}-\frac{1}{\Omega\rho^2}\nabla\rho\times\nabla\cdot\Pi}}_{\textit{Extended MHD}}=0$$

Gyro-viscosity:

$$(\nabla\cdot\Pi)_x=-(\rho_0\nu_0)'ikV_x+\rho_0\nu_0k^2V_y$$

$$(\nabla\cdot\Pi)_y=-(\rho_0\nu_0)'ikV_y-\rho_0\nu_0k^2V_x$$

G-mode stabilization

	Dispersion Relation	Solution	Stabilizing Wave Number
MHD $(\xi = 0,\ \zeta = 0)$	$\omega^2 + g\eta = 0$	$\omega = i\sqrt{g\eta}$	None
2-Fluid $(\xi = 0,\ \zeta = 1)$	$\omega^2 - \frac{gk}{\Omega_0}\omega + g\eta = 0$	$2\omega = \frac{gk}{\Omega_0} \pm \sqrt{\left(\frac{gk}{\Omega_0}\right)^2 - 4g\eta}$	$k^2 > \frac{4\eta\Omega_0^2}{g}$
Gyro-Viscosity $(\xi = 1,\ \zeta = 0)$	$\omega^2 - v_0\eta k\omega + g\eta = 0$	$2\omega = v_0\eta k \pm \sqrt{(v_0\eta k)^2 - 4g\eta}$	$k^2 > \frac{4g}{v_0^2\eta}$
Full Extended MHD $(\xi = 1,\ \zeta = 1)$	$\omega^2 - \left(\frac{gk}{\Omega_0} + v_0\eta k\right)\omega + g\eta = 0$	$2\omega = \frac{gk}{\Omega_0} + v_0\eta k \pm \sqrt{\left(\frac{gk}{\Omega_0} + v_0\eta k\right)^2 - 4g\eta}$	$k^2 > \frac{4g\eta}{\left(\frac{g}{\Omega_0} + v_0\eta\right)^2}$

Form of the Gyro-viscous Stress (Hooke's Law for a Magnetized Plasma)

- Braginskii: $$\Pi_{\wedge} = \Pi^{gv} = \frac{p}{4\Omega}\left[(\mathbf{b}\times\mathbf{W})\cdot(\mathbf{I}+3\mathbf{bb}) + transpose\right]$$

$$\mathbf{W} = \nabla\mathbf{V} + \nabla\mathbf{V}^T - \frac{2}{3}\mathbf{I}\nabla\cdot\mathbf{V}$$

- Suggested modifications for consistency (*Mikhailovskii and Tsypin, Hazeltine and Meiss, Simakov and Catto, Ramos*) involve adding term proportional to the ion heat rate of strain:

$$\Pi_{\wedge q} = \frac{2}{5\Omega}\left[\mathbf{b}\times\mathbf{W}_q\cdot(\mathbf{I}+3\mathbf{bb}) + transpose\right]$$

$$\mathbf{W}_q = \nabla\mathbf{q}_i + \nabla\mathbf{q}_i^T - \frac{2}{3}\mathbf{I}\nabla\cdot\mathbf{q}_i$$

- Implicit numerical treatment difficult
- What is the effect of this term on dispersion and stability?
  - *Does it introduce new normal modes?*
  - *Does it alter stability properties?*

Ion Heat Stress Has Little Effect on Important Dynamics

$$\Pi_{\wedge q} = \frac{2}{5\Omega}\left[\mathbf{b}\times\mathbf{W}_q\cdot(\mathbf{I}+3\mathbf{bb}) + transpose\right]$$

$$\mathbf{W}_q = \nabla\mathbf{q}_i + \nabla\mathbf{q}_i^T - \frac{2}{3}\mathbf{I}\nabla\cdot\mathbf{q}_i \qquad \mathbf{q} = -\kappa_{\|}\nabla_{\|}T - \kappa_{\perp}\nabla_{\perp}T - \kappa_{\wedge}\mathbf{b}\times\nabla_{\perp}T$$

$$\rho_0\frac{\partial\mathbf{V}}{\partial t} = -\nabla p - \nabla\cdot\Pi_{\wedge q}$$

$$\frac{\partial p}{\partial t} = -\gamma p_0\nabla\cdot\mathbf{V}$$

$$\omega^2 = C_s^2 k^2\left[1 + f(\theta)(\rho_i k)^2\right] \qquad f(0) = 0 \qquad f(\pi/2) = 1$$

- Dispersive effect on compressional waves, but……
- Negligible effect on *g*-mode stability
- Simplification: *ignore these terms (for now!)*

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Careful Computational Approach is Required

- Spatial approximation
  - Must capture anisotropy and global geometry
    - Flux aligned grids
    - High order finite elements
- Temporal approximation
  - Must compute for long times
    - Require implicit methods
    - Semi-implicit methods have proven useful

