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# ON THE RADIAL ION HEAT FLUX IN THE NEOCLASSICAL BANANA REGIME*

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**THE STANDARD EXPRESSIONS OF THE FLUX-SURFACE-AVERAGED RADIAL ION HEAT FLUX IN THE TOKAMAK NEOCLASSICAL BANANA REGIME [Rosenbluth, Hazeltine and Hinton, Phys. Fluids 15, 116 (1972)] ARE:**

$$\langle \mathbf{q}^{neo} \cdot \nabla\psi \rangle \; = \; \frac{1}{2} \, \langle \int d^3\mathbf{v} \; \mathbf{V}_d \cdot \nabla\psi \; (mv^2 - 5T) \; f \rangle \; = \; \frac{mI}{2e} \, \langle B^{-1} \int d^3\mathbf{v} \; (mv^2 - 5T) \; v_\parallel \; C[f] \rangle$$

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**THE FIRST EXPRESSION WAS OBTAINED DIRECTLY AND INVOLVES THE EVEN PART OF THE DISTRIBUTION FUNCTION**

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THE FIRST EXPRESSION WAS OBTAINED DIRECTLY AND INVOLVES THE EVEN PART OF THE DISTRIBUTION FUNCTION

THE SECOND EXPRESSION WAS OBTAINED VARIATIONALLY AND INVOLVES THE ODD PART OF THE DISTRIBUTION FUNCTION

## THE CONVENTIONAL TOKAMAK NEOCLASSICAL ANALYSIS:

- ASSUMES AN AXISYMMETRIC, ELECTROSTATIC QUASI-EQUILIBRIUM FROM THE START

- WORKS IN THE LABORATORY REFERENCE FRAME

- WRITES THE LOWEST-ORDER MAXWELLIAN IN TERMS OF PARTICLE CONSTANTS OF MOTION

**THE LOW-COLLISIONALITY, DYNAMICAL DESCRIPTION PROPOSED IN [Ramos, Phys. Plasmas 17, 082502 (2010) and Phys. Plasmas 18, 102506 (2011)]:**

- **IS FULLY 3-DIMENSIONAL AND ELECTROMAGNETIC**

- **WORKS IN THE REFERENCE FRAME OF THE MEAN MACROSCOPIC FLOW**

- **WRITES THE LOWEST-ORDER MAXWELLIAN IN TERMS OF THE PARTICLE RANDOM KINETIC ENERGY**

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- ITS STATIONARY AND AXISYMMETRIC LIMIT CONTAINS THE RESULTS OF THE TOKAMAK NEOCLASSICAL THEORY IN THE BANANA REGIME

- THIS FORMALISM WILL BE USED HERE TO DERIVE THE EXPRESSIONS OF THE FLUX-SURFACE-AVERAGED RADIAL ION HEAT FLUX

## 1. GENERAL EXPRESSION OF THE PERPENDICULAR HEAT FLUX FROM FINITE-LARMOR-RADIUS FLUID MOMENT THEORY

**THE** $\frac{1}{2}m|\mathbf{v}-\mathbf{u}|^2(\mathbf{v}-\mathbf{u})$ **MOMENT OF THE ION KINETIC EQUATION YIELDS**

$$\frac{\partial \mathbf{q}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{q} + (\nabla\cdot\mathbf{u})\mathbf{q} + (\nabla\mathbf{u}):\mathsf{Q} +$$

$$+ \frac{1}{m}\nabla\cdot\mathsf{R} - \frac{1}{mn}\left[\frac{3p}{2}\nabla\cdot\mathsf{P} + (\nabla\cdot\mathsf{P})\cdot\mathsf{P}\right] - \frac{e}{m}\mathbf{q}\times\mathbf{B} +$$

$$+ \frac{1}{mn}\left(\frac{3p}{2}\mathbf{F}^{coll} + \mathbf{F}^{coll}\cdot\mathsf{P}\right) - \mathbf{H}^{coll} = 0$$

