Derivation of Equation (2) in "Interpretation of Experiments on Collisional Drift Modes" in MATT-523, B. Coppi, H. Hendel, F. Perkins, and P. Politzer.

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Let $\vec{B}_0 = B_0\hat{z}, \quad n^0 = n^0(x)$

$$n^0 Mi\omega\vec{u}_{i\perp} - \mu\nabla_\perp^2\vec{u}_{i\perp} = -2Tk_B\nabla_\perp n_1 + \vec{J}_1 \times \vec{B}_0 \tag{1a}$$

$$-ik_\parallel(n_1k_BT - en^0\phi_1) + \frac{\nu_{ei}m}{e}J_\parallel = 0 \tag{1b}$$

$$i\omega n_1 + u_1^x n_x^0 - ik_y u_i^0 n_1 = 0 \tag{1c}$$

$$i\omega n_1 + u_{1e}^x n_x^0 - ik_\parallel \frac{1}{e}J_\parallel = 0 \tag{1d}$$

$$\begin{aligned} u_{1e}^x &= -ik_y \frac{1}{B}\phi_1 \\ ik_x u_1^x + ik_y u_1^y &= 0 \end{aligned} \tag{1e,f}$$

$$u_i^0 = \frac{-k_BT}{eBn}n_x^0\hat{y} \quad (\equiv v_d\hat{y} = -\underset{\sim}{u}_i \text{ in MATT-523}) \tag{1g}$$

> Sidebar: Note that instead of 1d, we could have used the relations:
>
> $$-\nabla_\perp\phi + \vec{u}_{e\perp}\times\vec{B} = 0$$
>
> $$\vec{u}_{e\perp} = \frac{\vec{B}\times\nabla\phi}{B^2} = \vec{u}_{i\perp} - \frac{1}{ne}\vec{J}_\perp$$
>
> $$\vec{J} = ne\left(\vec{u}_{i\perp} - \frac{\vec{B}\times\nabla\phi}{B^2}\right) + \vec{J}_\parallel$$
>
> $$\frac{1}{e}\nabla\bullet\vec{J} = \nabla n\bullet\left(\vec{u}_{i\perp} - \frac{\vec{B}\times\nabla\phi}{B^2}\right) + \frac{ik_\parallel}{e}J_\parallel$$
>
> $$0 = ik_y n_x^0\Omega_1 + \frac{n_x^0}{B_0}ik_y\phi_1 - ik_y u_i^0 n_1 + \frac{ik_\parallel}{e}J_\parallel$$
>
> using 1c to eliminate n_1 will yield 3c below

From Eq. (1f)

$$\vec{u}_{i\perp} = \nabla\Omega\times\hat{z}; \quad u_1^x = ik_y\Omega\,,\; u_1^y = -ik_x\Omega \tag{2}$$

Take the $\hat{z}$ component of the curl of (1a):

$$\left[i\omega n^0 M\left(k_x^2 + k_y^2\right) + \mu\left(k_x^2 + k_y^2\right)^2\right]\Omega = -B_0 ik_\parallel J_\parallel \tag{3a}$$

Eliminate n_1 from (1c,d):

$$\omega k_y n_x^0 \Omega + k_y \frac{1}{B} n_x^0 (\omega - k_y u_i^0)\phi_1 + (\omega - k_y u_i^0) k_\parallel \frac{1}{e} J_\parallel = 0 \tag{3b}$$

Eliminate n_1 from (1b,d):

$$-ik_\parallel \left(k_B T k_y n_x^0 \frac{1}{B} - en^0 \omega \right)\phi_1 + \left(\frac{\nu_{ei} m}{e}\omega - ik_\parallel^2 k_B T \frac{1}{e} \right) J_\parallel = 0 \tag{3c}$$

