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LOW-COLLISIONALITY REPRESENTATION OF THE HIGHER FLUID MOMENTS*

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EXACT SET OF FLUID EQUATIONS FOR A QUASI-NEUTRAL PLASMA

$$\sum_\alpha e_\alpha n_\alpha \ = \ 0 \ ,$$

$$\sum_\alpha e_\alpha n_\alpha \mathbf{u}_\alpha \ = \ \mathbf{j} \ = \ \nabla \times \mathbf{B} \ ,$$

$$\frac{\partial \mathbf{B}}{\partial t} \ + \ \nabla \times \mathbf{E} \ = \ 0,$$

$$\frac{\partial n_\alpha}{\partial t} \ + \ \nabla \cdot (n_\alpha \mathbf{u}_\alpha) \ = \ 0,$$

$$m_\alpha n_\alpha \Big[\frac{\partial \mathbf{u}_\alpha}{\partial t} + (\mathbf{u}_\alpha \cdot \nabla)\mathbf{u}_\alpha\Big] \ + \ \nabla \cdot \mathsf{\mathbf{P}}_\alpha \ - \ e_\alpha n_\alpha \Big(\mathbf{E} + \mathbf{u}_\alpha \times \mathbf{B}\Big) \ - \ \mathbf{F}_\alpha^{coll} \ = \ 0,$$

Closure of this system requires specification of the stress tensors $\mathsf{\mathbf{P}}_\alpha$ and the collisional friction forces $\mathbf{F}_\alpha^{coll}$.

I. INTRODUCTORY CONSIDERATIONS

Except in a high-collisionality regime, a rigorous closure of the set of fluid equations necessitates kinetic theory.

Because of their high dimensionality, any practical solutions of the kinetic equations involve restrictive approximations to a significant degree.

The fluid theory of the higher-rank moments of the distribution function provides exact or less restrictive constraints on those kinetic solutions, which should be worth exploiting when developing closure schemes.

Higher-rank fluid moment analyses rely on a number of asymptotic ordering choices. In particular it is worth reviewing the implications of different possible assumptions on:

$k_\perp \rho_i$ **ordering**

collisionality regime

ω/Ω_{ci} **and** u_i/v_{thi} **orderings**

reduction to eliminate the fast magnetosonic wave

$k_\perp \rho_i$ ORDERING

A) Gyrokinetic ordering

Allows arbitrary perpendicular fluctuation wavenumbers, $k_\perp \rho_i \sim 1$

Requires: small amplitude fluctuations, $\delta B/B \sim \delta F/F \sim \rho_i/L \ll 1$

small ratio, $k_\parallel/k_\perp \ll 1$**, of fluctuation wavenumbers**

small electric field, $E/(v_{thi}B) \sim \rho_i/L \ll 1$

reduced form of $\delta B_\parallel$ **that eliminates the fast magnetosonic wave**

isotropic lowest-order distribution functions in practical applications

$\Rightarrow$ **B) Drift-kinetic ordering**

Requires small perpendicular fluctuation wavenumbers, $k_\perp \rho_i \sim \rho_i/L \ll 1$

Allows: arbitrary amplitude fluctuations, $\delta B/B \sim \delta F/F \sim 1$

comparable parallel and perpendicular fluctuation wavenumbers

large electric field, $E/(v_{thi}B) \sim 1$

fully electromagnetic fluctuations including fast magnetosonic waves

strong anisotropies and far from Maxwellian distribution functions

COLLISIONALITY REGIME

A) High collisionality $(v_{th\alpha}/\nu_\alpha^{coll} \ll L)$

Allows a consistent closure of the fluid equations (Braginskii ...)

Little relevance to magnetic fusion conditions

$\Rightarrow$ **B) Low collisionality** $(v_{th\alpha}/\nu_\alpha^{coll} \gtrsim L)$

Most relevant to magnetic fusion conditions

Leads in general to: far from Maxwellian distribution functions

strong anisotropy $(p_{\alpha\|} - p_{\alpha\perp} \sim p_\alpha \quad ...)$

Near Maxwellian distribution functions can be justified only near closed flux surface equilibria, hence small amplitude fluctuations

ω/Ω_{ci} AND u_i/v_{thi} ORDERINGS

$\Rightarrow$ **A) MHD ordering (fast dynamics)**

$$\delta \sim \rho_i/L \ll 1$$

$$\omega/\Omega_{ci} \sim \delta$$

$$u_i/v_{thi} \sim 1$$

$\Rightarrow$ **B) Drift ordering (slow dynamics near equilibrium)**

$$\delta \sim \rho_i/L \ll 1$$

$$\omega/\Omega_{ci} \sim \omega_{*\alpha}/\Omega_{ci} \sim \delta^2$$

$$u_i/v_{thi} \sim u_{*\alpha}/v_{thi} \sim \delta$$

Consistent implementation of the drift ordering requires either a second order accurate evaluation of the CGL pressures (i.e. $p_{i\|,\perp} = O(m_i n_i v_{thi}^2) + O(\delta^2 m_i n_i v_{thi}^2)$ **which can only be accomplished kinetically), or the introduction of a small parallel gradient (b** $\cdot \nabla = O(\delta^2/L)$**) subsidiary ordering**

Calling $\mathsf{P}_i \equiv p_{i\perp}\mathsf{I} + (p_{i\|} - p_{i\perp})\mathrm{bb} + \hat{\mathsf{P}}_i$, with $\hat{\mathsf{P}}_i : \mathsf{I} = \hat{\mathsf{P}}_i : (\mathrm{bb}) = 0$, the parallel component of the ion momentum conservation equation is:

$$m_i n_i \ \mathbf{b} \cdot \Big[\frac{\partial \mathbf{u}_i}{\partial t} + (\mathbf{u}_i \cdot \nabla)\mathbf{u}_i\Big] \ + \ \mathbf{b} \cdot \nabla p_{i\|} \ - \ (p_{i\|} - p_{i\perp}) \ \mathbf{b} \cdot \nabla(\ln B) \ + \ \mathbf{b} \cdot (\nabla \cdot \hat{\mathsf{P}}_i - e_i n_i \mathbf{E} - \mathbf{F}_i^{coll}) \ = \ 0,$$

In the drift ordering, $m_i n_i \ \mathbf{b} \cdot [\partial \mathbf{u}_i/\partial t + (\mathbf{u}_i \cdot \nabla)\mathbf{u}_i] = O(\delta^2 m_i n_i v_{thi}^2/L)$. Therefore, to obtain the leading order $u_{i\|} = O(\delta v_{thi})$, the ion parallel momentum equation must be solved correct to $O(\delta^2 m_i n_i v_{thi}^2/L)$. This requires either knowing the CGL pressures correct to $O(\delta^2 m_i n_i v_{thi}^2)$ or ordering $\mathbf{b} \cdot \nabla = O(\delta^2/L)$.

CGL pressures correct to $O(\delta^2 m_i n_i v_{thi}^2)$ cannot be obtained from their fluid equations:

$$\frac{1}{2}\Big[\partial p_{i\|}/\partial t + \nabla \cdot (p_{i\|}\mathbf{u}_i)\Big] + \nabla \cdot \mathbf{q}_{iB} = ... \qquad \mathit{and} \qquad \partial p_{i\perp}/\partial t + \nabla \cdot (p_{i\perp}\mathbf{u}_i) + \nabla \cdot \mathbf{q}_{iT} = ...$$

since this would require knowing $u_{i\|}$ correct to $O(\delta^3 v_{thi})$ and knowing the heat fluxes $\mathbf{q}_{iB,T}$ correct to $O(\delta^3 m_i n_i v_{thi}^3)$.