**HENCE**

$$\mathbf{q}_\perp = \frac{\mathbf{b}}{eB}\times\left\{\nabla\cdot\mathsf{R} - \frac{1}{n}\left[\frac{3p}{2}\nabla\cdot\mathsf{P} + (\nabla\cdot\mathsf{P})\cdot\mathsf{P}\right]\right\} + \mathbf{q}_\perp^{pol} + \mathbf{q}_\perp^{coll}$$

**UNDER THE LARGE-SPATIAL-SCALE, LOW-COLLISIONALITY, SLOW-DYNAMICS AND NEAR-MAXWELLIAN ORDERINGS OF THE NEOCLASSICAL BANANA REGIME**

$$\delta = \rho/L \ll 1, \quad \delta \ll \nu_* = \nu L/v_{th} \ll 1, \quad \omega \lesssim \delta^2 \Omega_c, \quad u \sim \delta v_{th}, \quad f - f_M \sim \delta f_M,$$

$$\mathbf{q}_\perp^{neo} = \frac{\mathbf{b}}{eB} \times \left\{ \frac{5}{2} nT\nabla T + \frac{5}{6} T\nabla(p_\parallel - p_\perp) + T(p_\parallel - p_\perp)\left[\frac{1}{3}\nabla \ln(nT) - \frac{5}{2}\boldsymbol{\kappa}\right] + \nabla \hat{r}_\perp + (\hat{r}_\parallel - \hat{r}_\perp)\boldsymbol{\kappa} \right\}$$

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## WHERE

$$(p_\parallel - p_\perp) \;=\; \pi m \int dv'_\parallel \; dv'_\perp v'_\perp \; (2v'^2_\parallel - v'^2_\perp) \; \bar{f}_{NM}$$

$$\hat{r}_\parallel = \; \pi m^2 \int dv'_\parallel \; dv'_\perp v'_\perp \; v'^2_\parallel \; v'^2 \; \bar{f}_{NM} \;, \qquad \hat{r}_\perp = \; \frac{\pi m^2}{2} \int dv'_\parallel \; dv'_\perp v'_\perp \; v'^2_\perp \; v'^2 \; \bar{f}_{NM}$$

$$\mathbf{v}' \;=\; \mathbf{v} \;-\; \mathbf{u}(\mathbf{x}, t) \;=\; v'_\parallel \; \mathbf{b}(\mathbf{x}, t) \;+\; v'_\perp \; [\cos\alpha \; \mathbf{e}_1(\mathbf{x}, t) + \sin\alpha \; \mathbf{e}_2(\mathbf{x}, t)]$$

$$\bar{f}_{NM}(v'_\parallel, v'_\perp, \mathbf{x}, t) \;=\; (2\pi)^{-1} \oint d\alpha \; [f(v'_\parallel, v'_\perp, \alpha, \mathbf{x}, t) - f_M(v', \mathbf{x}, t)]$$

$$f_M(v', \mathbf{x}, t) \;=\; \left(\frac{m}{2\pi}\right)^{3/2} \frac{n}{T^{3/2}} \; \exp\left(-\frac{mv'^2}{2T}\right)$$

## 2. SPECIALIZATION TO AN AXISYMMETRIC EQUILIBRIUM

$$\mathbf{B} \;=\; \nabla\psi \times \nabla\zeta \;+\; RB_\zeta \nabla\zeta \;, \qquad\qquad \mathbf{E} = -\nabla\phi - V_0 \nabla\zeta$$

**Neglecting corrections of $O(\delta\nu_*)$:**

$$RB_\zeta = I(\psi), \qquad \phi = \phi^{(0)}(\psi), \qquad n = N^{(0)}(\psi), \qquad T = T^{(0)}(\psi)$$

$$\nabla\psi \cdot (\mathbf{b} \times \boldsymbol{\kappa}) \;=\; \nabla\psi \cdot (\mathbf{b} \times \nabla \ln B) \;=\; I(\psi)\; \mathbf{b} \cdot \nabla \ln B$$

$$\mathbf{u} \;=\; U(\psi)\; \mathbf{B} \;+\; R^2 \left[ \frac{d\phi^{(0)}}{d\psi} + \frac{1}{eN^{(0)}} \frac{d(N^{(0)} T^{(0)})}{d\psi} \right] \nabla\zeta$$

