Reduced Quintic Finite Element

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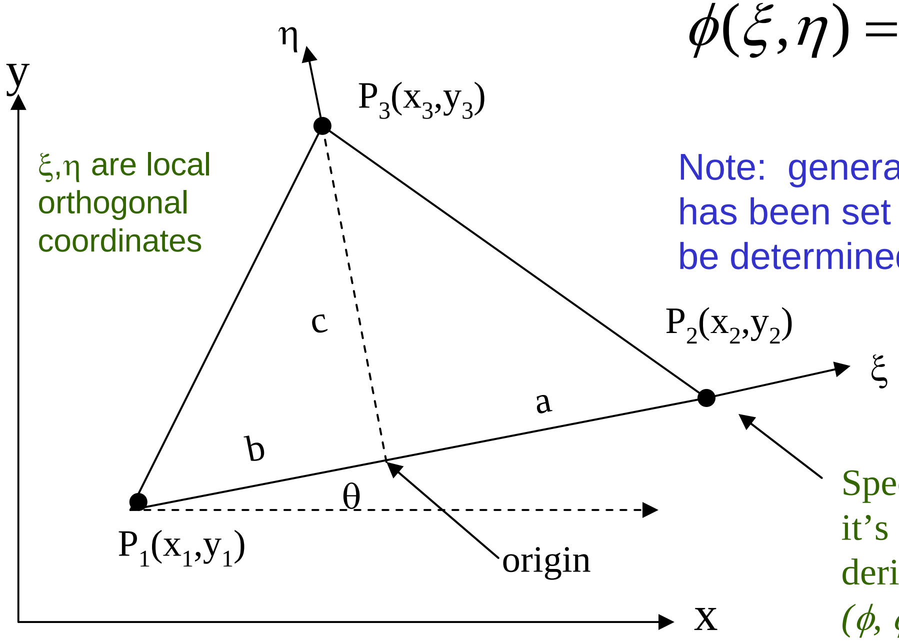
Reduced Quintic 2D Triangular Finite Element



$$\phi(\xi,\eta)=\sum_{k=1}^{20} a_k \xi^{m_k}\eta^{n_k}$$

Note: general quintic has 21 terms. 1 has been set to zero, and 2 others will be determined by constraints.

Specify the function and it's first and second derivatives at the 3 nodes (ϕ, ϕ_x, ϕ_y, ϕ_{xx}, ϕ_{xy}, ϕ_{yy})

Guarantees C^0 continuity

k	m_k	n_k
1	0	0
2	1	0
3	0	1
4	2	0
5	1	1
6	0	2
7	3	0
8	2	1
9	1	2
10	0	3
11	4	0
12	3	1
13	2	2
14	1	3
15	0	4
16	5	0
17	3	2
18	2	3
19	1	4
20	0	5

For C^1, require that the normal slope along the edges ϕ_n have only cubic variation:

$$5b^4ca_{16} + (3b^2c^3 - 2b^4c)a_{17} + (2bc^4 - 3b^3c^2)\,a_{18} + (c^5 - 4b^2c^3)a_{19} - 5bc^4a_{20} = 0$$

$$5a^4ca_{16} + (3a^2c^3 - 2a^4c)a_{17} + (-2ac^4 - 3a^3c^2)\,a_{18} + (c^5 - 4a^2c^3)a_{19} - 5ac^4a_{20} = 0$$

20 – 2 = 18 unknowns:

These are determined in terms of [ϕ, ϕ_x, ϕ_y, ϕ_{xx}, ϕ_{xy}, ϕ_{yy}] at P_1,P_2,P_3

Implies C^1 continuity at edges and C^2 at nodes !