REDUCTION TO ELIMINATE THE FAST MAGNETOSONIC WAVE

⇒ **A) Full systems:**

No small parameters other than ρ_i/L, ω/Ω_{ci} (possibly u_i/v_{thi}), m_e/m_i and $\rho_i\nu_\alpha^{coll}/v_{th\alpha}$

General geometry and strongly inhomogeneous magnetic field

No distinction between parallel and perpendicular length scales

Include the fast magnetosonic wave

⇒ **B) Reduced systems:**

Separate parallel and perpendicular length scales in large aspect ratio geometry

Subsidiary expansion parameter $\epsilon \sim L_\perp/L_\parallel \sim k_\parallel/k_\perp \ll 1$ besides $\delta \sim \rho_i/L_\perp \ll 1$

Weakly inhomogeneous magnetic field ($||\nabla \mathbf{B}|| \sim B/L_\parallel$)

Eliminate the fast magnetosonic wave

Allow a consistent fluid treatment of the $\omega \sim \omega_{*\alpha}, u_i \sim u_{*\alpha}$ drift ordering

II. GENERAL FLUID FORMALISM

The underlying kinetic description is assumed given by

$$\frac{\partial f_\alpha(\mathbf{v}_\alpha, \mathbf{x}, t)}{\partial t} \; + \; \mathbf{v}_\alpha \cdot \frac{\partial f_\alpha(\mathbf{v}_\alpha, \mathbf{x}, t)}{\partial \mathbf{x}} \; + \; \frac{e_\alpha}{m_\alpha}\Big(\mathbf{E} + \mathbf{v}_\alpha \times \mathbf{B}\Big) \cdot \frac{\partial f_\alpha(\mathbf{v}_\alpha, \mathbf{x}, t)}{\partial \mathbf{v}_\alpha} \; = \; C_\alpha(\mathbf{v}_\alpha, \mathbf{x}, t) \ ,$$

with the Fokker-Plank collision operator:

$$C_\alpha(\mathbf{v}_\alpha, \mathbf{x}, t) \; = \; -\sum_\beta \; \frac{c^4 e_\alpha^2 e_\beta^2 \ln \Lambda_{\alpha\beta}}{8\pi m_\alpha} \; \Gamma_{\alpha\beta}(\mathbf{v}_\alpha, \mathbf{x}, t) \ ,$$

$$\Gamma_{\alpha\beta}(\mathbf{v}_\alpha, \mathbf{x}, t) \; = \; \frac{\partial}{\partial \mathbf{v}_\alpha} \cdot \int d^3\mathbf{v}_\beta \; \mathsf{W}(\mathbf{v}_\alpha, \mathbf{v}_\beta) \cdot \Big[\frac{f_\alpha(\mathbf{v}_\alpha, \mathbf{x}, t)}{m_\beta} \; \frac{\partial f_\beta(\mathbf{v}_\beta, \mathbf{x}, t)}{\partial \mathbf{v}_\beta} \; - \; \frac{f_\beta(\mathbf{v}_\beta, \mathbf{x}, t)}{m_\alpha} \; \frac{\partial f_\alpha(\mathbf{v}_\alpha, \mathbf{x}, t)}{\partial \mathbf{v}_\alpha}\Big] \ ,$$

$$\mathsf{W}(\mathbf{v}_\alpha, \mathbf{v}_\beta) \; = \; \frac{|\mathbf{v}_\alpha - \mathbf{v}_\beta|^2 \; \mathsf{I} \; - \; (\mathbf{v}_\alpha - \mathbf{v}_\beta)(\mathbf{v}_\alpha - \mathbf{v}_\beta)}{|\mathbf{v}_\alpha - \mathbf{v}_\beta|^3} \ .$$

For each species and dropping the species index α, define the following fluid moments:

$$n(\mathbf{x},t) \;=\; \int d^3\mathbf{v}\; f(\mathbf{v},\mathbf{x},t),$$

$$n(\mathbf{x},t)\;\mathbf{u}(\mathbf{x},t) \;=\; \int d^3\mathbf{v}\;\mathbf{v}\; f(\mathbf{v},\mathbf{x},t),$$

$$\mathbf{F}^{coll}(\mathbf{x},t) \;=\; m\int d^3\mathbf{v}\;(\mathbf{v}-\mathbf{u})\;C(\mathbf{v},\mathbf{x},t),$$

$$\mathsf{P}(\mathbf{x},t) \;=\; m\int d^3\mathbf{v}\;(\mathbf{v}-\mathbf{u})(\mathbf{v}-\mathbf{u})\;f(\mathbf{v},\mathbf{x},t),$$

$$\mathsf{G}^{coll}(\mathbf{x},t) \;=\; m\int d^3\mathbf{v}\;(\mathbf{v}-\mathbf{u})(\mathbf{v}-\mathbf{u})\;C(\mathbf{v},\mathbf{x},t),$$

$$\mathsf{Q}(\mathbf{x},t) \;=\; m\int d^3\mathbf{v}\;(\mathbf{v}-\mathbf{u})(\mathbf{v}-\mathbf{u})(\mathbf{v}-\mathbf{u})\;f(\mathbf{v},\mathbf{x},t),$$

$$\mathsf{H}^{coll}(\mathbf{x},t) \;=\; m\int d^3\mathbf{v}\;(\mathbf{v}-\mathbf{u})(\mathbf{v}-\mathbf{u})(\mathbf{v}-\mathbf{u})\;C(\mathbf{v},\mathbf{x},t),$$

$$\mathsf{R}(\mathbf{x},t) \;=\; m^2\int d^3\mathbf{v}\;(\mathbf{v}-\mathbf{u})(\mathbf{v}-\mathbf{u})(\mathbf{v}-\mathbf{u})(\mathbf{v}-\mathbf{u})\;f(\mathbf{v},\mathbf{x},t).$$

Further define

$$\mathsf{P}_{jk} \;=\; p_\perp \delta_{jk} \;+\; (p_\parallel - p_\perp) b_j b_k \;+\; \hat{\mathsf{P}}_{jk} \;=\; \mathsf{P}^{CGL}_{jk} \;+\; \hat{\mathsf{P}}_{jk}$$

$$with \qquad \hat{\mathsf{P}}_{jj} \;=\; \hat{\mathsf{P}}_{jk} b_j b_k \;=\; 0 \;,$$

or

$$\mathsf{P}_{jk} \;=\; p\,\delta_{jk} \;+\; \Pi_{jk} \;=\; p\,\delta_{jk} \;+\; (p_\parallel - p_\perp)\,(b_j b_k - \delta_{jk}/3) \;+\; \hat{\mathsf{P}}_{jk}$$

$$with \qquad p \;=\; (p_\parallel + 2p_\perp)/3 \qquad and \qquad \Pi_{jj} \;=\; 0 \;,$$

and

$$\mathsf{Q}_{jkl} \;=\; q_{T\parallel} \delta_{[jk} b_{l]} \;+\; (2q_{B\parallel} - 3q_{T\parallel}) b_j b_k b_l \;+\; \hat{\mathsf{Q}}_{jkl} \;=\; \mathsf{Q}^{CGL}_{jkl} \;+\; \hat{\mathsf{Q}}_{jkl}$$

$$with \qquad \hat{\mathsf{Q}}_{jjk} b_k \;=\; \hat{\mathsf{Q}}_{jkl} b_j b_k b_l \;=\; 0 \;,$$

or

$$\mathsf{Q}_{jkl} \;=\; (2q_\parallel/5)\;\delta_{[jk} b_{l]} \;+\; \Theta_{jkl} \;=\; (2q_\parallel/5)\;\delta_{[jk} b_{l]} \;+\; (2q_{B\parallel} - 3q_{T\parallel})\;(b_j b_k b_l - \delta_{[jk} b_{l]}/5) \;+\; \hat{\mathsf{Q}}_{jkl}$$

$$with \qquad q_\parallel \;=\; q_{B\parallel} + q_{T\parallel} \qquad and \qquad \Theta_{jjk} b_k \;=\; 0 \;.$$