Next, eliminate Ω and ϕ_1 from (3a-c)

$$\begin{aligned} &\omega k_y n_x^0 B_0 ik_\parallel ik_\parallel \left(k_B T k_y n_x^0 \frac{1}{B_0} - en^0\omega \right) \\ &-k_y \frac{1}{B} n_x^0 (\omega - k_y u_i^0)\left[i\omega n^0 M \left(k_x^2 + k_y^2\right) + \mu\left(k_x^2 + k_y^2\right)^2 \right]\left(\frac{\nu_{ei} m}{e}\omega - ik_\parallel^2 k_B T \frac{1}{e} \right) \\ &-(\omega - k_y u_i^0) k_\parallel \frac{1}{e}\left[i\omega n^0 M \left(k_x^2 + k_y^2\right) + \mu\left(k_x^2 + k_y^2\right)^2 \right] ik_\parallel \left(k_B T k_y n_x^0 \frac{1}{B_0} - en^0\omega \right) = 0 \end{aligned} \tag{4}$$

$$\begin{aligned} &-\omega k_y n_x^0 k_\parallel^2 \left(k_B T k_y n_x^0 - B_0 e n^0 \omega \right) \\ &-(\omega - k_y u_i^0)\left[i\omega n^0 M \left(k_x^2 + k_y^2\right) + n^0 k_B T \frac{\nu_{ii}}{4\Omega_i^2 h_\pi^2}\left(k_x^2 + k_y^2\right)^2 \right] \\ &\times \left\{ k_y n_x^0 \frac{\nu_{ei} m}{e B_0}\omega - ik_\parallel^2 n^0 \omega \right\} = 0 \end{aligned} \tag{5}$$

let $(k_x^2 + k_y^2) = \dfrac{b^2 M \Omega_i^2}{k_B T}$ (note, this differs from MATT-523 by a factor of 2)

$$i\left(\omega + k_y u_i^0\right)\frac{k_\parallel^2 k_B T}{\nu_{ei} m} = b^2(\omega - k_y u_i^0)\left[\omega - \frac{i\nu_{ii}}{4h_\pi^2} b^2 \right]\left\{ 1 + i\frac{k_\parallel^2 k_B T}{\nu_{ei} m}\frac{1}{k_y u_i^0} \right\} \tag{6}$$

$$\begin{aligned} i\left(\omega + k_y u_i^0\right)\frac{k_\parallel^2 k_B T}{\nu_{ei} m} &= b^2 \omega(\omega - k_y u_i^0)\left\{ 1 + i\frac{k_\parallel^2 k_B T}{\nu_{ei} m}\frac{1}{k_y u_i^0} \right\} \\ &\quad - \frac{i\nu_{ii}}{4h_\pi^2} b^4 (\omega - k_y u_i^0)\left\{ 1 + i\frac{k_\parallel^2 k_B T}{\nu_{ei} m}\frac{1}{k_y u_i^0} \right\} \end{aligned} \tag{7}$$

$$i\left(\omega+k_y u_i^0\right)\frac{k_\parallel^2 k_B T}{\nu_{ei} m} = b^2\omega(\omega-k_y u_i^0)\left\{1+i\frac{k_\parallel^2 k_B T}{\nu_{ei} m}\frac{1}{k_y u_i^0}\right\}$$
$$-\omega b^2\frac{2i\nu_{ii}}{4h_\pi^2}b^2\left\{1+i\frac{k_\parallel^2 k_B T}{\nu_{ei} m}\frac{1}{k_y u_i^0}\right\}$$
$$+\frac{i\nu_{ii}}{4h_\pi^2}b^4(\omega)\left\{1+i\frac{k_\parallel^2 k_B T}{\nu_{ei} m}\frac{1}{k_y u_i^0}\right\} \qquad (8)$$
$$+k_y u_i^0\frac{i\nu_{ii}}{4h_\pi^2}b^4\left\{1+i\frac{k_\parallel^2 k_B T}{\nu_{ei} m}\frac{1}{k_y u_i^0}\right\}$$