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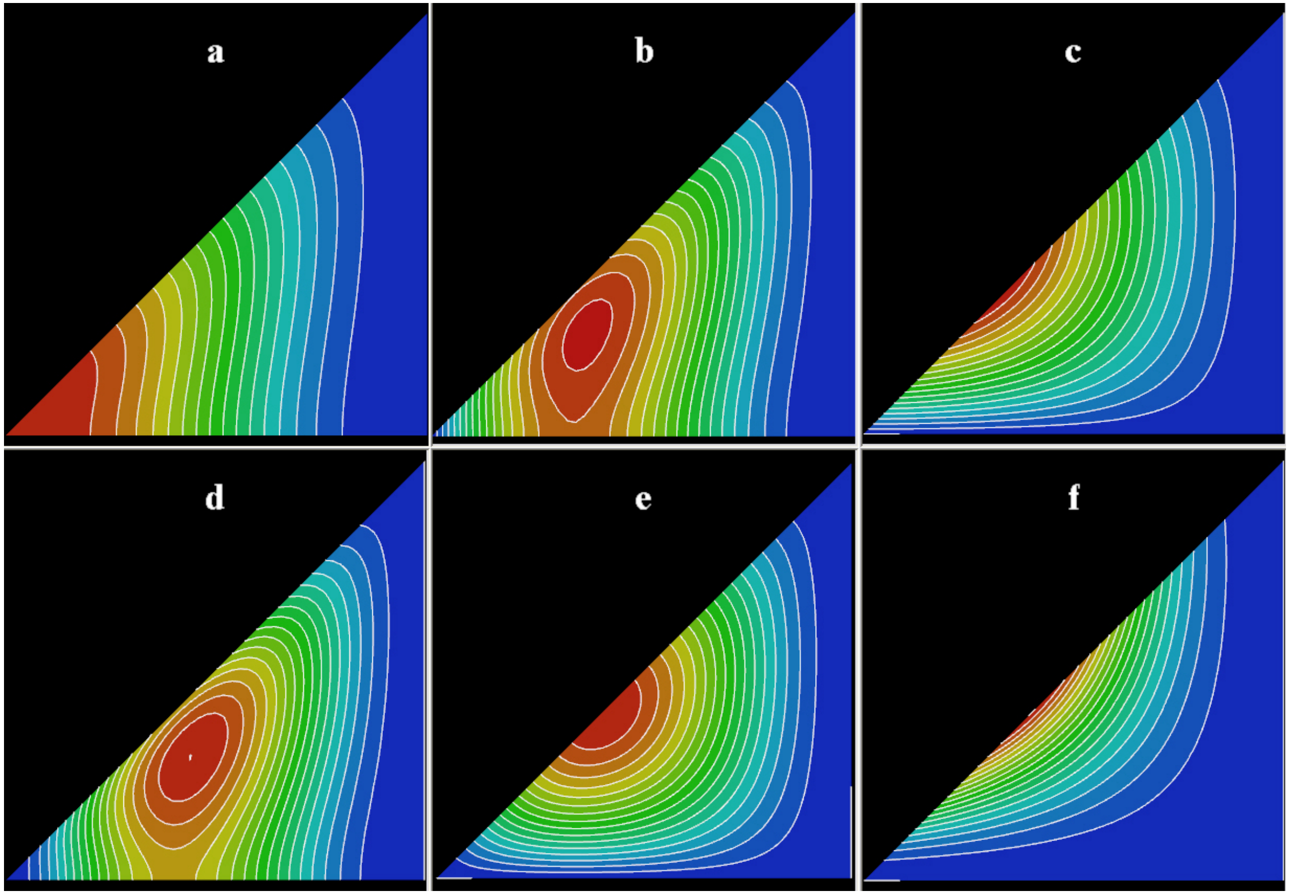
$a_i = g_{ij}\, \Phi_j$

The Trial Functions:

$$\phi = \sum_{i=1}^{20} a_i\, \xi^{m_i} \eta^{n_i} = \sum_{i=1}^{20} \sum_{j=1}^{18} g_{ij} \Phi_j \xi^{m_i} \eta^{n_i} = \sum_{j=1}^{18} v_j \Phi_j$$

$$v_j = \sum_{i=1}^{20} \xi^{m_i} \eta^{n_i} g_{ij}$$



These are the trial functions. There are 18 for each triangle.

The 6 shown here correspond to one node, and vanish at the other nodes, along with their derivatives

Each of the six has value 1 for the function or one of it's derivatives at the node, zero for the others.

Note that the function and it's derivatives (through 2nd) play the role of the amplitudes

Element Order

If an element with typical size *h* contains a complete polynomial of order *M*, then the error will be at most of order h^{M+1}

This follows directly from a local Taylor series expansion:

$$\phi(x,y) = \sum_{k=0}^{M} \sum_{l=0}^{k} \frac{1}{l!(k-l)!} \left[\frac{\partial^k \phi}{\partial x^l \partial z^{k-l}} \right]_{x_0, z_0} (x-x_0)^l (z-z_0)^{k-l} + O(h^{M+1})$$

Thus, linear elements will be $O(h^2)$

quadratic elements will be $O(h^3)$

cubic elements will be $O(h^4)$

quartic elements will be $O(h^5)$

complete quintic elements will be $O(h^6)$

Reduced quintic contains a complete quartic and thus its error is $O(h^5)$

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Element Continuity

Theorem: A finite element with continuity C^{k-1} belongs to Hilbert space H^k, and hence can be used for differential operators with order up to $2k$

H^k means that derivatives exist up to order k

Continuity	Hilbert Space	Applicability
C^0	H^1	second order equations
C^1	H^2	fourth order equations

The vast majority of the literature concerns C^0 elements, (including Spectral Elements, NIMROD elements)

The reduced quintic elements are C^1 and thus can be used on spatial derivatives up to 4th order.