Solenoidal Constraint

- Faraday: $$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E} \quad \Rightarrow \quad \frac{\partial}{\partial t} \nabla \cdot \mathbf{B} = 0$$
- Depends on $\nabla \cdot \nabla \times = 0$
- Different discrete approximations
  - Modified wave system $$\frac{\partial \mathbf{U}}{\partial t} + \nabla \cdot \mathbf{F} = \mathbf{R} \nabla \cdot \mathbf{B}$$
  - Projection $$\mathbf{B}' = \mathbf{B} + \nabla \phi \quad \nabla^2 \phi = -\nabla \cdot \mathbf{B}$$
  - Diffusion $$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E} + \kappa \nabla \nabla \cdot \mathbf{B} \;, \quad \frac{\partial}{\partial t} \nabla \cdot \mathbf{B} = \nabla \cdot \kappa \nabla \nabla \cdot \mathbf{B}$$
  - Grid properties $\overline{\nabla}_a \cdot \overline{\nabla}_b \times \equiv 0$ E.g., staggered grid, “dual mesh”

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Galerkin Methods

- Finite differences and finite volumes minimize error locally
  - Based on Taylor series expansion
- Galerkin methods minimize weighted error
  - Based on expansion in basis functions
- Solve "weak form" of problem

$$\frac{\partial u(x,t)}{\partial t} = Lu(x,t) \quad \rightarrow \quad \int v\left(\frac{\partial u}{\partial t} - Lu\right)dx = 0$$

Minimize error by expansion in basis functions and determining coefficients

Galerkin Discrete Approximation

$$M_{ij}\frac{du_j}{dt} = L_{ij}u_j \qquad M_{ij} = \underbrace{\int dV \beta_i \alpha_j}_{Mass\ Matrix} \qquad L_{ij} = \underbrace{\int dV \beta_i L \alpha_j}_{Response\ Matrix}$$

- Solution generally requires inverting the mass matrix, even for “explicit” methods
- Different basis functions give different methods
  - Usually: $\beta_i = \alpha_i$
    - α_i=exp(ikx) => Fourier spectral methods
    - α_i=localized polynomial => finite element methods

Finite Elements

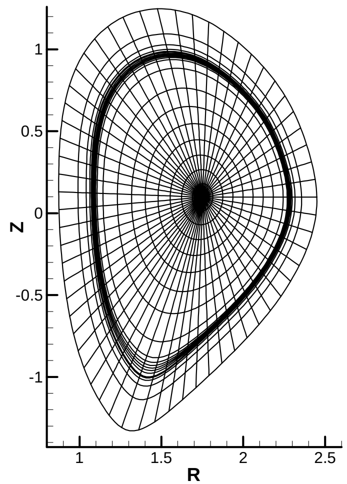
- Project onto basis of locally defined polynomials of degree p
- Polynomials of degree p can converge as fast as h^{p+1}
- Integrate by parts:

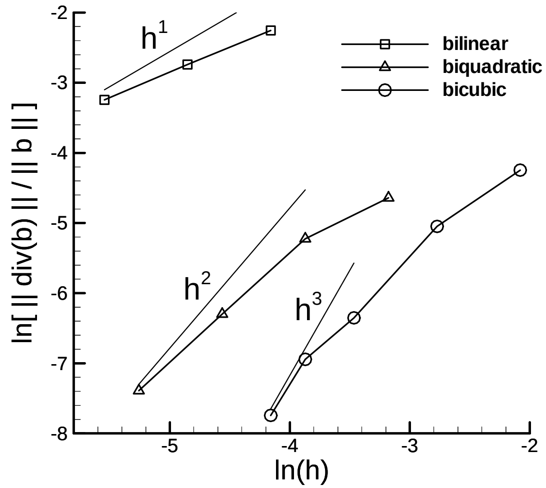
$$\int \alpha_i \alpha_j \frac{d\mathbf{V}_j}{dt} dV = -\int \alpha_i \nabla \cdot \Pi(\alpha_j \mathbf{V}_j) dV = \underbrace{\int \nabla \alpha_i \cdot \Pi(\alpha_j \mathbf{V}_j) dV}_{Evaluate\ \Pi\ only} - \underbrace{\int_S \alpha_j \mathbf{n} \cdot \Pi(\alpha_j \mathbf{V}_j) dS}_{Boundary\ \ condition}$$

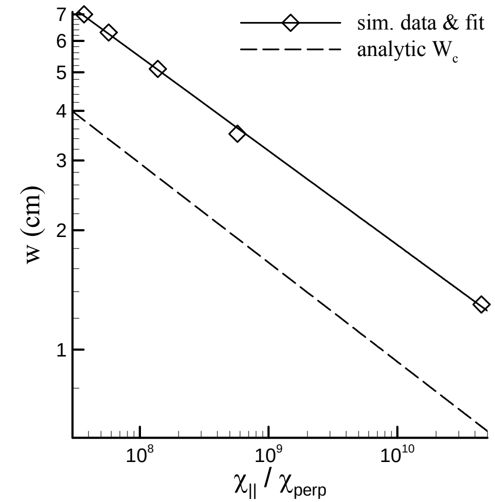
  - Simplifies implementation of complex closure relations
  - Natural implementation of boundary conditions