For the fourth-rank moment, define

$$\mathsf{R}_{jklm} \;=\; \frac{1}{n}\,\mathsf{P}_{[jk}\,\mathsf{P}_{lm]} \;+\; \tilde{\mathsf{R}}_{jklm}$$

and

$$\tilde{\mathsf{R}}_{jklm} \;=\; (2\tilde{r}_\perp/5 - \tilde{r}_\Delta/10)\;\delta_{[jk}\delta_{lm]} \;+\; \tilde{r}_\Delta\;\delta_{[jk}b_l b_{m]}/2 \;+\; (2\tilde{r}_\parallel - 2\tilde{r}_\perp - 7\tilde{r}_\Delta/2)\;b_j b_k b_l b_m \;+\; \hat{\tilde{\mathsf{R}}}_{jklm}$$

$$\textit{with} \qquad \hat{\tilde{\mathsf{R}}}_{jjll} \;=\; \hat{\tilde{\mathsf{R}}}_{jjlm} b_l b_m \;=\; \hat{\tilde{\mathsf{R}}}_{jklm} b_j b_k b_l b_m \;=\; 0\;.$$

The anisotropic generalization of Grad's thirteen moment collisionless closure (now twenty moment) corresponds to setting $\tilde{\mathbf{R}}$=0. This yields a model that includes all the convective and diamagnetic fluid effects, but neglects the purely kinetic effects such as wave-particle resonances and collisionless dissipation.

EXACT DYNAMIC EVOLUTION EQUATIONS FOR THE CGL PART OF THE STRESS TENSOR

$$\frac{3}{2}\Big[\frac{\partial p}{\partial t} + \nabla\cdot(p\mathbf{u})\Big] \;+\; \mathsf{P} : (\nabla\mathbf{u}) \;+\; \nabla\cdot\mathbf{q} \;-\; g^{coll} \;=\; 0$$

and

$$\frac{1}{2}\Big[\frac{\partial p_\parallel}{\partial t} + \nabla\cdot(p_\parallel\mathbf{u})\Big] \;-\; \mathbf{b}\cdot\mathsf{P}\cdot\Big[\frac{\partial\mathbf{b}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{b} - (\mathbf{b}\cdot\nabla)\mathbf{u} - \mathbf{b}\times(\nabla\times\mathbf{u})\Big] -$$

$$-\;\mathbf{b}\cdot\mathsf{Q} : (\nabla\mathbf{b}) \;+\; \nabla\cdot\mathbf{q}_B \;-\; g_B^{coll} \;=\; 0\ .$$

Here we have defined the heat flux vectors and collisional heat generation rates:

$$\mathbf{q} \;=\; q_\parallel\mathbf{b} + \mathbf{q}_\perp \;=\; \mathsf{Q} : \mathsf{I}/2 \;=\; (m/2)\int d^3\mathbf{v}\ (\mathbf{v}-\mathbf{u})\ |\mathbf{v}-\mathbf{u}|^2\ f(\mathbf{v},\mathbf{x},t)\ ,$$

$$\mathbf{q}_B \;=\; q_{B\parallel}\mathbf{b} + \mathbf{q}_{B\perp} \;=\; \mathsf{Q} : (\mathbf{b}\mathbf{b})/2 \;=\; (m/2)\int d^3\mathbf{v}\ (\mathbf{v}-\mathbf{u})\ [(\mathbf{v}-\mathbf{u})\cdot\mathbf{b}]^2\ f(\mathbf{v},\mathbf{x},t)\ ,$$

$$g^{coll} \;=\; \mathsf{G}^{coll} : \mathsf{I}/2 \;=\; (m/2)\int d^3\mathbf{v}\ |\mathbf{v}-\mathbf{u}|^2\ C(\mathbf{v},\mathbf{x},t)\ ,$$

$$g_B^{coll} \;=\; \mathsf{G}^{coll} : (\mathbf{b}\mathbf{b})/2 \;=\; (m/2)\int d^3\mathbf{v}\ [(\mathbf{v}-\mathbf{u})\cdot\mathbf{b}]^2\ C(\mathbf{v},\mathbf{x},t)\ .$$

PERTURBATIVE EXPRESSIONS IN POWERS OF δ FOR THE PERPENDICULAR PARTS OF THE STRESS AND STRESS-FLUX TENSORS, $\hat{\mathsf{P}}$ AND $\hat{\mathsf{Q}}$

$\hat{\mathsf{P}}$ and $\hat{\mathsf{Q}}$ can be split into terms that do and do not depend explicitly on the collision frequencies:

$$\hat{\mathsf{P}} = \hat{\mathsf{P}}_{\perp}^{coll} + \hat{\mathsf{P}}^{gyr} \qquad and \qquad \hat{\mathsf{Q}} = \hat{\mathsf{Q}}_{\perp}^{coll} + \hat{\mathsf{Q}}^{gyr} \ .$$

CONSIDERING THE MHD (FAST DYNAMICS) ORDERING:

$$||\hat{\mathsf{P}}_{\perp}^{coll}|| \ = \ O\Big(\frac{\rho_i \nu^{coll} \ m^{3/2}}{v_{th} \ m_i^{3/2}} \, \delta \ p \Big) \ \ll ||\hat{\mathsf{P}}^{gyr}|| \ = \ O\Big(\frac{m}{m_i} \, \delta \ p \Big) \ ,$$

$$\hat{\mathsf{P}}_{jk}^{gyr} \ = \ \frac{1}{4} \, \epsilon_{[jlm} b_l (\delta_{nk]} + 3 b_n b_{k]}) \ \mathsf{K}_{mn}^{gyr} \ = \ O\Big(\frac{m}{m_i} \, \delta \ p \Big)$$

and

$$\mathsf{K}_{mn}^{gyr} \ = \ \frac{m}{eB} \Big\{ \, p_{\perp} \frac{\partial u_{n]}}{\partial x_{[m}} \ + \ \frac{\partial (q_{T\|} b_{n]})}{\partial x_{[m}} \ + \ b_{[m} \Big[(2 q_{B\|} - 3 q_{T\|}) \ b_l \frac{\partial b_{n]}}{\partial x_l} \ + \ 2 (p_{\|} - p_{\perp}) \ b_l \frac{\partial u_{n]}}{\partial x_l} \Big] \, \Big\} \ = \ O\Big(\frac{m}{m_i} \, \delta \ p \Big) \ .$$