$$i\left(\omega+k_y u_i^0\right)\left[\frac{k_\parallel^2 k_B T}{\nu_{ei} m}-\frac{\nu_{ii} b^4}{4h_\pi^2}\left\{1+i\frac{k_\parallel^2 k_B T}{\nu_{ei} m}\frac{1}{k_y u_i^0}\right\}\right] = b^2\omega(\omega-k_y u_i^0-\frac{i\nu_{ii}}{2h_\pi^2}b^2)\left\{1+i\frac{k_\parallel^2 k_B T}{\nu_{ei} m}\frac{1}{k_y u_i^0}\right\} \qquad (9)$$

This is the same as Eq. (2) in MATT-523 if we let the term in the brackets go to 1, let $h_\pi=1$, and let $b^2 \to b$

Appendix 1: Determinant method

Consider the three equations:
Take the $\hat{z}$ component of the curl of (1a):

$$\left[i\omega n^0 M\left(k_x^2+k_y^2\right)+\mu\left(k_x^2+k_y^2\right)^2\right]\Omega = -B_0 i k_\parallel J_\parallel \qquad (3a)$$

Eliminate n_1 from (1c,d):

$$\omega k_y n_x^0\Omega + k_y\frac{1}{B}n_x^0(\omega-k_y u_i^0)\phi_1+(\omega-k_y u_i^0)k_\parallel\frac{1}{e}J_\parallel = 0 \qquad (3b)$$

Eliminate n_1 from (1b,d):

$$-ik_\parallel\left(k_B T k_y n_x^0\frac{1}{B_0}-en^0\omega\right)\phi_1+\left(\frac{\nu_{ei} m}{e}\omega-ik_\parallel^2 k_B T\frac{1}{e}\right)J_\parallel = 0 \qquad (3c)$$

form the determinant:

$$\begin{vmatrix} \left[i\omega n^0 M\left(k_x^2+k_y^2\right)+\mu\left(k_x^2+k_y^2\right)^2\right] & B_0 i k_\parallel & 0 \\ \omega k_y n_x^0 & (\omega-k_y u_i^0)k_\parallel\frac{1}{e} & k_y\frac{1}{B}n_x^0(\omega-k_y u_i^0) \\ 0 & \left(\frac{\nu_{ei} m}{e}\omega-ik_\parallel^2 k_B T\frac{1}{e}\right) & -ik_\parallel\left(k_B T k_y n_x^0\frac{1}{B_0}-en^0\omega\right) \end{vmatrix} = 0$$

or

$$\begin{vmatrix} b^2 i\left[\omega - i\nu_{ii}\dfrac{b^2}{4h_\pi^2}\right] & 1 & 0 \\ \omega & \dfrac{i}{u_i^0 k_y}\left(\omega - k_y u_i^0\right) & (\omega - k_y u_i^0) \\ 0 & -\dfrac{\nu_{ei} m\omega}{k_\parallel^2 k_B T} + i & \left(\omega + k_y u_i^0\right) \end{vmatrix} = 0 \qquad \text{(A1:a)}$$

This gives back equation (6) as expected.

Appendix 2: can we take k_x=*0* ?

$$in^0 M\omega k_y^2 \Omega + \mu k_y^4 \Omega = -iB_0 k_\parallel J_\parallel \qquad \text{(1a)'}$$

$$-ik_\parallel (n_1 k_B T - en^0 \phi_1) + \frac{\nu_{ei} m}{e} J_\parallel = 0 \qquad \text{(1b)'}$$

$$\begin{aligned} &i(\omega - k_y u_i^0) n_1 + u_1^x n_x^0 = 0 \\ &i\omega n_1 - ik_y \frac{1}{B}\phi_1 n_x^0 - ik_\parallel \frac{1}{e} J_\parallel = 0 \end{aligned} \qquad \text{(1c,d)'}$$

$$\begin{aligned} u_{1e}^x &= -ik_y \frac{1}{B}\phi_1 \\ u_1^y &= 0 \\ u_1^x &= ik_y \Omega \end{aligned} \qquad \text{(1e,f)'}$$