This applicability is made possible by performing integration by parts in the Galerkin method, shifting derivatives from the unknown to the trial function

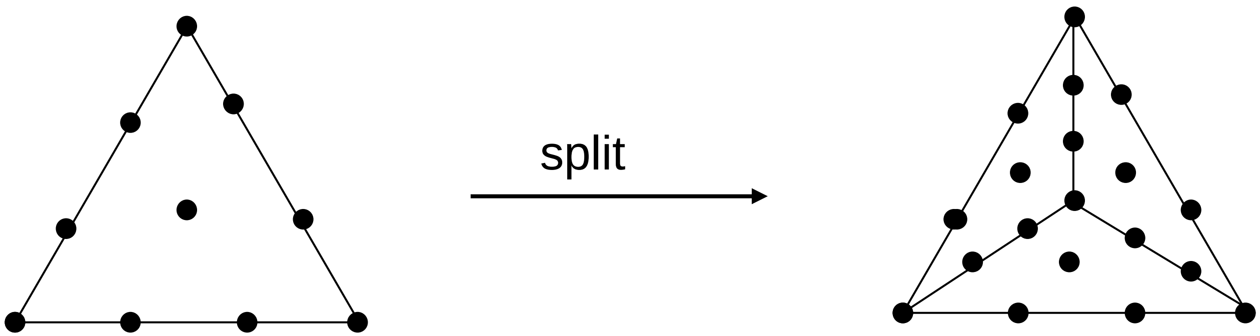
recall:

$$\iint_{domain} v_i \left[\nabla \bullet f(x,y) \nabla \phi\right] dxdy = -\iint_{domain} f(x,y) \nabla v_i \bullet \nabla \phi \, dxdy$$

$$\iint_{domain} v_i \left[\nabla^2 f(x,y) \nabla^2 \phi\right] dxdy = \iint_{domain} f(x,y) \nabla^2 v_i \nabla^2 \phi \, dxdy$$

NOTE: requires the trial function have appropriate boundary conditions

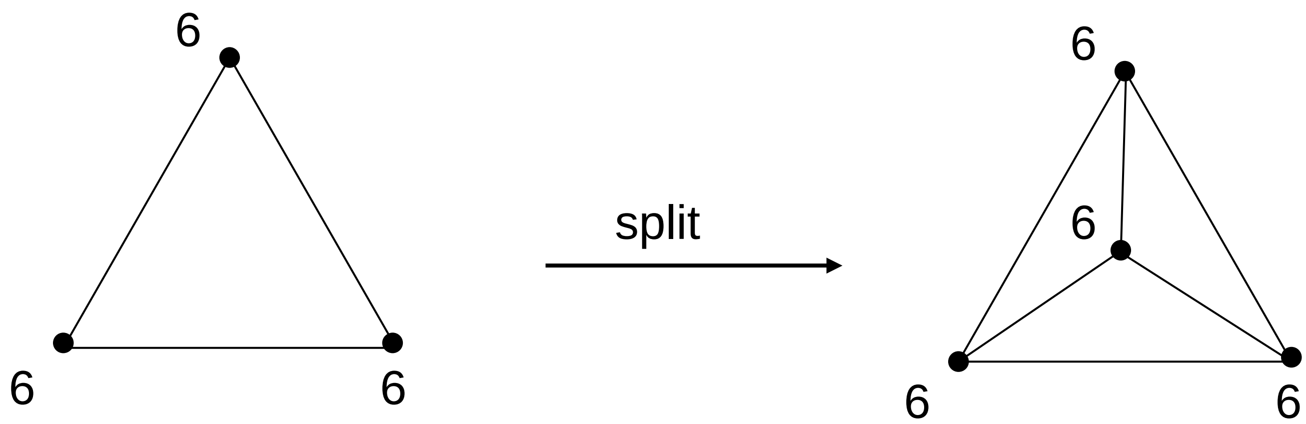
Comparison with a popular C^0 Element



Lagrange Cubic: C^0, h^4

9 new unknowns: 2 new triangles

9/2 = $4^{1/2}$ unknowns/ triangle



Reduced Quintic: C^1, h^5

6 new unknowns: 2 new triangles

6/2 = 3 unknowns/ triangle

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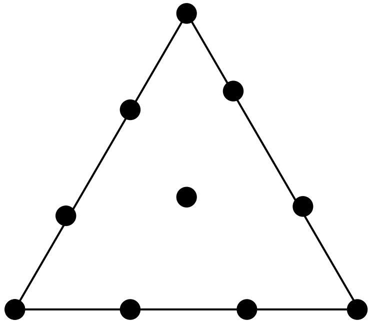
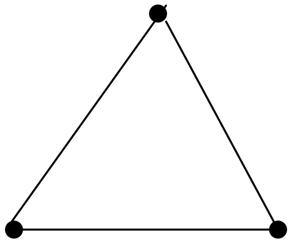
Comparison of reduced quintic to other popular triangular elements