Similarly retaining the leading terms in the MHD ordering:

$$||\hat{\mathbf{Q}}_{\perp}^{coll}|| \;=\; O\Big(\;\frac{\rho_i \nu^{coll}\;m^{1/2}}{v_{th}\;m_i^{1/2}}\;\delta\;p\;v_{thi}\;\Big)\;\ll ||\hat{\mathbf{Q}}^{gyr}|| \;=\; O(\delta\;p\;v_{thi})\;,$$

$$\mathbf{q}_{\perp}^{gyr} = \frac{1}{2}\hat{\mathbf{Q}}^{gyr} : \mathbf{I} = \frac{1}{eB}\mathbf{b}\times\Big[p_{\perp}\nabla\Big(\frac{p_{\parallel}+4p_{\perp}}{2n}\Big) + \frac{p_{\parallel}(p_{\parallel}-p_{\perp})}{n}\boldsymbol{\kappa} + 2m(q_{B\parallel}+2q_{T\parallel})(\mathbf{b}\cdot\nabla)\mathbf{u} + mq_{T\parallel}\mathbf{b}\times\boldsymbol{\omega}\Big] + \tilde{\mathbf{q}}_{\perp}$$

$$\mathbf{q}_{B\perp}^{gyr} \;=\; \frac{1}{2}\;\hat{\mathbf{Q}}^{gyr} : (\mathbf{bb}) \;=\; \frac{1}{eB}\mathbf{b}\times\Big[p_{\perp}\nabla\Big(\frac{p_{\parallel}}{2n}\Big) + \frac{p_{\parallel}(p_{\parallel}-p_{\perp})}{n}\boldsymbol{\kappa} + 2mq_{B\parallel}(\mathbf{b}\cdot\nabla)\mathbf{u} + mq_{T\parallel}\mathbf{b}\times\boldsymbol{\omega}\Big] \;+\; \tilde{\mathbf{q}}_{B\perp}$$

$$\mathbf{b}\cdot\hat{\mathbf{Q}}^{gyr} : (\nabla\mathbf{b}) \;=\; 2\;\mathbf{q}_{B\perp}^{gyr}\cdot\boldsymbol{\kappa} \;-\; q_{T\parallel}\;\sigma^{gyr}$$

and

$$\sigma^{gyr} \;=\; \frac{m}{4eB}\epsilon_{jkl}b_j\Big(\frac{\partial b_k}{\partial x_m} + \frac{\partial b_m}{\partial x_k}\Big)(\delta_{mn} - b_m b_n)\Big(\frac{\partial u_l}{\partial x_n} + \frac{\partial u_n}{\partial x_l}\Big)\;,$$

where

$$\boldsymbol{\kappa} = (\mathbf{b}\cdot\nabla)\mathbf{b} \qquad \textit{and} \qquad \boldsymbol{\omega} = \nabla\times\mathbf{u}\;.$$

CONSIDERING THE DRIFT ORDERING WITH SMALL PARALLEL GRADIENTS AND WEAKLY INHOMOGENEOUS MAGNETIC FIELD:

Introduce the subsidiary ordering

$$\mathbf{b}\cdot\nabla \sim L_{\parallel}^{-1} \sim \epsilon L_{\perp}^{-1} \ll L_{\perp}^{-1} \qquad \textit{and} \qquad ||\nabla\mathbf{B}|| \sim BL_{\parallel}^{-1} \sim \epsilon BL_{\perp}^{-1} \ll BL_{\perp}^{-1}\ .$$

Also, order the heat fluxes as $q \sim up \sim \delta v_{thi}p$. Then, keeping the lowest significant order terms, we have:

$$||\hat{\mathsf{P}}_{\perp}^{coll}|| \ = \ O\Big(\ \frac{\rho_i \nu^{coll}\ m^{3/2}}{v_{th}\ m_i^{3/2}}\ \delta\ p\ \Big)\ , \qquad\qquad ||\hat{\mathsf{P}}^{gyr}|| \ = \ O\Big(\ \frac{m}{m_i}\ \delta^2\ p\ \Big)\ ,$$

$$\hat{\mathsf{P}}_{\perp jk}^{coll} \ = \ -\ \frac{m}{4eB}\ \epsilon_{[jlm}b_l(\delta_{nk]} + 3b_n b_{k]})\ \mathsf{G}_{mn}^{coll}\ ,$$

$$\hat{\mathsf{P}}_{jk}^{gyr} \ = \ \frac{1}{4}\ \epsilon_{[jlm}b_l(\delta_{nk]} + 3b_n b_{k]})\ \mathsf{K}_{mn}^{gyr}\ ,$$

$$\mathsf{K}_{mn}^{gyr} \ = \ \frac{m}{eB}\Big\{\ p_\perp \frac{\partial u_{n]}}{\partial x_{[m}} \ + \ \frac{\partial}{\partial x_{[m}}\Big(q_{T\parallel}b_{n]} + \frac{1}{2}q_{T\perp n]}^{gyr}\Big) \ + \ \frac{1}{eB}\ b_{[m}\ \frac{\partial p_\perp}{\partial x_k}\ \frac{\partial[(p_\parallel - p_\perp)/n]}{\partial x_l}\ \epsilon_{kln]}\Big\}\ .$$

For the perpendicular part of the stress-flux tensor in the drift ordering with small parallel gradients and weakly inhomogeneous magnetic field:

$$||\hat{\mathbf{Q}}_{\perp}^{coll}|| \;=\; O\Big(\frac{\rho_i \nu^{coll}\; m^{1/2}}{v_{th}\; m_i^{1/2}}\; \delta\; p\; v_{thi} \Big) \;\ll ||\hat{\mathbf{Q}}^{gyr}|| \;=\; O(\delta\; p\; v_{thi}) \;,$$

$$\mathbf{q}_{\perp}^{gyr} \;=\; \frac{1}{2}\,\hat{\mathbf{Q}}^{gyr} : \mathbf{I} \;=\; \frac{p_\perp}{2eB}\,\mathbf{b}\times\nabla\Big(\frac{p_\parallel + 4p_\perp}{n}\Big) \;+\; \tilde{\mathbf{q}}_\perp$$

$$\mathbf{q}_{B\perp}^{gyr} \;=\; \frac{1}{2}\,\hat{\mathbf{Q}}^{gyr} : (\mathbf{b}\mathbf{b}) \;=\; \frac{p_\perp}{2eB}\,\mathbf{b}\times\nabla\Big(\frac{p_\parallel}{n}\Big) \;+\; \tilde{\mathbf{q}}_{B\perp}$$

$$\mathbf{b}\cdot\hat{\mathbf{Q}}^{gyr} : (\nabla\mathbf{b}) \;=\; O(\epsilon\delta\; p\; v_{thi}L_\perp^{-1}) \;\ll\; (\partial/\partial t + \mathbf{u}\cdot\nabla)p_\parallel \;=\; O(\delta\; p\; v_{thi}L_\perp^{-1}) \;.$$

COLLISIONLESS CLOSURE TERMS

Besides the moments of the collision operator, the still unspecified terms needed to close the fluid equations are:

The two independent parallel heat fluxes ($q_{T\|} = q_\| - q_{B\|}$):

$$q_\| \;=\; \mathbf{b}\cdot\mathbf{Q} : \mathbf{I}/2 \;=\; (m/2)\int d^3\mathbf{v}\;[(\mathbf{v}-\mathbf{u})\cdot\mathbf{b}]\;|\mathbf{v}-\mathbf{u}|^2\; f(\mathbf{v},\mathbf{x},t)\ ,$$

$$q_{B\|} \;=\; \mathbf{b}\cdot\mathbf{Q} : (\mathbf{bb})/2 \;=\; (m/2)\int d^3\mathbf{v}\;[(\mathbf{v}-\mathbf{u})\cdot\mathbf{b}]^3\; f(\mathbf{v},\mathbf{x},t)\ .$$