**Keeping corrections of $O(\delta\nu_*)$ in $\phi$, $n$ and $T$:**

$$\phi = \phi^{(0)}(\psi) + \Delta\phi, \qquad n = N^{(0)}(\psi) + \Delta n, \qquad T = T^{(0)}(\psi) + \Delta T$$

IN THE AXISYMMETRIC EQUILIBRIUM, THE RADIAL COMPONENT OF THE HEAT FLUX BECOMES

$$\mathbf{q}^{neo} \cdot \nabla\psi = \frac{I}{eB} \left\{ \mathbf{b} \cdot \nabla \left[ \frac{5}{2} N^{(0)} T^{(0)} \Delta T + \frac{5}{6} T^{(0)} (p_\parallel - p_\perp) + \hat{r}_\perp \right] - (\mathbf{b} \cdot \nabla \ln B) \left[ \frac{5}{2} T^{(0)} (p_\parallel - p_\perp) - (\hat{r}_\parallel - \hat{r}_\perp) \right] \right\}$$

AND ITS FLUX SURFACE AVERAGE

$$\langle \mathbf{q}^{neo} \cdot \nabla\psi \rangle = \left\langle \frac{I}{eB} \, (\mathbf{b} \cdot \nabla \ln B) \, \left[ 5 N^{(0)} T^{(0)} \Delta T - \frac{5}{6} T^{(0)} (p_\parallel - p_\perp) + \hat{r}_\parallel + \hat{r}_\perp \right] \right\rangle$$

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WHERE

$$-\frac{5}{6} T^{(0)} (p_{\parallel} - p_{\perp}) + \hat{r}_{\parallel} + \hat{r}_{\perp} \;=\; \pi m \int dv'_{\parallel} \; dv'_{\perp} v'_{\perp} \; \left( m v'^2 - 5 T^{(0)} \right) \, \left( v'^2_{\parallel} + \frac{1}{2} v'^2_{\perp} \right) \, \bar{f}_{NM}$$

## 3. AXISYMMETRIC EQUILIBRIUM DRIFT-KINETIC EQUATION FOR $\bar{f}_{NM}(v'_\parallel, v'_\perp, \theta, \psi)$

$$v'_\parallel (\mathbf{b}\cdot\nabla\theta)\frac{\partial \bar{f}_{NM}}{\partial\theta} \;+\; \frac{v'_\perp}{2}(\mathbf{b}\cdot\nabla\ln B)\left(v'_\parallel \frac{\partial \bar{f}_{NM}}{\partial v'_\perp} - v'_\perp \frac{\partial \bar{f}_{NM}}{\partial v'_\parallel}\right) \;-\; \mathcal{C}[\bar{f}_{NM}] \;=$$

$$= \;-\; \left[\mathbf{b}\cdot\nabla\left(\frac{e\phi}{T^{(0)}} + \frac{n}{N^{(0)}}\right) \;+\; \left(\frac{mv'^2}{2T^{(0)}} - \frac{3}{2}\right)\mathbf{b}\cdot\nabla\left(\frac{T}{T^{(0)}}\right)\right] v'_\parallel \; f_M^{(0)}(v', \psi) \;-$$

$$-\;(\mathbf{b}\cdot\nabla\ln B)\;\left\{\frac{m(2v'^2_\parallel - v'^2_\perp)}{2T^{(0)}}\,UB \;+\; \frac{m(2v'^2_\parallel + v'^2_\perp)}{4T^{(0)}}\left(\frac{mv'^2}{T^{(0)}} - 5\right)\frac{I}{eB}\;\frac{dT^{(0)}}{d\psi}\right\}\; f_M^{(0)}(v', \psi)$$

**where**

$$\mathcal{C}[\bar{f}_{NM}] \;=\; (2\pi)^{-1}\oint d\alpha\,\left(C[f_M, f_{NM}] + C[f_{NM}, f_M]\right)$$