- Automatically preserves self-adjointness
- Works well with arbitrary grid shapes

Three Examples of Favorable Properties of High Order Elements







Grid used for ELM studies

Non-uniform meshes retain high-order convergence rate

Magnetic divergence constraint

Scalings show expected convergence rates

Critical island width for temperature flattening

Dealing with extreme anisotropy

Agreement on scaling

Multiple Time Scales
(Parasitic Waves)

- MHD contains widely separated time scales (eigenvalues)

$$\frac{\partial u}{\partial t} = \underbrace{\Omega u}_{\text{Full MHD operator}} = \underbrace{Fu}_{\substack{\text{Fast time scales:}\\ \text{Alfvén waves, soundwaves, etc}\\ \text{(parasitic waves)}}} + \underbrace{Su}_{\substack{\text{Slow time scales:}\\ \text{Resistive instabilities, island evolution,}\\ \text{(interesting physics)}}}$$

- "Parasitic" waves are properties of the physics problem but are not the dynamics of interest
- Treat only "fast" part of operator implicitly to avoid time step restriction

$$\frac{u^{n+1} - u^n}{\Delta t} = Fu^{n+1} + Su^n$$

- Precise decomposition of Ω for complex nonlinear system is often difficult or impractical to achieve algebraicly

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Dealing with Parasitic Waves

- Original idea from André Robert (1971)
  - Gravity waves in climate modeling
- F and Ω are often known, but an expression for S is difficult to achieve
  - Ω: full MHD operator
  - F: linearized MHD operator
- Use operator splitting: $\Omega = F + S \quad \Rightarrow \quad S = \Omega - F$

$$\frac{u^{n+1}-u^n}{\Delta t} = Fu^{n+1} + (\Omega - F)u^n = \Omega u^n + \Delta t F\left(\frac{u^{n+1}-u^n}{\Delta t}\right)$$

- Expression for S not needed

Semi-Implicit Method

$$\frac{u^{n+1}-u^{n}}{\Delta t}=Fu^{n+1}+(\Omega-F)u^{n}=\Omega u^{n}+\Delta tF\left(\frac{u^{n+1}-u^{n}}{\Delta t}\right)$$

- Recognize that the operator F is completely arbitrary!!

$$(I-\underbrace{\Delta t G}_{\substack{S.\,I.\\ operator}})u^{n+1}=\underbrace{(I-\Delta t\Omega)u^{n}}_{Explicit}-\underbrace{\Delta t G u^{n}}_{\substack{S.\,I.\\ operator}}$$

- G can be chosen for accuracy and ease of inversion
  - G should be easier to invert than F (or Ω, e.g., toroidal coupling)
  - G should approximate F for “modes of interest”
  - Some choices are better than others!
- Has proven to be very useful for resistive and extended MHD
  - Used for spheromak, RFP, tokamak, and solar corona modeling

SI Operator for MHD

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{V} \times \mathbf{B} - \eta \mathbf{J})$$

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot \rho \mathbf{V}$$

$$\frac{\partial p}{\partial t} = -\nabla \cdot p\mathbf{V} - (\gamma - 1) p \nabla \cdot \mathbf{V}$$

$$\rho \frac{\partial \mathbf{V}}{\partial t} = -\rho \mathbf{V} \cdot \nabla \mathbf{V} - \nabla p + \mathbf{J} \times \mathbf{B} + \boxed{\alpha \Delta t^2 \left[\underbrace{\nabla \times \nabla \times \left(\frac{\partial \mathbf{V}}{\partial t} \times \mathbf{B}_0 \right) \times \mathbf{B}_0}_{\text{Alfvén waves}} + \underbrace{\nabla \gamma P_0 \nabla \cdot \frac{\partial \mathbf{V}}{\partial t}}_{\text{Sound waves}} \right]}$$

- Ideal MHD operator (Lerbinger and Luciani)
- Anisotropic, self-adjoint
- Avoids implicit toroidal coupling (great simplification)
- Accurate linear results for CFL ~ $10^{4\text{-}5}$ (=> Condition number ~ 10^{10}!!)