The $\tilde{\mathbf{q}}_\perp$ and $\tilde{\mathbf{q}}_{B\perp}$ terms in the perpendicular heat fluxes :

$$\tilde{\mathbf{q}}_\perp = \frac{1}{eB}\mathbf{b}\times\left[\nabla\tilde{r}_\perp^{(0)} + (\tilde{r}_\|^{(0)} - \tilde{r}_\perp^{(0)})\boldsymbol{\kappa}\right]\ ,$$

$$\tilde{\mathbf{q}}_{B\perp} = \frac{1}{eB}\mathbf{b}\times\left[\nabla(\tilde{r}_\perp^{(0)} + \tilde{r}_\Delta^{(0)})/5 + (\tilde{r}_\|^{(0)} - \tilde{r}_\perp^{(0)} - \tilde{r}_\Delta^{(0)})\boldsymbol{\kappa}\right]\ .$$

The three scalars $\tilde{r}_\perp^{(0)}$, $\tilde{r}_\parallel^{(0)}$ and $\tilde{r}_\Delta^{(0)}$ (or combinations thereof), are the three independent components of the fourth-rank tensor $\tilde{\mathsf{R}}$ in its zero-Larmor-radius limit. They are moments of the difference between the actual zeroth-order distribution function and a two-temperature Maxwellian, and therefore are well suited for a Landau-fluid closure approximation. Specifically, they are:

$$\begin{aligned}
\tilde{r}_\perp^{(0)} &= (m^2/4)\int d^3\mathbf{v}\ |\mathbf{v}-\mathbf{u}|^4\ \cos^2\lambda\ (f^{(0)} - f_{2M}), \\
\tilde{r}_\parallel^{(0)} &= (m^2/2)\int d^3\mathbf{v}\ |\mathbf{v}-\mathbf{u}|^4\ \sin^2\lambda\ (f^{(0)} - f_{2M}), \\
\tilde{r}_\Delta^{(0)} &= (m^2/4)\int d^3\mathbf{v}\ |\mathbf{v}-\mathbf{u}|^4\ \cos^2\lambda\ (5\sin^2\lambda - 1)\ (f^{(0)} - f_{2M}).
\end{aligned}$$

Here, $f^{(0)} = f^{(0)}(m|\mathbf{v}-\mathbf{u}|^2/2, \lambda, \mathbf{x}, t)$ is the zero-Larmor-radius distribution function which depends on the velocity space coordinates through the fluid-rest-frame energy, $m|\mathbf{v}-\mathbf{u}|^2/2$, and the magnetic pitch angle, $\sin\lambda = (\mathbf{v}-\mathbf{u})\cdot\mathbf{b}/|\mathbf{v}-\mathbf{u}|$, but is independent of the gyrophase. The two-temperature Maxwellian is

$$f_{2M}(m|\mathbf{v}-\mathbf{u}|^2/2, \lambda, \mathbf{x}, t) = \Big(\frac{m}{2\pi}\Big)^{3/2}\ \frac{n^{5/2}}{p_\perp\ p_\parallel^{1/2}}\ \ \exp\Big[-\frac{m\ n\ |\mathbf{v}-\mathbf{u}|^2}{2}\Big(\frac{\cos^2\lambda}{p_\perp} + \frac{\sin^2\lambda}{p_\parallel}\Big)\Big].$$

Dynamic evolution equations for the parallel heat fluxes $q_\parallel$ and $q_{B\parallel}$ are also available. However, their finite-Larmor-radius terms involve two further unknown (albeit well defined) scalars stemming from the FLR part of the $\tilde{\mathbf{R}}$ tensor. Thus we are left with three possible options in practice:

a) Evaluate $q_\parallel$ and $q_{B\parallel}$, only in their zero-Larmor-radius limit for the fast MHD ordering ($q_\parallel \sim q_{B\parallel} \sim pv_{thi}$), using the corresponding fluid evolution equations. In this case, the only collisionless closure variables left are $\tilde{r}_\perp^{(0)}$, $\tilde{r}_\parallel^{(0)}$ and $\tilde{r}_\Delta^{(0)}$ to account for the collisionless disssipation and other purely kinetic effects.

b) Use truncated FLR fluid equations for $q_\parallel$ and $q_{B\parallel}$ that include the known convective and diamagnetic terms, ignoring the further unknown FLR terms, and leaving again only $\tilde{r}_\perp^{(0)}$, $\tilde{r}_\parallel^{(0)}$ and $\tilde{r}_\Delta^{(0)}$ as collisionless closure variables.

c) Evaluate $q_\parallel$ and $q_{B\parallel}$ kinetically.

The available dynamic evolution equation for $q_{B\|}$ is

$$
\frac{\partial q_{B\|}}{\partial t} \; + \; \nabla \cdot (q_{B\|}\mathbf{u}) \; + \; 3q_{B\|}\mathbf{b} \cdot [(\mathbf{b} \cdot \nabla)\mathbf{u}] \; + \; \frac{3p_\|}{2m}\mathbf{b} \cdot \nabla\Big(\frac{p_\|}{n}\Big) \; +
$$

$$
+ \; \frac{1}{m}\Big[\mathbf{b} \cdot \nabla\Big(\tilde{r}_\|^{(0)} - 2\tilde{r}_\perp^{(0)}/5 - 2\tilde{r}_\Delta^{(0)}/5\Big) - \Big(\tilde{r}_\|^{(0)} - \tilde{r}_\perp^{(0)} - \tilde{r}_\Delta^{(0)}\Big)\mathbf{b} \cdot \nabla(\ln B)\Big] + \; \frac{3p_\|}{2mn}\mathbf{b} \cdot \mathbf{F}^{coll} \; - \; h_B^{coll} \; +
$$

$$
+ \; \frac{3}{2m}\mathbf{b} \cdot \hat{\mathbf{P}} \cdot \Big[\nabla\Big(\frac{p_\|}{n}\Big) - \frac{2p_\|}{n}\boldsymbol{\kappa}\Big] \; - \; \frac{3}{2}\Big[\frac{\partial \mathbf{b}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{b} - (\mathbf{b} \cdot \nabla)\mathbf{u} - \mathbf{b} \times (\nabla \times \mathbf{u})\Big] \cdot \hat{\mathbf{Q}} : (\mathbf{b}\mathbf{b}) \; = \; 0 \; ,
$$

where the first two lines contain the zero-Larmor-radius terms retained in option a), and the third line contains the FLR terms added in option b).

Here, the new collisional term is

$$
h_B^{coll} \; = \; \mathbf{b} \cdot \mathbf{H}^{coll} : (\mathbf{b}\mathbf{b})/2 \; = \; (m/2) \int d^3\mathbf{v} \; [(\mathbf{v} - \mathbf{u}) \cdot \mathbf{b}]^3 \; C(\mathbf{v}, \mathbf{x}, t) \; .
$$