## THE AXISYMMETRIC EQUILIBRIUM DRIFT-KINETIC EQUATION HAS THE SOLUTION

$$\bar{f}_{NM} \;=\; \Big[g_0(v',\theta,\psi) + v'_{\parallel} g_1(v',\theta,\psi)\Big] f_M^{(0)}(v',\psi) \;+\; h(v',\lambda,\theta,\psi)$$

## WHERE

$$g_0(v',\theta,\psi) \;=\; -\left[\frac{e\Delta\phi}{T^{(0)}} \;+\; \frac{\Delta n}{N^{(0)}} \;+\; \left(\frac{mv'^2}{T^{(0)}} - 3\right)\frac{\Delta T}{2T^{(0)}}\right]$$

$$g_1(v',\theta,\psi) \;=\; \frac{mI}{2eBT^{(0)}}\left(\frac{mv'^2}{T^{(0)}} - 5\right)\frac{dT^{(0)}}{d\psi} \;-\; \frac{mUB}{T^{(0)}}$$

$$\lambda \;=\; \frac{v_{\perp}'^2 B_{max}(\psi)}{v'^2 B(\psi,\theta)}$$

$$v'_{\parallel}\;(\mathbf{b}\cdot\nabla\theta)\;\frac{\partial h}{\partial\theta}\Big|_{v',\lambda,\psi} \;-\; \mathcal{C}[h] \;=\; \mathcal{C}[v'_{\parallel} g_1 f_M^{(0)}]$$

## 4. FLUX-SURFACE-AVERAGED RADIAL HEAT FLUX IN AXISYMMETRIC EQUILIBRIUM

$$\langle \mathbf{q}^{neo} \cdot \nabla\psi \rangle = \Big\langle \frac{I}{eB} \left(\mathbf{b} \cdot \nabla \ln B\right) \Big[ 5N^{(0)}T^{(0)}\Delta T + \pi m \int dv'_\parallel \; dv'_\perp v'_\perp \; \left(m v'^2 - 5T^{(0)}\right) \left(v'^2_\parallel + \frac{1}{2} v'^2_\perp\right) \; \bar{f}_{NM} \Big] \Big\rangle =$$

$$= \Big\langle \frac{I}{eB} \left(\mathbf{b} \cdot \nabla \ln B\right) \Big[ 5N^{(0)}T^{(0)}\Delta T + \pi m \int dv'_\parallel \; dv'_\perp v'_\perp \; \left(m v'^2 - 5T^{(0)}\right) \left(v'^2_\parallel + \frac{1}{2} v'^2_\perp\right) \left(g_0 f_M + h^{even}\right) \Big] \Big\rangle =$$

$$= \Big\langle \frac{\pi m I}{eB} \left(\mathbf{b} \cdot \nabla \ln B\right) \int dv'_\parallel \; dv'_\perp v'_\perp \; \left(m v'^2 - 5T^{(0)}\right) \left(v'^2_\parallel + \frac{1}{2} v'^2_\perp\right) \; h^{even} \Big\rangle$$

## 4. FLUX-SURFACE-AVERAGED RADIAL HEAT FLUX IN AXISYMMETRIC EQUILIBRIUM

$$\langle \mathbf{q}^{neo} \cdot \nabla\psi \rangle = \Big\langle \frac{I}{eB} \, (\mathbf{b} \cdot \nabla \ln B) \, \Big[ 5N^{(0)} T^{(0)} \Delta T + \pi m \int dv'_\parallel \; dv'_\perp v'_\perp \; (m v'^2 - 5T^{(0)}) \, \Big( v'^2_\parallel + \frac{1}{2} v'^2_\perp \Big) \; \bar{f}_{NM} \Big] \Big\rangle =$$

$$= \Big\langle \frac{I}{eB} \, (\mathbf{b} \cdot \nabla \ln B) \, \Big[ 5N^{(0)} T^{(0)} \Delta T + \pi m \int dv'_\parallel \; dv'_\perp v'_\perp \; (m v'^2 - 5T^{(0)}) \, \Big( v'^2_\parallel + \frac{1}{2} v'^2_\perp \Big) \; (g_0 f_M + h^{even}) \Big] \Big\rangle =$$

$$= \Big\langle \frac{\pi m I}{eB} \, (\mathbf{b} \cdot \nabla \ln B) \; \int dv'_\parallel \; dv'_\perp v'_\perp \; (m v'^2 - 5T^{(0)}) \, \Big( v'^2_\parallel + \frac{1}{2} v'^2_\perp \Big) \; h^{even} \Big\rangle$$