$$kV\Delta t < 1$$

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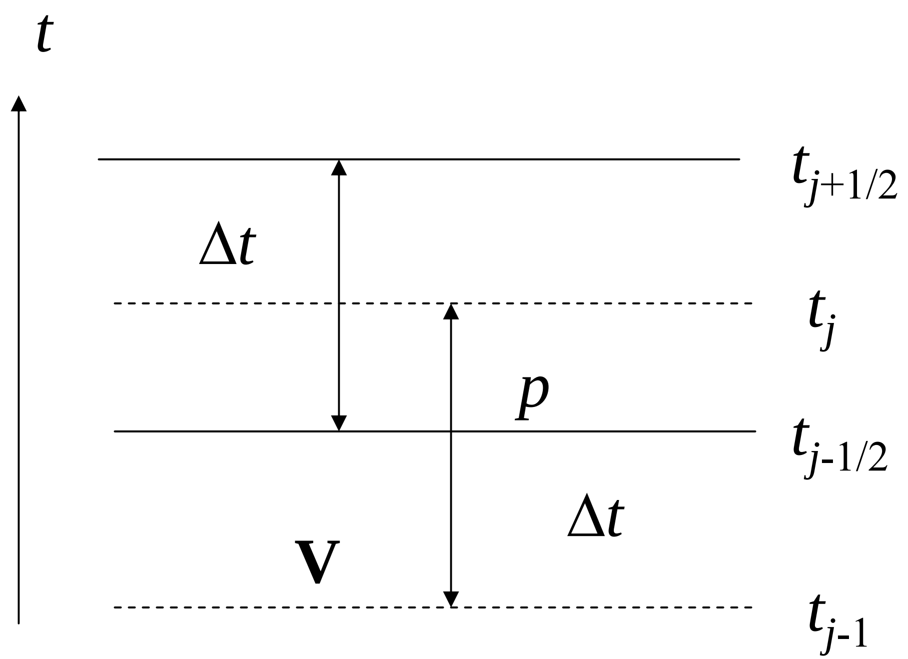
Semi-Implicit Leap Frog

- Variables staggered at different time levels
- SI operator on velocity



$$\Delta V = V^{j} - V^{j-1}$$

$$\Delta p = p^{j+1/2} - p^{j-1/2}$$

$$\frac{\Delta V}{\Delta t} = -\nabla p^{j-1/2} + \alpha \Delta t S\left(\frac{\Delta V}{\Delta t}\right)$$

$$V^{j} = V + \Delta V$$

$$\frac{\Delta p}{\Delta t} = -\gamma P_0 \nabla \cdot V^{j}$$

$$p^{j+1/2} = p^{j-1/2} + \Delta p$$

Extended MHD Time Advance

- "Implicit leap-frog" (also used in MHD)
  - Maintains numerical stability without introducing numerical dissipation
- MHD advance unchanged (semi-implicit self-adjoint operators)
- Need to invert *non-self-adjoint* operators at each step for dispersive modes
- Requires high performance parallel linear algebra software

Implicit Leap Frog for Extended MHD

Momentum

$$m_i n^{j+1/2}\left(\frac{\Delta \mathbf{V}}{\Delta t}+\underbrace{\frac{1}{2}\mathbf{V}^{j}\cdot\nabla\Delta\mathbf{V}+\frac{1}{2}\Delta\mathbf{V}\cdot\nabla\mathbf{V}^{j}}_{\textit{Implicit advection}}\right)-\underbrace{\Delta t L^{j+1/2}(\Delta\mathbf{V})}_{\textit{SI MHD}}+\underbrace{\nabla\cdot\Pi_i(\Delta\mathbf{V})}_{\textit{Includes ALL stresses}}=$$

$$\mathbf{J}^{j+1/2}\times\mathbf{B}^{j+1/2}-m_i n^{j+1/2}\mathbf{V}^{j}\cdot\nabla\mathbf{V}^{j}-\nabla p^{j+1/2}-\nabla\cdot\Pi_i\left(\mathbf{V}^{j}\right)$$

Continuity

$$\frac{\Delta n}{\Delta t}+\frac{1}{2}\mathbf{V}^{j+1}\cdot\nabla\Delta n=-\nabla\cdot\left(\mathbf{V}^{j+1}\cdot n^{j+1/2}\right)$$

Energy

$$\frac{3n}{2}\left(\frac{\Delta T_\alpha}{\Delta t}+\frac{1}{2}\mathbf{V}_\alpha^{j+1}\cdot\nabla\Delta T_\alpha\right)+\frac{1}{2}\underbrace{\nabla\cdot\mathbf{q}_\alpha(\Delta T_\alpha)}_{\textit{Anisotropic thermal conduction}}=$$

$$-\frac{3n}{2}\mathbf{V}_\alpha^{j+1}\cdot\nabla T_\alpha^{j+1/2}-nT_\alpha^{j+1/2}\nabla\cdot\mathbf{V}_\alpha^{j+1}-\nabla\cdot\mathbf{q}_\alpha\left(T_\alpha^{j+1/2}\right)+Q_\alpha^{j+1/2}$$