The available dynamic evolution equation for $q_{T\|} = q_{\|} - q_{B\|}$ is

$$\frac{\partial q_{T\|}}{\partial t} \ + \ \nabla\cdot(q_{T\|}\mathbf{u}) \ + \ q_{T\|}\nabla\cdot\mathbf{u} \ + \ \frac{p_\|}{m}\mathbf{b}\cdot\nabla\Big(\frac{p_\perp}{n}\Big) \ - \ \frac{p_\perp(p_\| - p_\perp)}{mn}\mathbf{b}\cdot\nabla(\ln B) \ +$$

$$+ \ \frac{1}{m}\Big[2\mathbf{b}\cdot\nabla\Big(\tilde{r}_\perp^{(0)} + \tilde{r}_\Delta^{(0)}\Big)/5 + \tilde{r}_\Delta^{(0)}\mathbf{b}\cdot\nabla(\ln B)\Big] + \ \frac{p_\perp}{mn}\mathbf{b}\cdot\mathbf{F}^{coll} \ + \ h_B^{coll} \ - \ h^{coll} \ +$$

$$+ \ \frac{1}{m}\mathbf{b}\cdot\hat{\mathbf{P}}\cdot\Big[\nabla\Big(\frac{p_\perp}{n}\Big) + \Big(\frac{p_\| + 2p_\perp}{n}\Big)\boldsymbol{\kappa}\Big] \ + \ \Big(\frac{p_\| - 2p_\perp}{mn}\Big)\hat{\mathbf{P}} : (\nabla\mathbf{b}) \ + \ \Big(\frac{p_\perp}{m}\Big)\nabla\cdot\Big(\frac{1}{n}\mathbf{b}\cdot\hat{\mathbf{P}}\Big) \ +$$

$$+ \ \mathbf{b}\cdot\hat{\mathbf{Q}} : (\nabla\mathbf{u}) \ - \ \frac{1}{2}\Big[\frac{\partial\mathbf{b}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{b} - (\mathbf{b}\cdot\nabla)\mathbf{u} - \mathbf{b}\times(\nabla\times\mathbf{u})\Big]\cdot\hat{\mathbf{Q}} : (\mathbf{I} - 3\mathbf{b}\mathbf{b}) \ = \ 0 \ ,$$

where the first two lines contain the zero-Larmor-radius terms retained in option a), and the last two lines contain the FLR terms added in option b).

The last collisional term is

$$h^{coll} \ = \ \mathbf{b}\cdot\mathbf{H}^{coll} : \mathbf{I}/2 \ = \ (m/2)\int d^3\mathbf{v}\ [(\mathbf{v} - \mathbf{u})\cdot\mathbf{b}]\ |\mathbf{v} - \mathbf{u}|^2\ C(\mathbf{v}, \mathbf{x}, t) \ .$$

IN THE DRIFT ORDERING WITH SMALL PARALLEL GRADIENTS AND WEAKLY INHOMOGENEOUS MAGNETIC FIELD :

Assuming also the slow ordering $q_{B,T\|} \sim up \sim \delta v_{thi} p$ **for the parallel heat fluxes and** $\tilde{r}^{(0)}_{\|,\perp,\Delta} \sim p^2/n \sim m_i v_{thi}^2 p$ **,**

$$\nabla \cdot (q_{B,T\|}\mathbf{b}) \;=\; \mathbf{b}\cdot\nabla q_{B,T\|} \;+\; q_{B,T\|}\nabla\cdot\mathbf{b} \;=\; O(\epsilon\delta\; p\; v_{thi} L_\perp^{-1}) \;\ll\; (\partial/\partial t + \mathbf{u}\cdot\nabla)p \;=\; O(\delta\; p\; v_{thi} L_\perp^{-1}) \;,$$

$$\nabla\cdot\tilde{\mathbf{q}}_\perp \;=\; \nabla\tilde{r}^{(0)}_\perp \cdot \Big[\nabla\times\Big(\frac{1}{eB}\mathbf{b}\Big)\Big] \;+\; \nabla\cdot\Big[\Big(\frac{\tilde{r}^{(0)}_\| - \tilde{r}^{(0)}_\perp}{eB}\Big)\mathbf{b}\times\boldsymbol{\kappa}\Big] \;=\; O(\epsilon\delta\; p\; v_{thi} L_\perp^{-1}) \;,$$

$$\nabla\cdot\tilde{\mathbf{q}}_{B\perp} \;=\; \nabla(\tilde{r}^{(0)}_\perp + \tilde{r}^{(0)}_\Delta)\cdot\Big[\nabla\times\Big(\frac{1}{5eB}\mathbf{b}\Big)\Big] \;+\; \nabla\cdot\Big[\Big(\frac{\tilde{r}^{(0)}_\| - \tilde{r}^{(0)}_\perp - \tilde{r}^{(0)}_\Delta}{eB}\Big)\mathbf{b}\times\boldsymbol{\kappa}\Big] \;=\; O(\epsilon\delta\; p\; v_{thi} L_\perp^{-1}) \;.$$

Similarly, the needed contribution of $q_{T\|}$ **and** $\tilde{\mathbf{q}}_{T\perp} = \tilde{\mathbf{q}}_\perp - \tilde{\mathbf{q}}_{B\perp}$ **to the gyroviscous force in the ion momentum equation is** $O(\epsilon)$ **smaller than the leading order terms.**

Therefore, this ordering yields a formally consistent collisionless closure.

III. SIMPLIFIED SYSTEMS WITH SINGLE ION SPECIES AND NEGLIGIBLE ELECTRON MASS

Neglecting $m_e/m_i \ll 1$, assuming for simplicity that the ions have unit charge and reintroducing the species indices, the basic set of quasi-neutral fluid equations becomes:

$$\mathbf{u}_e \;=\; \mathbf{u}_i \;-\; \frac{1}{en}\mathbf{j} \;,$$

$$\mathbf{j} \;=\; \nabla \times \mathbf{B} \;,$$

$$\frac{\partial \mathbf{B}}{\partial t} \;+\; \nabla \times \mathbf{E} \;=\; 0 \;,$$

$$\mathbf{E} \;=\; -\mathbf{u}_i \times \mathbf{B} \;+\; \frac{1}{en}\Big(\mathbf{j} \times \mathbf{B} - \nabla \cdot \mathsf{P}_e^{CGL} + \mathbf{F}_e^{coll}\Big) \;,$$

$$\frac{\partial n}{\partial t} \;+\; \nabla \cdot (n\mathbf{u}_i) \;=\; 0 \;,$$

$$m_i n\Big[\frac{\partial \mathbf{u}_i}{\partial t} + (\mathbf{u}_i \cdot \nabla)\mathbf{u}_i\Big] \;+\; \nabla \cdot \Big(\mathsf{P}_e^{CGL} + \mathsf{P}_i^{CGL} + \hat{\mathsf{P}}_i^{gyr} + \hat{\mathsf{P}}_{i\perp}^{coll}\Big) \;-\; \mathbf{j} \times \mathbf{B} \;=\; 0 \;.$$

III. 1. FULL TWO-FLUID SYSTEM IN THE MHD ORDERING

In the fast dynamics, MHD ordering we have:

$$\delta \sim \rho_i/L \ll 1 \ , \qquad \omega \sim \delta\, \Omega_{ci} \ , \qquad u_\alpha \sim v_{thi} \ , \qquad q_{\alpha\|} \sim v_{thi}\, p_\alpha \ , \qquad q_{\alpha\perp} \sim \delta\, v_{thi}\, p_\alpha \ .$$

Also, assume comparable ion and electron pressures and strong anisotropies:

$$p_i \sim p_e \ , \qquad (p_{\alpha\|} - p_{\alpha\perp}) \sim p_\alpha \ .$$

The ion terms are as given in the previous Section II.

The electron terms are simplified as the result of the small mass ratio assumption.