**Calling** $\mathbf{V}_d \equiv \dfrac{m}{eB} \mathbf{b} \times \left( v'^2_\parallel \boldsymbol{\kappa} + \dfrac{v'^2_\perp}{2} \nabla \ln B \right)$, **hence** $\mathbf{V}_d \cdot \nabla\psi = \dfrac{mI}{eB} (\mathbf{b} \cdot \nabla \ln B) \left( v'^2_\parallel + \dfrac{v'^2_\perp}{2} \right)$ :

$$\langle \mathbf{q}^{neo} \cdot \nabla\psi \rangle = \Big\langle \pi \int dv'_\parallel \; dv'_\perp v'_\perp \; (m v'^2 - 5T^{(0)}) \; (\mathbf{V}_d \cdot \nabla\psi) \; h^{even} \Big\rangle$$

**SPLIT $\langle \mathbf{q}^{neo} \cdot \nabla\psi \rangle$ IN TWO PIECES:**

$$\langle \mathbf{q}^{neo} \cdot \nabla\psi \rangle \;=\; \Big\langle \frac{\pi m I}{eB} \; (\mathbf{b}\cdot\nabla \ln B) \; \int dv'_{\|} \; dv'_{\perp} v'_{\perp} \; (m v'^2 - 5T^{(0)}) \; \Big( v'^2_{\|} + \frac{1}{2} v'^2_{\perp} \Big) \; h^{even} \Big\rangle \;=$$

$$= \; \Big\langle \frac{2\pi m I}{eB} \; (\mathbf{b}\cdot\nabla \ln B) \; \int dv'_{\|} \; dv'_{\perp} v'_{\perp} \; (m v'^2 - 5T^{(0)}) \; v'^2_{\|} \; h^{even} \Big\rangle +$$

$$+ \; \Big\langle \frac{\pi m I}{eB} \; (\mathbf{b}\cdot\nabla \ln B) \; \int dv'_{\|} \; dv'_{\perp} v'_{\perp} \; (m v'^2 - 5T^{(0)}) \; \Big( \frac{1}{2} v'^2_{\perp} - v'^2_{\|} \Big) \; h^{even} \Big\rangle$$

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$$+ \ \Big\langle \frac{\pi m I}{eB} \ (\mathbf{b} \cdot \nabla \ln B) \ \int dv'_\parallel \ dv'_\perp v'_\perp \ (m v'^2 - 5T^{(0)}) \ \Big( \frac{1}{2} v'^2_\perp - v'^2_\parallel \Big) \ h^{even} \Big\rangle$$

**INTEGRATE THE FIRST PIECE BY PARTS WITH RESPECT TO $\theta$:**

$$\Big\langle \frac{2\pi m I}{eB} \ (\mathbf{b} \cdot \nabla \ln B) \ \int dv'_\parallel \ dv'_\perp v'_\perp \ (m v'^2 - 5T^{(0)}) \ v'^2_\parallel \ h^{even}\Big( v', \frac{v'^2_\perp B_{max}}{v'^2 B}, \theta, \psi \Big) \Big\rangle \ =$$

$$= \ \Big\langle \frac{\pi m I}{eB} \ \int dv'_\parallel \ dv'_\perp v'_\perp \ (m v'^2 - 5T^{(0)}) \ v'^2_\parallel \ \Big[ (\mathbf{b} \cdot \nabla\theta) \frac{\partial h^{even}}{\partial\theta}\Big|_{v',\lambda,\psi} - \frac{v'^2_\perp B_{max}}{v'^2 B} (\mathbf{b} \cdot \nabla \ln B) \frac{\partial h^{even}}{\partial\lambda}\Big|_{v',\theta,\psi} \Big] \Big\rangle$$