	Vertex nodes	Line nodes	Interior nodes	accuracy order h^p	UK/T[1]	continuity
linear element	3	0	0	2	½	C^0
Lagrange quadratic	3	3	0	3	2	C^0
Lagrange cubic	3	6	1	4	4½	C^0
Lagrange quartic	3	9	3	5	8	C^0
reduced quintic	18	0	0	5	3	C^1 *



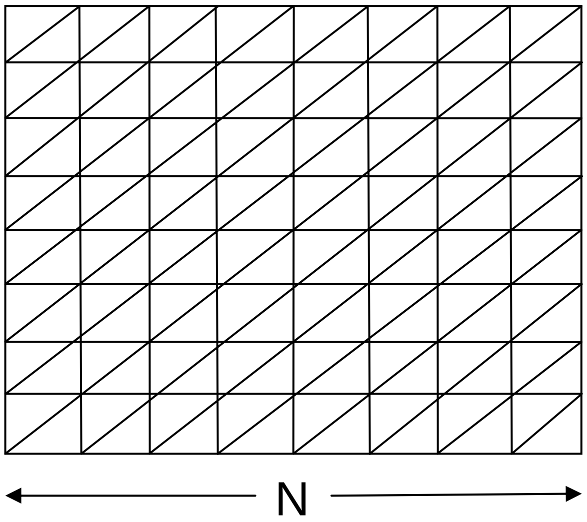


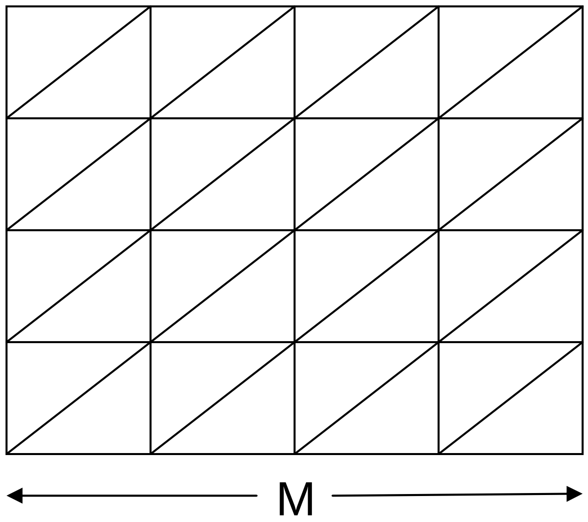
UK/T[1] is number of unknowns (or Degrees of Freedom) per triangle

Second order equation: $$\nabla^2\Phi = S$$

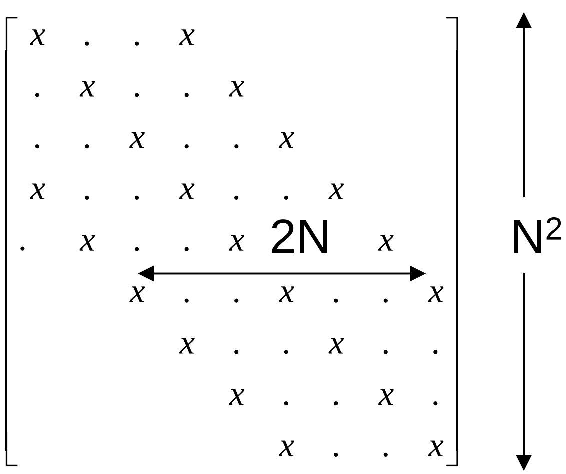
Linear Elements $E_L = a\dfrac{1}{N^2}$

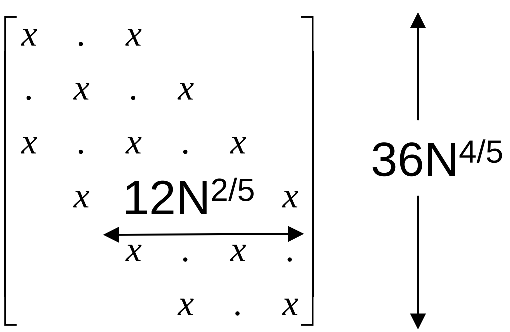
Reduced Quintic Elements $E_Q = b\dfrac{1}{M^5}$





same error $\Rightarrow E_L = E_Q \Rightarrow M = \left(\dfrac{b}{a}\right)^{1/5} N^{2/5} \sim N^{2/5}$