SMALL MASS RATIO ELECTRON PRESSURE EQUATIONS:

$$\frac{3}{2}\Big[\frac{\partial p_e}{\partial t} + \nabla\cdot\Big(p_e\mathbf{u}_e\Big)\Big] \;+\; p_e\nabla\cdot\mathbf{u}_e \;+\; (p_{e\|} - p_{e\perp})\Big\{\mathbf{b}\cdot\Big[(\mathbf{b}\cdot\nabla)\mathbf{u}_e\Big] - \nabla\cdot\mathbf{u}_e/3\Big\} +$$

$$+\;\nabla\cdot\Big(q_{e\|}\mathbf{b} + \mathbf{q}_{e\perp}\Big) \;-\; g_e^{coll} \;=\; 0$$

and

$$\frac{1}{2}\Big[\frac{\partial p_{e\|}}{\partial t} + \nabla\cdot\Big(p_{e\|}\mathbf{u}_e\Big)\Big] \;+\; p_{e\|}\mathbf{b}\cdot\Big[(\mathbf{b}\cdot\nabla)\mathbf{u}\Big] \;+\; \nabla\cdot\Big(q_{eB\|}\mathbf{b} + \mathbf{q}_{eB\perp}\Big) \;+\; q_{eT\|}\mathbf{b}\cdot\nabla(\ln\; B) \;-\; g_{eB}^{coll} \;=\; 0\;,$$

where the small mass ratio electron perpendicular heat fluxes are:

$$\mathbf{q}_{e\perp} \;=\; -\,\frac{1}{eB}\mathbf{b}\times\Big[p_{e\perp}\nabla\Big(\frac{p_{e\|} + 4p_{e\perp}}{2n}\Big) + \frac{p_{e\|}(p_{e\|} - p_{e\perp})}{n}\boldsymbol{\kappa}\Big] \;+\; \tilde{\mathbf{q}}_{e\perp}$$

and

$$\mathbf{q}_{eB\perp} \;=\; -\,\frac{1}{eB}\mathbf{b}\times\Big[p_{e\perp}\nabla\Big(\frac{p_{e\|}}{2n}\Big) + \frac{p_{e\|}(p_{e\|} - p_{e\perp})}{n}\boldsymbol{\kappa}\Big] \;+\; \tilde{\mathbf{q}}_{eB\perp}\;.$$

In the limit of negligible mass ratio, the electron parallel heat flux equations yield two time independent constraints on the parallel gradients of the electron temperatures:

$$\frac{3p_{e\|}}{2}\mathbf{b}\cdot\nabla\Big(\frac{p_{e\|}}{n}\Big) \;+\; \mathbf{b}\cdot\nabla\Big(\tilde{r}^{(0)}_{e\|} - \frac{2}{5}\tilde{r}^{(0)}_{e\perp} - \frac{2}{5}\tilde{r}^{(0)}_{e\Delta}\Big) \;-\; \Big(\tilde{r}^{(0)}_{e\|} - \tilde{r}^{(0)}_{e\perp} - \tilde{r}^{(0)}_{e\Delta}\Big)\mathbf{b}\cdot\nabla(\ln B) \;+\; \frac{3p_{e\|}}{2n}\mathbf{b}\cdot\mathbf{F}^{coll}_{e} \;=\; 0$$

and

$$p_{e\|}\mathbf{b}\cdot\nabla\Big(\frac{p_{e\perp}}{n}\Big) - \frac{p_{e\perp}(p_{e\|} - p_{e\perp})}{n}\mathbf{b}\cdot\nabla(\ln B) \;+\; \frac{2}{5}\mathbf{b}\cdot\nabla\Big(\tilde{r}^{(0)}_{e\perp} + \tilde{r}^{(0)}_{e\Delta}\Big) \;+\; \tilde{r}^{(0)}_{e\Delta}\mathbf{b}\cdot\nabla(\ln B) + \frac{p_{e\perp}}{n}\mathbf{b}\cdot\mathbf{F}^{coll}_{e} \;=\; 0\,.$$

These should determine implicitly the electron parallel heat fluxes and provide an improvement over the adiabatic electron response model.

III. 2. REDUCED TWO-FLUID SYSTEM IN THE DRIFT ORDERING

Assume the slow dynamics, drift ordering:

$$\delta \sim \rho_i/L \ll 1 \;, \qquad \omega \sim \delta^2\,\Omega_{ci} \;, \qquad u_\alpha \sim \delta\; v_{thi} \;, \qquad q_{\alpha\|} \sim q_{\alpha\perp} \sim \delta\; v_{thi}\; p_\alpha \;.$$

Introduce the weakly inhomogenous magnetic field and small parallel gradient ordering, in a large aspect ratio toroidal background geometry:

$$\epsilon \sim (R - R_0)/R_0 \sim L_\perp/L_\| \gtrsim \delta^2 \;, \qquad k_\| \sim \mathbf{b}\cdot\nabla \sim \mathbf{e}_\zeta\cdot\nabla \sim 1/R_0 \sim \epsilon k_\perp \;,$$

$$\mathbf{B} \;=\; B_0\mathbf{e}_\zeta + \mathbf{B}_1 \;, \qquad |\mathbf{B}_1| \sim \epsilon B_0 \;.$$

Assume comparable ion and electron pressures with $\beta = O(\epsilon)$ and strong anisotropies:

$$p_i \sim p_e \;\sim \epsilon B_0^2, \qquad (p_{\alpha\|} - p_{\alpha\perp}) \sim p_\alpha \;.$$

In their lowest significant order, the magnetic field and current density are:

$$\mathbf{B} \;=\; (B_0 + B_{1\zeta})\,\mathbf{e}_\zeta \;-\; \mathbf{e}_\zeta \times \nabla\psi \;+\; O(\epsilon^2 B_0)$$

and

$$\mathbf{j} \;=\; \frac{B_0}{R_0}\,\mathbf{e}_Z \;-\; \mathbf{e}_\zeta \times \nabla B_{1\zeta} \;-\; \nabla_\perp^2\psi\,\mathbf{e}_\zeta \;+\; O(\epsilon B_0/R_0)\;,$$

with

$$B_{1\zeta} \;=\; -B_0\Big(\frac{R-R_0}{R_0}\Big) \;-\; \frac{1}{B_0}\,(p_{i\perp}+p_{e\perp}) \;+\; O(\epsilon^2 B_0)\;.$$

Here, we use the notations:

$$\nabla_\perp^2 f \;\equiv\; \nabla_\perp\cdot(\nabla_\perp f)\;, \qquad \nabla_\perp f \;\equiv\; \frac{\partial f}{\partial R}\mathbf{e}_R + \frac{\partial f}{\partial Z}\mathbf{e}_Z \quad \textit{and} \quad \nabla_\perp\cdot\mathbf{h} \;\equiv\; \frac{\partial(\mathbf{e}_R\cdot\mathbf{h})}{\partial R} + \frac{\partial(\mathbf{e}_Z\cdot\mathbf{h})}{\partial Z}\;.$$

The lowest-order ion flow velocity is:

$$\mathbf{u}_i \;=\; u_\zeta\,\mathbf{e}_\zeta \;+\; \frac{1}{B_0}\,\mathbf{e}_\zeta \times \Big(\nabla\Phi + \frac{1}{en}\nabla p_{i\perp}\Big) \;+\; O(\epsilon\delta v_{thi})\;,$$

where Φ is the electric potential. We further introduce the following notations:

$$\frac{d'f}{dt} \;\equiv\; \frac{\partial f}{\partial t} + \frac{1}{B_0}[\Phi, f]\;, \qquad \nabla_\parallel f \;\equiv\; \frac{1}{R_0}\frac{\partial f}{\partial\zeta} - \frac{1}{B_0}[\psi, f] \quad \textit{and} \quad [g,f] \;\equiv\; \mathbf{e}_\zeta\cdot(\nabla g\times\nabla f)\;.$$