**CARRY OUT THE VELOCITY-SPACE INTEGRAL OF THE SECOND PIECE IN $(v', \lambda)$ VARIABLES AND INTEGRATE BY PARTS WITH RESPECT TO $\lambda$:**

$$\Big\langle \frac{\pi m I}{eB} \, (\mathbf{b}\cdot\nabla \ln B) \, \int dv'_\| \, dv'_\perp v'_\perp \, (m v'^2 - 5T^{(0)}) \, \Big(\frac{1}{2} v'^2_\perp - v'^2_\|\Big) \, h^{even} \Big\rangle \; =$$

$$= \; \Big\langle -\frac{\pi m I}{2eB_{max}} \, (\mathbf{b}\cdot\nabla \ln B) \, \int_0^\infty dv' \; v'^4 (m v'^2 - 5T^{(0)}) \, \int_0^{B_{max}/B} d\lambda \, \frac{2 - 3\lambda B/B_{max}}{(1-\lambda B/B_{max})^{1/2}} \, h^{even} \Big\rangle$$

**CARRY OUT THE VELOCITY-SPACE INTEGRAL OF THE SECOND PIECE IN $(v', \lambda)$ VARIABLES AND INTEGRATE BY PARTS WITH RESPECT TO $\lambda$:**

$$\Big\langle \frac{\pi m I}{eB} \, (\mathbf{b} \cdot \nabla \ln B) \, \int dv'_\parallel \, dv'_\perp v'_\perp \, (m v'^2 - 5T^{(0)}) \, \Big(\frac{1}{2} v'^2_\perp - v'^2_\parallel\Big) \, h^{even} \Big\rangle \; =$$

$$= \; \Big\langle - \frac{\pi m I}{2 e B_{max}} \, (\mathbf{b} \cdot \nabla \ln B) \, \int_0^\infty dv' \; v'^4 (m v'^2 - 5T^{(0)}) \, \int_0^{B_{max}/B} d\lambda \, \frac{2 - 3\lambda B/B_{max}}{(1 - \lambda B/B_{max})^{1/2}} \, h^{even} \Big\rangle \; =$$

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**CARRY OUT THE VELOCITY-SPACE INTEGRAL OF THE SECOND PIECE IN $(v', \lambda)$ VARIABLES AND INTEGRATE BY PARTS WITH RESPECT TO $\lambda$:**

$$\Big\langle \frac{\pi m I}{eB}\, (\mathbf{b}\cdot\nabla \ln B)\ \int dv'_{\parallel}\ dv'_{\perp} v'_{\perp}\ (m v'^2 - 5T^{(0)})\ \Big(\frac{1}{2} v'^2_{\perp} - v'^2_{\parallel}\Big)\ h^{even} \Big\rangle\ =$$

$$=\ \Big\langle -\frac{\pi m I}{2eB_{max}}\ (\mathbf{b}\cdot\nabla \ln B)\ \int_0^{\infty} dv'\ v'^4 (m v'^2 - 5T^{(0)})\ \int_0^{B_{max}/B} d\lambda\ \frac{2 - 3\lambda B/B_{max}}{(1 - \lambda B/B_{max})^{1/2}}\ h^{even} \Big\rangle\ =$$

$$=\ \Big\langle -\frac{\pi m I}{eB_{max}}\ (\mathbf{b}\cdot\nabla \ln B)\ \int_0^{\infty} dv'\ v'^4 (m v'^2 - 5T^{(0)})\ \int_0^{B_{max}/B} d\lambda\ \frac{\partial}{\partial \lambda}\Big[\lambda (1 - \lambda B/B_{max})^{1/2}\Big]\ h^{even} \Big\rangle\ =$$

$$=\ \Big\langle \frac{\pi m I}{eB_{max}}\ (\mathbf{b}\cdot\nabla \ln B)\ \int_0^{\infty} dv'\ v'^4 (m v'^2 - 5T^{(0)})\ \int_0^{B_{max}/B} d\lambda\ \lambda (1 - \lambda B/B_{max})^{1/2}\ \frac{\partial h^{even}}{\partial \lambda} \Big\rangle$$