Maxwell/Ohm

$$\frac{\Delta\mathbf{B}}{\Delta t}+\frac{1}{2}\mathbf{V}^{j+1}\cdot\nabla\Delta\mathbf{B}+\underbrace{\frac{1}{2}\nabla\times\frac{1}{ne}\left(\mathbf{J}^{j+1/2}\times\Delta\mathbf{B}+\Delta\mathbf{J}\times\mathbf{B}^{j+1/2}\right)}_{\textit{Implicit HALL term}}+\underbrace{\frac{1}{2}\nabla\times\eta\Delta\mathbf{J}}_{\textit{Implicit resistive term}}=$$

$$-\nabla\times\left[\frac{1}{ne}\left(\mathbf{J}^{j+1/2}\times\mathbf{B}^{j+1/2}-\nabla p_e\right)-\mathbf{V}^{j+1}\times\mathbf{B}^{j+1/2}+\eta\mathbf{J}^{j+1/2}\right]$$

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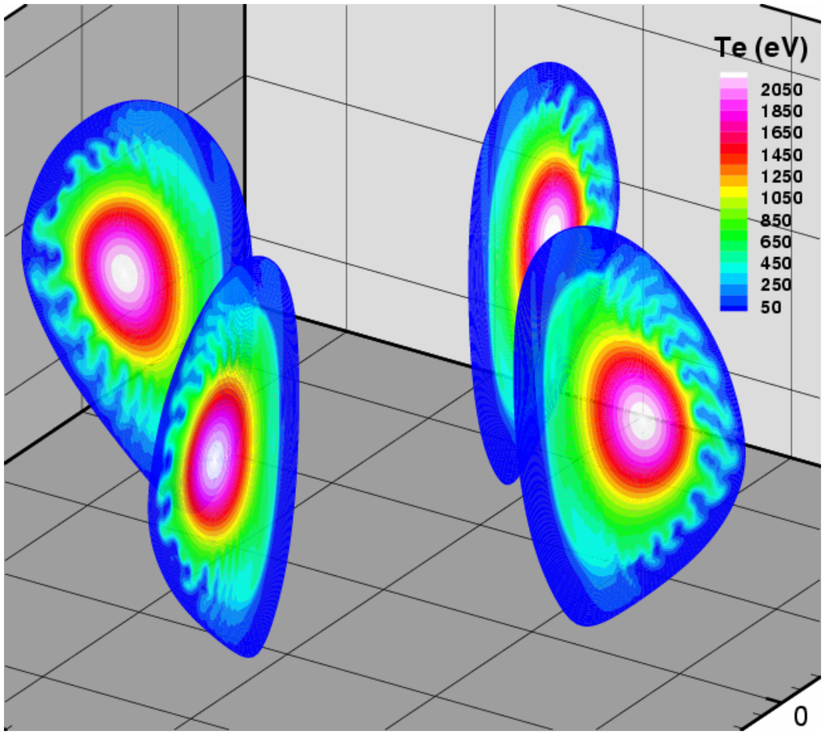
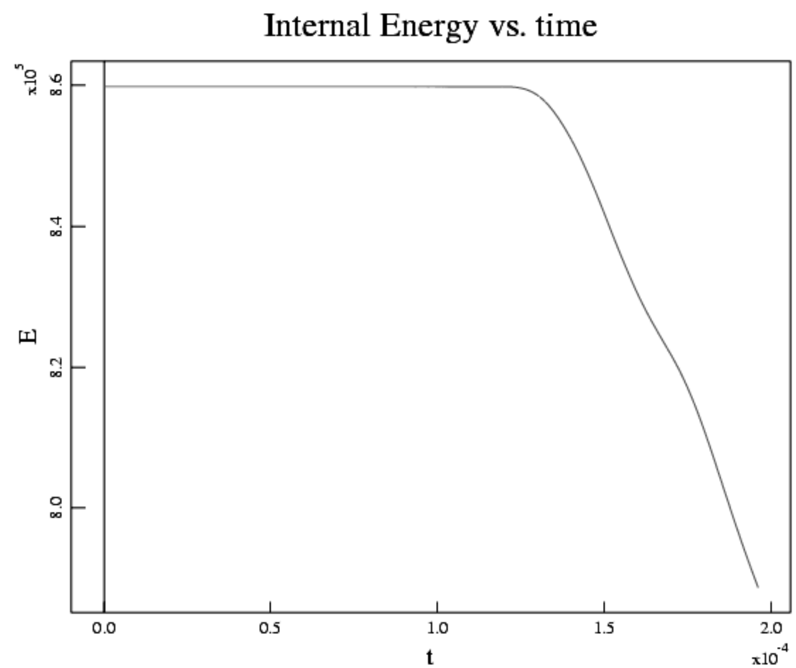
Nonlinear ELM Evolution