Leads to the dispersion relation:

$$\begin{bmatrix} in^0 M\omega k_y^2 + \mu k_y^4 & iB_0 k_\parallel & 0 \\ \omega k_y n_x^0 & (\omega - k_y u_i^0)\dfrac{1}{e} k_\parallel & (\omega - k_y u_i^0) k_y \dfrac{1}{B} n_x^0 \\ 0 & \left(\dfrac{\nu_{ei} m}{e}\omega - ik_\parallel^2 k_B T \dfrac{1}{e}\right) & -ik_\parallel \left(k_B T k_y n_x^0 \dfrac{1}{B_0} - en^0 \omega\right) \end{bmatrix} \bullet \begin{bmatrix} \Omega \\ J_\parallel \\ \phi_1 \end{bmatrix} = 0$$

Comparing with the previous determinant, it looks as though it should be ok.

Appendix 3: Nonlinear equations:

We will assume a particular value of $k_{||}$ and derive a set of 2D time evolution equations that reduce to (1a)'-(1f)' ? Assume all quantities vary as ~ $\exp(ik_{||}z)$ and that we take the real part at the end.

Let the vector potential be given by: $\vec{A} = -\psi\hat{z} + \hat{z}\times\nabla f$. This corresponds to a particular gauge condition. Then, the magnetic field can be written as:

$\vec{B} = \nabla\times\vec{A} + F_0\hat{z} = \hat{z}\times\nabla\psi + F\hat{z} - ik_{||}\nabla_\perp f$, where $F \equiv F_0 + \nabla_\perp^2 f$, or,

$$B_x = -\frac{\partial\psi}{\partial y} - ik_{||}\frac{\partial f}{\partial x}, \quad B_y = \frac{\partial\psi}{\partial x} - ik_{||}\frac{\partial f}{\partial y}, \quad B_z = F_0 + \nabla_\perp^2 f\ . \tag{A3:1}$$

The current density is given by: $\vec{J} = \nabla\times\vec{B} = -ik_{||}\nabla_\perp\psi + \nabla\left(F - k_{||}^2 f\right)\times\hat{z} + \nabla_\perp^2\psi\hat{z}$, or

$$J_x = \frac{\partial F}{\partial y} - k_{||}^2\frac{\partial f}{\partial y} - ik_{||}\frac{\partial\psi}{\partial x}, \quad J_y = -\frac{\partial F}{\partial x} + k_{||}^2\frac{\partial f}{\partial x} - ik_{||}\frac{\partial\psi}{\partial y}, \quad J_z = \nabla_\perp^2\psi\ . \tag{A3:2}$$

We can assume the flow to be incompressible in the plane perpendicular to the magnetic field, so that the fluid velocity is: $\vec{u} = \hat{z}\times\nabla\Omega + V_z\hat{z}$. Let all quantities consist of a 2D equilibrium part and a perturbed part: ie $n(x,y,z) = \bar{n}(x,y) + \tilde{n}(x,y,z,t)e^{ik_{||}z}$. For the initial equilibrium, we can take:

$$\frac{\partial\Omega^0}{\partial x} = \frac{k_B T}{Fen}n_x^0; \qquad \frac{\partial F}{\partial x} = -\frac{k_B T}{F}n_x^0 = -en\frac{\partial\Omega^0}{\partial x}; \quad u_i^0 = -\frac{k_B T}{Fen}n_x^0\ .$$