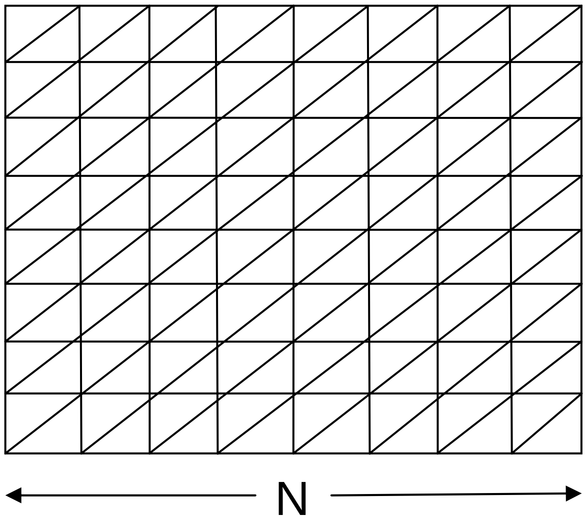


Win for N > 20 !

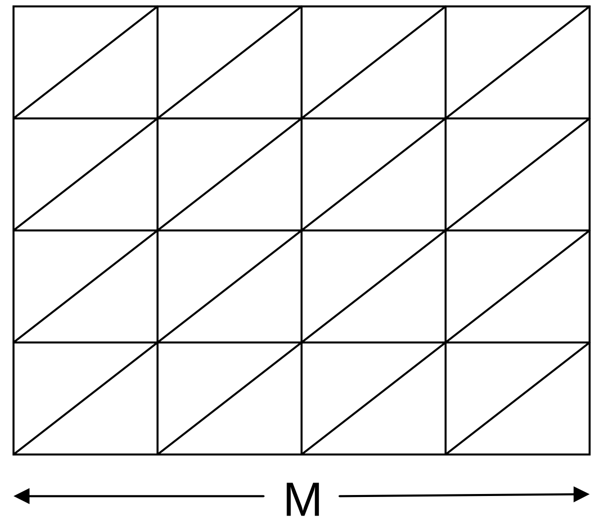
Fourth order equation: $\nabla^4\Phi = S$

$$\nabla^2\Phi = \Psi$$
$$\nabla^2\Psi = S$$

Linear Elements $E_L = a\dfrac{1}{N^2}$



Reduced Quintic Elements $E_Q = b\dfrac{1}{M^5}$



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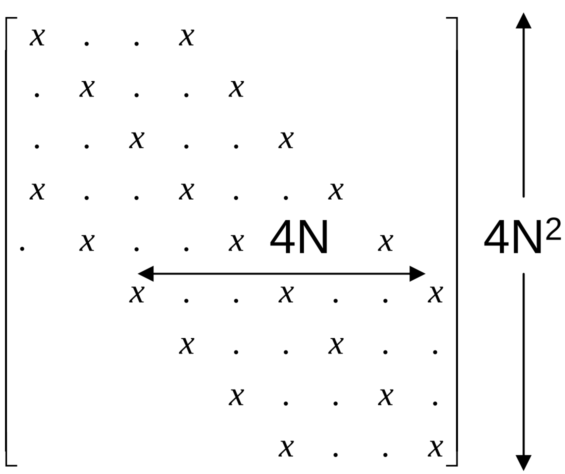

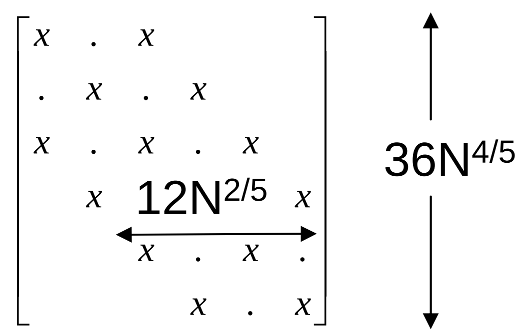

Win for N > 6 !

Summary

- Triangular finite element with error $O(h^5)$ and C^1 continuity
- Advantages
  - Minimum number of DoF per triangle for a given accuracy
  - Because it can treat up to 4th order spatial derivatives, does not require intermediate variables such as vorticity and current density
- Both of these advantages lead to smaller matrices for implicit solution
- Question: are there new numerical stability issues associated with this element?