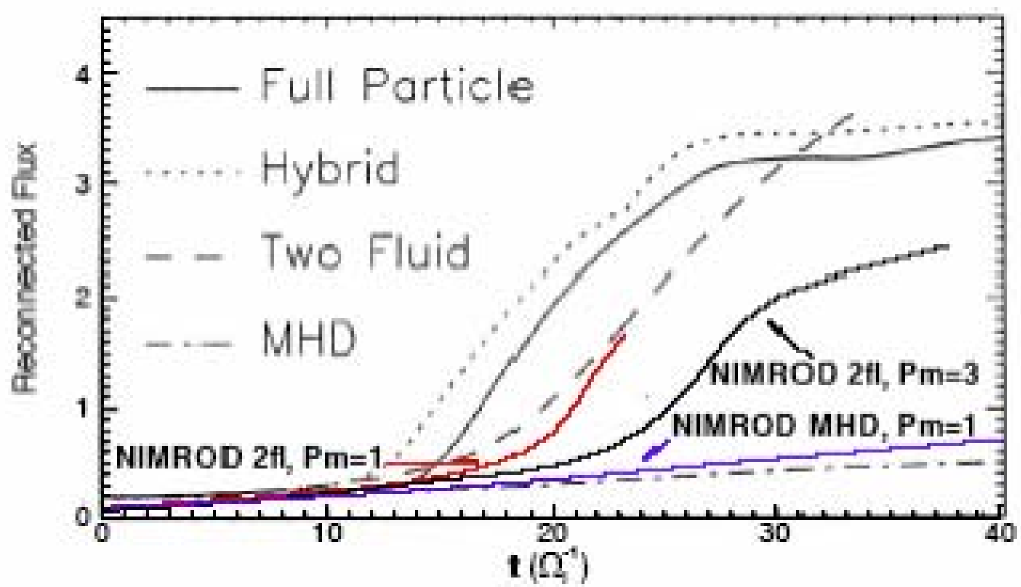




- Anisotropic thermal conduction
- ELM interaction with wall

- 70 kJ lost in 60 μsec
- 2-fluid and gyro-viscosity have little effect on linear properties

Two-fluid Reconnection
GEM Problem



- 2-D slab
- $\eta = 0.005$
- Good agreement with many other calculations
- Computed with same code used for tokamaks, spheromaks, RFPs

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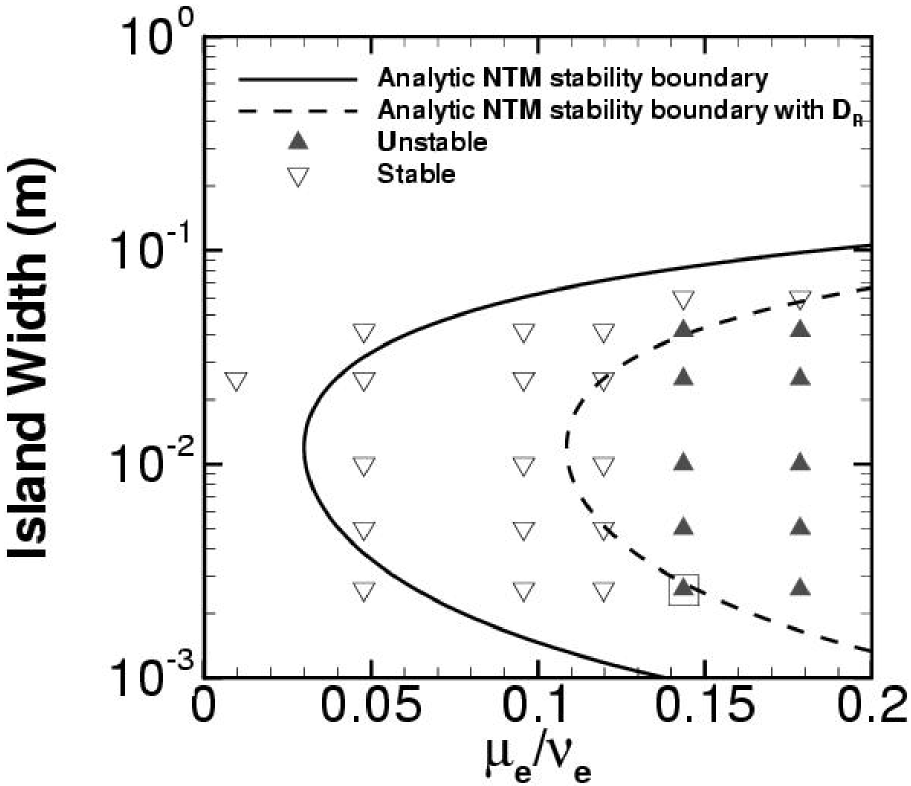
“Heuristic Closure” Captures Essential Neoclassical Physics



(Gianakon et.al., Phys. Plasmas **9**, 536 (2002)

Neo-classical theory gives flux surface average

Local form for stress tensor forces:

$$\nabla \cdot \Pi_\alpha = \rho_\alpha \mu_\alpha \left\langle B^2 \right\rangle \frac{\mathbf{V}_\alpha \cdot \mathbf{e}_\theta}{\left(B_\alpha \cdot \mathbf{e}_\theta\right)^2} \mathbf{e}_\theta$$