The following reduced system is obtained for the seven coupled scalar fields n , $p_{\alpha\|}$, $p_{\alpha\perp}$ ($with \ \ \alpha = i, e \ \ and \ \ e_i = -e_e = e$), ψ and Φ :

$$\frac{d'n}{dt} \ = \ 0 \ ,$$

$$\frac{1}{2}\frac{d'p_{\alpha\|}}{dt} \ - \ h^{coll}_{\alpha B} \ = \ 0 \ ,$$

$$\frac{d'p_{\alpha\perp}}{dt} \ + \ \Big(\frac{p_{\alpha\|} - p_{\alpha\perp}}{3e_\alpha B_0 n^2}\Big) \ [n, p_{\alpha\perp}] \ - \ h^{coll}_{\alpha} \ + \ h^{coll}_{\alpha B} \ = \ 0 \ ,$$

$$\frac{\partial\psi}{\partial t} \ + \ \nabla_\| \Phi \ - \ \frac{1}{en}\nabla_\| \ p_{e\|} \ + \ \frac{1}{en} \ \mathbf{e}_\zeta \cdot \mathbf{F}^{coll}_e \ = \ 0 \ ,$$

$$\frac{1}{n}\nabla_\perp \cdot (n\nabla_\perp\Phi) \ = \ U \ - \ \frac{1}{en} \ \nabla^2_\perp p_{i\perp} \ ,$$

$$\frac{d'U}{dt} - \frac{1}{2B_0 n} \ [n, |\nabla_\perp\Phi|^2] + \frac{B_0^2}{m_i n}\nabla_\|(\nabla^2_\perp\psi) \ - \ \frac{\Lambda(p_{i\perp}, \Phi)}{eB_0 n} \ + \ \frac{B_0}{m_i R_0 n} \ [R, (p_{i\|} + p_{i\perp} + p_{e\|} + p_{e\perp})] + w^{coll} = 0.$$

In the vorticity equation, we defined

$$\Lambda(p_{i\perp}, \Phi) \; \equiv \; \frac{\partial^2 p_{i\perp}}{\partial R \partial Z}\Big(\frac{\partial^2 \Phi}{\partial R^2} - \frac{\partial^2 \Phi}{\partial Z^2}\Big) \; - \; \frac{\partial^2 \Phi}{\partial R \partial Z}\Big(\frac{\partial^2 p_{i\perp}}{\partial R^2} - \frac{\partial^2 p_{i\perp}}{\partial Z^2}\Big)$$

and

$$w^{coll} \; \equiv \; \frac{1}{m_i n} \, \mathbf{e}_\zeta \cdot \Big[\nabla \times (\nabla \cdot \hat{\mathsf{P}}_{i\perp}^{coll})\Big] \; .$$

The decoupled equation for the toroidal component of the ion velocity, u_ζ**, is:**

$$\frac{d' u_\zeta}{dt} \; + \; \frac{1}{m_i n} \, \nabla_\| (p_{i\|} + p_{e\|}) \; + \; \frac{1}{m_i n} \, \mathbf{e}_\zeta \cdot (\nabla \cdot \hat{\mathsf{P}}_{i\perp}^{coll}) \; = \; 0 \; .$$

This reduced system takes into account all the two-fluid effects associated with the generalized Ohm's law, the ion gyroviscosity, the ion and electron pressure anisotropies and the diamagnetic perpendicular heat fluxes, within the assumed orderings. Unlike other reduced two-fluid systems, it is valid for arbitrary density and temperature fluctuations. As discussed before, it is consistently closed except for the collisional terms.

IV. FORMAL REPRESENTATION OF THE COLLISIONAL MOMENTS FOR SINGLE ION SPECIES AND SMALL MASS RATIO

Consider the simple plasma with one ion species of unit charge.

Retain only leading-order terms in the $m_e/m_i \to 0$ limit.

Formal manipulations on the moments of the Fokker-Plank operator (integrations by parts and expansions in m_e/m_i) yield simplified expressions which are still applicable to any collisionality regime and do not require the distribution functions to be close to Maxwellians.

These simplified expressions are well suited for Chapman-Enskog, neoclassical or other kinds of approximations.

For each species, define the thermal speed $v_{th\alpha} \equiv \sqrt{p_\alpha/(m_\alpha n)}$, the dimensionless phase space coordinate

$$\boldsymbol{\xi} \equiv \frac{\mathbf{v}_\alpha - \mathbf{u}_\alpha(\mathbf{x},t)}{v_{th\alpha}(\mathbf{x},t)} \ ,$$

and the dimensionless distribution function

$$\hat{f}_\alpha(\boldsymbol{\xi},\mathbf{x},t) \ \equiv \ \frac{v_{th\alpha}^3}{n} \, f_\alpha(\mathbf{u}_\alpha + v_{th\alpha}\boldsymbol{\xi}, \ \mathbf{x}, \ t) \ ,$$

so that

$$\int d^3\boldsymbol{\xi} \ \hat{f}_\alpha(\boldsymbol{\xi},\mathbf{x},t) \ = \ 1 \qquad and \qquad \int d^3\boldsymbol{\xi} \ \boldsymbol{\xi} \ \hat{f}_\alpha(\boldsymbol{\xi},\mathbf{x},t) \ = \ 0 \ .$$

Define also the collision frequencies:

$$\nu_e^{coll} \ \equiv \ \frac{c^4 e^4 n \ln \Lambda_{ei}}{m_e^2 v_{the}^3} \qquad and \qquad \nu_i^{coll} \ \equiv \ \frac{c^4 e^4 n \ln \Lambda_{ei}}{m_i^2 v_{thi}^3}$$

FOR THE COLLISIONAL FRICTION FORCE:

$$\mathbf{F}_e^{coll} \;=\; -\mathbf{F}_i^{coll} \;=\; \frac{\nu_e^{coll}\; p_e}{v_{the}}\Big(\frac{1}{env_{the}}\hat{\mathbf{S}}\cdot\mathbf{j} \;+\; \hat{\mathbf{T}}\Big)$$

where

$$\hat{\mathbf{S}} \;=\; \frac{1}{4\pi}\int d^3\boldsymbol{\xi}\;\frac{\partial}{\partial\boldsymbol{\xi}}\Big(\frac{1}{|\boldsymbol{\xi}|}\Big)\;\;\frac{\partial\hat{f}_e}{\partial\boldsymbol{\xi}} \qquad\quad and \qquad\quad \hat{\mathbf{T}} \;=\; \frac{1}{4\pi}\int d^3\boldsymbol{\xi}\;\frac{\partial}{\partial\boldsymbol{\xi}}\Big(\frac{1}{|\boldsymbol{\xi}|}\Big)\;\;\hat{f}_e\;.$$

If the electron distribution function is expanded in a series of spherical harmonics, $\hat{f}_e = \Sigma_{lm}\,\hat{f}_{e,lm}(|\boldsymbol{\xi}|)Y_{lm}(\lambda,\varphi)$, $\hat{\mathbf{S}}$ depends only on the $\hat{f}_{e,00}$ and $\hat{f}_{e,2m}$ components, and $\hat{\mathbf{T}}$ depends only on the $\hat{f}_{e,1m}$ components.

Notice that $||\hat{\mathbf{S}}|| \sim 1$, $\;|\mathbf{j}/(env_{the})| \sim (m_e/m_i)^{1/2}\delta$ and $|\hat{\mathbf{T}}| \sim q_e/(p_e v_{the}) \sim (m_e/m_i)^{1/2} q_e/(p_e v_{thi})$.

SIMILAR EXPRESSIONS CAN BE OBTAINED FOR THE OTHER MOMENTS OF THE COLLISION OPERATOR.