The evolution equations are given by the following. Take the curl of the momentum equation:

$$\begin{aligned}\bar{n}M\left(\frac{\partial}{\partial t}\nabla_\perp^2\Omega + \left[\Omega,\nabla_\perp^2\Omega\right]\right) - \mu\nabla_\perp^4\Omega &= ik_{||}F_0\nabla_\perp^2\psi \\ &+ \left[\psi,\nabla_\perp^2\psi\right] - ik_{||}\left(f,\nabla_\perp^2\psi\right) + k_{||}^2\left[F,f\right] + ik_{||}\left(\psi,F\right)\end{aligned} \tag{A3:3}$$

Take the z component of the momentum equation:

$$\begin{aligned}&\bar{n}M\left(\frac{\partial V_z}{\partial t} + \left[\Omega,V_z\right] + \tfrac{1}{2}ik_{||}\left\{\left(\Omega,\Omega\right) + V_z^2\right\}\right) - \mu\nabla_\perp{}^2V_z + ik_{||}p \\ &= -ik_{||}\left(\psi,\psi\right) + 2k_{||}^2\left[f,\psi\right] + \left[\psi,F\right] - ik_{||}\left(f,F\right) + ik_{||}^3(f,f)\end{aligned} \tag{A3:4}$$

Consider the Ohm's law:

$$\vec{E} + \vec{u}\times\vec{B} = \eta\vec{J} + \frac{1}{ne}\left(\vec{J}\times\vec{B} - \nabla p_e\right). \tag{A3:5}$$

Faraday's law gives: $\dfrac{\partial\vec{A}}{\partial t} = -\vec{E} + \nabla\Phi$. By taking appropriate projections, we obtain the 3 scalar equations:

$$\frac{\partial \psi}{\partial t} = -[\Omega,\psi] - ik_{\|}(\Omega, f) + \eta\nabla_{\perp}^{2}\psi + \frac{1}{ne}\begin{pmatrix} -ik_{\|}(\psi,\psi) + 2k_{\|}^{2}[f,\psi] + [\psi,F] \\ -ik_{\|}(f,F) + ik_{\|}^{3}(f,f) - ik_{z}p_{e} \end{pmatrix} - ik_{z}\Phi$$

(A3:6)

$$\begin{aligned} \nabla_{\perp}^{2}\Phi &= -(F,\Omega) - F\nabla_{\perp}^{2}\Omega + (V_{z},\psi) + V_{z}\nabla_{\perp}^{2}\psi - k_{\|}^{2}[V_{z}, f] - ik_{\|}\eta\nabla_{\perp}^{2}\psi \\ &+ \frac{1}{n^{2}e}F\{(n,F) - k_{\|}^{2}(n, f) - ik_{\|}[\psi,n]\} + \frac{1}{n^{2}e}\nabla_{\perp}^{2}\psi\{(n,\psi) + ik_{\|}(f,n)\} \\ &+ \frac{1}{n^{2}e}(n, p_{e}) - \frac{1}{ne}\nabla_{\perp}^{2}p_{e} - \frac{1}{ne}\{(F,F) - k_{\|}^{2}(F, f) + ik_{\|}[F,\psi]\} \\ &- \frac{F}{ne}\{\Delta_{\perp}^{2}F - k_{\|}^{2}\Delta_{\perp}^{2}f\} - \frac{1}{ne}\{(\nabla_{\perp}^{2}\psi,\psi) + ik_{\|}[f,\nabla_{\perp}^{2}\psi]\} - \frac{1}{ne}[\nabla_{\perp}^{2}\psi]^{2} \end{aligned} \qquad (A3:7)$$

$$\begin{aligned} &\frac{\partial n}{\partial t} - \frac{\partial \Omega}{\partial y}\frac{\partial n}{\partial x} + \frac{\partial \Omega}{\partial x}\frac{\partial n}{\partial y} + ik_{\|}\bar{n}u_{z} = 0 \\ &i\omega n - ik_{y}\Omega n_{x}^{0} - u_{i}^{0}ik_{y}n_{1} + ik_{\|}\bar{n}u_{z} = 0 \end{aligned} \qquad (A3:8)$$