- Valid for both ion and electrons
- Energy conserving and entropy producing
- Gives:
    - bootstrap current
    - neoclassical resistivity
    - polarization current enhancement

Beyond Extended MHD: Parallel Kinetic closures

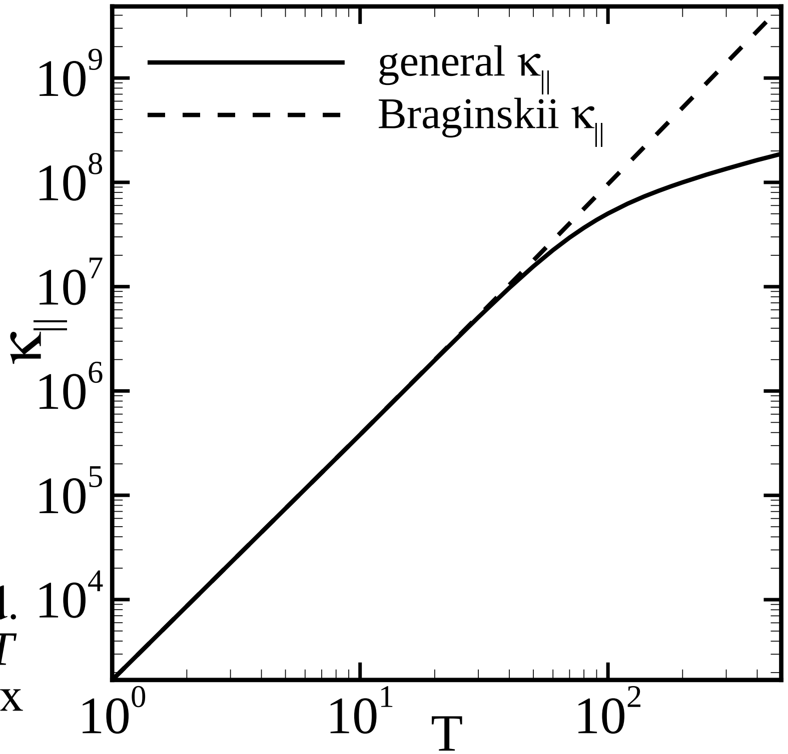
- Parallel closures for $q_{\|}$ and $\Pi_{\|}$ derived using Chapman-Enskog-like approach.

- Non-local; requires integration along perturbed field lines.

- General closures map continuously from collisional to nearly collisionless regime.

- General $q_{\|}$ closure predicts collisional response for heat flow inside magnetic island. As plasma becomes moderately collisional (T > 50 eV), general closure predicts correct flux limited response.

- Incorporated into global extended MHD algorithms.



Thermal diffusivity as function of T showing $T^{5/2}$ response of Braginskii and general closure.

Beyond Extended MHD: Kinetic Minority Species

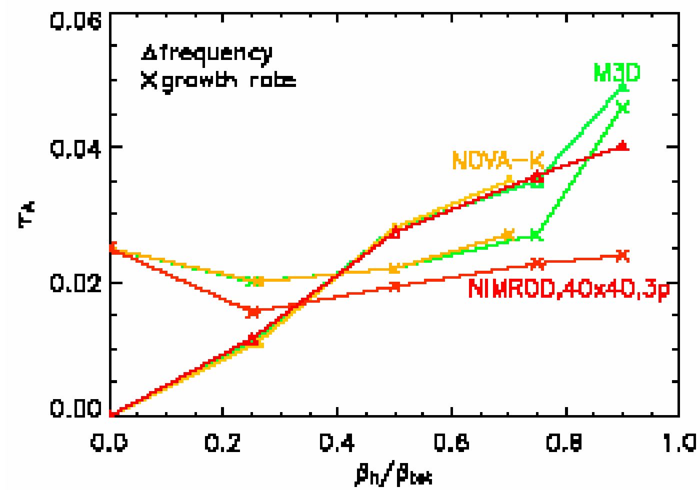
- Minority ions species affects bulk evolution:

$$n_h << n_0 \quad , \qquad \beta_h \sim \beta_0$$

$$Mn\frac{d\mathbf{V}}{dt} = \mathbf{J}\times\mathbf{B} - \underbrace{\nabla}_{\textit{Bulk Plasma}}\cdot\Pi$$

$$- \underbrace{\nabla\cdot\Pi_h}_{\textit{Hot Minority Ion Species}}$$

$$\delta\Pi_h = \int M(\mathbf{v}-\mathbf{V}_h)(\mathbf{v}-\mathbf{V}_h)\delta f(\mathbf{x},\mathbf{v})d^3\mathbf{v}$$



- δf determined by kinetic particle simulation in evolving fields
- Demonstrated transition from internal kink to fishbone
- Benchmark of three codes