**CARRY OUT THE VELOCITY-SPACE INTEGRAL OF THE SECOND PIECE IN $(v', \lambda)$ VARIABLES AND INTEGRATE BY PARTS WITH RESPECT TO $\lambda$:**

$$\Big\langle \frac{\pi m I}{eB} \, (\mathbf{b}\cdot\nabla \ln B) \, \int dv'_\parallel \, dv'_\perp v'_\perp \, (mv'^2 - 5T^{(0)}) \, \Big(\frac{1}{2} v'^2_\perp - v'^2_\parallel\Big) \, h^{even} \Big\rangle \; =$$

$$= \Big\langle -\frac{\pi m I}{2eB_{max}} \, (\mathbf{b}\cdot\nabla \ln B) \, \int_0^\infty dv' \; v'^4 (mv'^2 - 5T^{(0)}) \, \int_0^{B_{max}/B} d\lambda \, \frac{2 - 3\lambda B/B_{max}}{(1 - \lambda B/B_{max})^{1/2}} \, h^{even} \Big\rangle \; =$$

$$= \Big\langle -\frac{\pi m I}{eB_{max}} \, (\mathbf{b}\cdot\nabla \ln B) \, \int_0^\infty dv' \; v'^4 (mv'^2 - 5T^{(0)}) \, \int_0^{B_{max}/B} d\lambda \, \frac{\partial}{\partial\lambda}\Big[\lambda (1 - \lambda B/B_{max})^{1/2}\Big] \, h^{even} \Big\rangle \; =$$

$$= \Big\langle \frac{\pi m I}{eB_{max}} \, (\mathbf{b}\cdot\nabla \ln B) \, \int_0^\infty dv' \; v'^4 (mv'^2 - 5T^{(0)}) \, \int_0^{B_{max}/B} d\lambda \; \lambda (1 - \lambda B/B_{max})^{1/2} \, \frac{\partial h^{even}}{\partial\lambda} \Big\rangle \; =$$

$$= \Big\langle \frac{\pi m I}{eB} \, (\mathbf{b}\cdot\nabla \ln B) \, \int dv'_\parallel \, dv'_\perp v'_\perp \, (mv'^2 - 5T^{(0)}) \, v'^2_\parallel \, \frac{v'^2_\perp B_{max}}{v'^2 B} \, \frac{\partial h^{even}}{\partial\lambda} \Big\rangle$$

## COMBINING THE TWO PIECES OF $\langle \mathbf{q}^{neo} \cdot \nabla\psi \rangle$

$$\langle \mathbf{q}^{neo} \cdot \nabla\psi \rangle \; = \; \Big\langle \frac{\pi m I}{eB} \int dv'_\parallel \; dv'_\perp v'_\perp \; (m v'^2 - 5T^{(0)}) \; v'^2_\parallel \; (\mathbf{b} \cdot \nabla\theta) \frac{\partial h^{even}}{\partial \theta}\Big|_{v',\lambda,\psi} \Big\rangle$$

## AND USING THE EQUATION FOR $h^{even}$

$$v'_\parallel \; (\mathbf{b} \cdot \nabla\theta) \frac{\partial h^{even}}{\partial \theta}\Big|_{v',\lambda,\psi} \; = \; \mathcal{C}[h^{odd}] \; + \; \mathcal{C}[v'_\parallel g_1 f_M^{(0)}] \; = \; \mathcal{C}[f_{NM}^{odd}]$$

## ONE GETS THE FINAL RESULT

$$\langle \mathbf{q}^{neo} \cdot \nabla\psi \rangle \; = \; \Big\langle \frac{\pi m I}{eB} \int dv'_\parallel \; dv'_\perp v'_\perp \; (m v'^2 - 5T^{(0)}) \; v'_\parallel \; \mathcal{C}[f_{NM}^{odd}] \Big\rangle \; =$$

$$= \; \Big\langle \frac{m I}{2eB} \int d^3\mathbf{v}' \; (m v'^2 - 5T^{(0)}) \; v'_\parallel \; \Big( C[f_M, f_{NM}] + C[f_{NM}, f_M] \Big) \Big\rangle$$