Note that in steady state, (A3:5) becomes:

$$\nabla\Phi + \vec{u}\times\vec{B} = \eta\vec{J} + \frac{1}{ne}\left(\vec{J}\times\vec{B} - \nabla p_{e}\right) \qquad (A3:9)$$

Multiply (A3:9) by n and operate on with $\hat{z}\bullet\nabla\times$

$$\begin{aligned} &\hat{z}\bullet\nabla n^{0}\times\nabla\phi + F\hat{z}\bullet\nabla n^{0}\times\nabla_{\perp}\Omega - \nabla n\bullet\vec{u}_{i}^{0}F = F_{0}ik_{\|}\nabla_{\perp}^{2}\psi \\ &\quad + [\psi,\nabla_{\perp}^{2}\psi] - ik_{\|}(f,\nabla_{\perp}^{2}\psi) + k_{\|}^{2}[F, f] + ik_{\|}(\psi,F) \end{aligned} \qquad (A3:10)$$

Take the dot product with B,

$$\vec{B}\bullet\nabla\Phi = \eta\vec{B}\bullet\vec{J} - \frac{\vec{B}\bullet\nabla p_{e}}{ne} \qquad (A3:11)$$

The linearized forms of Eqns (A3:3), (A3:10), (A3:11), (A3:8) can be written as a matrix equation:

$$\begin{bmatrix} \frac{-ib^2 M\Omega_i^2 n^0 M}{k_B T}\left[(\omega - k_y u_i^0) - i\frac{\nu_{ii} b^2}{4h_\pi^2}\right] & 0 & 0 & ik_{\|} F_0 \left(k_x^2 + k_y^2\right) \\ iF_0 k_y n_x^0 & ik_y n_x^0 & u_i^0 ik_y F & -k_{\|} F_0 \left(k_x^2 + k_y^2\right) \\ 0 & -ik_z & -\frac{k_B T}{n_0 e} ik_z & -\eta\left(k_x^2 + k_y^2\right) \\ -ik_y n_x^0 & 0 & i\omega - u_i^0 ik_y & 0 \end{bmatrix} \bullet \begin{bmatrix} \Omega \\ \phi_1 \\ n_1 \\ \Psi \end{bmatrix} = 0$$

Appendix 5: Useful Identities:

$$B_x = -\frac{\partial \psi}{\partial y} - ik_{\|}\frac{\partial f}{\partial x}, \quad B_y = \frac{\partial \psi}{\partial x} - ik_{\|}\frac{\partial f}{\partial y}, \quad B_z = F_0 + \nabla_{\perp}^2 f$$

$$J_x = \frac{\partial F}{\partial y} - k_{\|}^2\frac{\partial f}{\partial y} - ik_{\|}\frac{\partial \psi}{\partial x}, \quad J_y = -\frac{\partial F}{\partial x} + k_{\|}^2\frac{\partial f}{\partial x} - ik_{\|}\frac{\partial \psi}{\partial y}, \ J_z = \nabla_{\perp}^2\psi$$

$$\left[\hat{z}\bullet\nabla n\times(\vec{J}\times\vec{B})\right] = \nabla_{\perp}^2\psi\left\{[\psi,n] - ik_{\|}(n,f)\right\} - F\left\{[n,F] + k_{\|}^2[f,n] - ik_{\|}(n,\psi)\right\}$$

$$\nabla_{\perp}\bullet\vec{J} = -ik_{\|}\nabla_{\perp}^2\psi$$

$$\hat{z}\bullet\nabla\times\vec{J} = -\nabla_{\perp}^2 F + k_{\|}^2\nabla_{\perp}^2 f$$

$$J\times B = \left(J_y B_z - J_z B_y\right)\hat{x} + \left(J_z B_x - J_x B_z\right)\hat{y} + \left(J_x B_y - J_y B_x\right)\hat{z}$$