Constraints on Modeling

Balance of algorithm performance and problem requirements with available cycles

$$\underbrace{\frac{N^{\alpha}Q}{\Delta t}}_{Algorithms} = 3\times10^{7}\ \underbrace{\frac{\varepsilon P}{CT}}_{Constraints}$$

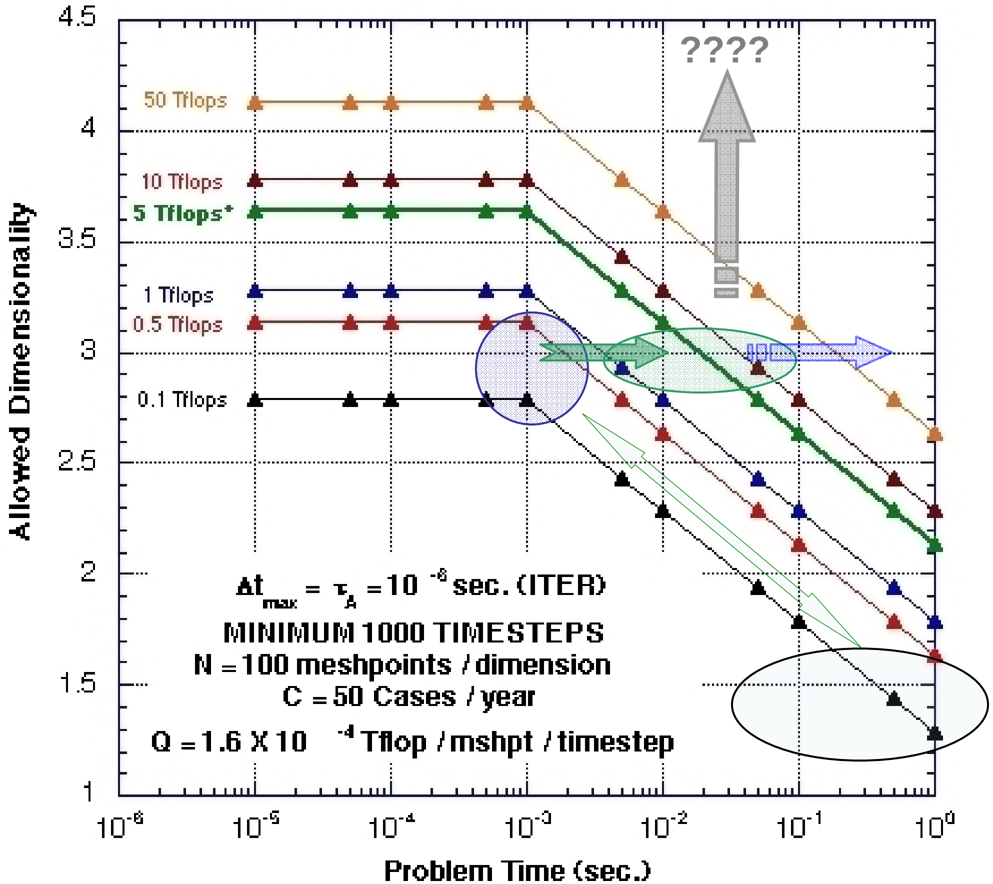
Algorithms:

- N - # of meshpoints for each dimension
- α - # of dimensions
  - 1.5 - transport
  - 3 (spatial) fluid
  - 5-6 kinetic (spatial + velocity)
- Q - code-algorithm requirements (Tflop / meshpoint / timestep)
- Δt - time step (seconds)

Constraints:

- P - peak hardware performance (Tflop/sec)
- ε - hardware efficiency
  - εP - delivered sustained performance
- T - problem time duration (seconds)
- C - # of cases / year
  - 1 case / week ==> C ~ 50

“The Future”



$$\underbrace{\frac{N^{\alpha} Q}{\Delta t}}_{Algorithms} = 3 \times 10^{7} \underbrace{\frac{\varepsilon P}{CT}}_{Constraints}$$

Assumptions:

- Performance is *delivered*
- Implicit algorithm
- Q ind. of Δt (!!)

Requirements:

- At least 3-D physics required
- Required problem time: 1 msec - 1 sec

Conclusions:

- 3-D (i.e., fluid) calculations for times of ~ 10 msec within reach
- Longer times require next generation computers (or better algorithms)
- *Higher dimensional (kinetic) long time calculations unrealistic*
- *Integrated kinetic effects must come through low dimensionality fluid closures*

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Summary

- Fluid models are an approximation to the plasma kinetic equation, but are *required* for modeling low frequency response of hot, magnetized plasmas with global geometry
  - Direct kinetic calculations are impractical
- Primitive equations and implicit methods have proven successful in modeling a variety of plasmas
- Implicit methods are required for handling the dispersive terms of MHD. An understanding of the dispersive characteristics of discretized equations needed.
- "Kinetic" effects must be captured through fluid closures
  - "Best" form of fluid equations still unknown
  - Often problem dependent
- Next step is direct coupling of kinetic/fluid/transport models