$$\hat{z}\bullet\nabla\times\left(J\times B\right) = \frac{\partial}{\partial x}\left(J_z B_x - J_x B_z\right) - \frac{\partial}{\partial y}\left(J_y B_z - J_z B_y\right)$$

$$= \left[\psi,\nabla_{\perp}^2\psi\right] - ik_{\|}\left(f,\nabla_{\perp}^2\psi\right) + ik_{\|}F_0\nabla_{\perp}^2\psi + ik_{\|}\left(\psi,F\right) - k_{\|}^2\left[f,F\right]$$

$$B_x = -\frac{\partial \psi}{\partial y} - ik_{\|}\frac{\partial f}{\partial x}, \quad B_y = \frac{\partial \psi}{\partial x} - ik_{\|}\frac{\partial f}{\partial y}, \quad B_z = F_0 + \nabla_{\perp}^2 f$$

$$J_x = \frac{\partial F}{\partial y} - k_{\|}^2\frac{\partial f}{\partial y} - ik_{\|}\frac{\partial \psi}{\partial x}, \quad J_y = -\frac{\partial F}{\partial x} + k_{\|}^2\frac{\partial f}{\partial x} - ik_{\|}\frac{\partial \psi}{\partial y}, \ J_z = \nabla_{\perp}^2\psi$$

$$\hat{z}\bullet\nabla\times\left(J\times B\right) = \left[\psi,\nabla_{\perp}^2\psi\right] - ik_{\|}\left(f,\nabla_{\perp}^2\psi\right) + k_{\|}^2\left[F,f\right] + ik_{\|}\left(\psi,F\right)$$

$$\hat{z}\bullet\left(J\times B\right) = \left(J_x B_y - J_y B_x\right) = -ik_{\|}\left(\psi,\psi\right) + 2k_{\|}^2\left[f,\psi\right] + \left[\psi,F\right] - ik_{\|}\left(f,F\right) + ik_{\|}^3(f,f)$$

$$\nabla_{\perp}\bullet\left[\frac{1}{ne}\left(\vec{J}\times\vec{B} - \nabla p_e\right)\right] = -\frac{1}{n^2 e}\nabla_{\perp}n\bullet\left[\left(\vec{J}\times\vec{B} - \nabla p_e\right)\right] + \frac{1}{ne}\nabla_{\perp}\bullet\left[\left(\vec{J}\times\vec{B} - \nabla p_e\right)\right]$$

$$= \frac{1}{n^2 e}F\left\{(n,F) - k_{\|}^2(n,f) - ik_{\|}[\psi,n]\right\} + \frac{1}{n^2 e}\nabla_{\perp}^2\psi\left\{(n,\psi) + ik_{\|}(f,n)\right\} + \frac{1}{n^2 e}(n,p_e) - \frac{1}{ne}\nabla_{\perp}^2 p_e$$

$$-\frac{1}{ne}\left\{(F,F) - k_{\|}^2(F,f) + ik_{\|}[F,\psi]\right\} - \frac{F}{ne}\left\{\Delta_{\perp}^2 F - k_{\|}^2\Delta_{\perp}^2 f\right\}$$

$$-\frac{1}{ne}\left\{\left(\nabla_{\perp}^2\psi,\psi\right) + ik_{\|}\left[f,\nabla_{\perp}^2\psi\right]\right\} - \frac{1}{ne}\left[\nabla_{\perp}^2\psi\right]^2$$

$$\vec{u} = \hat{z}\times\nabla\Omega + V_z\hat{z} \qquad \vec{B} = \nabla\times\vec{A} = \hat{z}\times\nabla\psi + F\hat{z} - ik_{\|}\nabla_{\perp}f$$

$$u\times B = \left[\Omega,\psi\right]\hat{z} + ik_{\|}\left(\Omega,f\right)\hat{z} + F\nabla_{\perp}\Omega - V_z\nabla_{\perp}\psi - ik_{\|}V_z\hat{z}\times\nabla_{\perp}f$$

$$-\hat{z}\bullet\vec{u}\times\vec{B} = -\left[\Omega,\psi\right] - ik_{\parallel}\left(\Omega,\ f\right)$$
$$-\nabla_{\perp}\bullet\vec{u}\times\vec{B} = -\left(F,\Omega\right) - F\nabla_{\perp}^{2}\Omega + \left(V_z,\psi\right) + V_z\nabla_{\perp}^{2}\psi - k_{\parallel}^{2}\left[V_z,\ f\right]$$
$$\hat{z}\bullet\nabla\times\vec{u}\times\vec{B} = \left[F,\Omega\right] + \left[\psi,V_z\right] - ik_{\parallel}V_zF - ik_{\parallel}\left(V_z,\ f\right)$$

$$\vec{u} = \hat{z}\times\nabla\Omega + V_z\hat{z}$$
$$u^2 = \left(\Omega,\Omega\right) + V_z^2$$

$$\nabla\times\vec{u} = \nabla_{\perp}^{2}\Omega\hat{z} + \nabla V_z\times\hat{z}$$
$$\vec{u}\times\nabla\times\vec{u} = \hat{z}\times\nabla\Omega\times\left(\nabla_{\perp}^{2}\Omega\hat{z} + \nabla V_z\times\hat{z}\right) + V_z\hat{z}\times\left(\nabla_{\perp}^{2}\Omega\hat{z} + \nabla V_z\times\hat{z}\right)$$
$$= \nabla_{\perp}\Omega\nabla_{\perp}^{2}\Omega - \hat{z}\times\nabla\Omega\bullet\nabla V_z\hat{z} + V_z\nabla_{\perp}V_z$$

$$\vec{u}\bullet\nabla\vec{u} = -\vec{u}\times\nabla\times\vec{u} + \tfrac{1}{2}\nabla u^2$$
$$= -\nabla_{\perp}\Omega\nabla_{\perp}^{2}\Omega - V_z\nabla_{\perp}V_z + \left[\Omega,V_z\right]\hat{z} + \tfrac{1}{2}\nabla\left\{\left(\Omega,\Omega\right) + V_z^2\right\}$$
$$\hat{z}\bullet\vec{u}\bullet\nabla\vec{u} = \left[\Omega,V_z\right] + \tfrac{1}{2}ik_{\parallel}\left\{\left(\Omega,\Omega\right) + V_z^2\right\}$$

$$\hat{z}\bullet\nabla\times(\vec{u}\bullet\nabla\vec{u}) = -\hat{z}\bullet\nabla\times\left\{\nabla_{\perp}^{2}\Omega\nabla_{\perp}\Omega + V_z\nabla_{\perp}V_z\right\}$$
$$= -\hat{z}\bullet\nabla\nabla_{\perp}^{2}\Omega\times\left\{\nabla_{\perp}\Omega\right\} - \hat{z}\bullet\nabla V_z\times\left\{\nabla_{\perp}V_z\right\}$$
$$= \left[\Omega,\nabla_{\perp}^{2}\Omega\right]$$

Appendix 6: Corrected Derivation

Let $\vec{B}_0 = B_0\hat{z},\quad n^0 = n^0(x)$. The equilibrium is given by: $\vec{J}_0\times\vec{B}_0 = \nabla p_0$. If we assume that $\vec{J}_0 = ne(\vec{u}_i^0 - \vec{u}_e^0)$, then the equilibrium condition implies an equilibrium velocity given by: $\vec{u}_i^0 = \dfrac{k_BT}{n_0eB_0}n_x^0\hat{y} \equiv -u_i^0\hat{y},\quad \vec{u}_e^0 = -\dfrac{k_BT}{n_0eB_0}n_x^0\hat{y} \equiv u_i^0\hat{y}$. The equations governing the first order linear quantities are then:

$$n^0Mi\left(\omega - k_yu_i^0\right)\vec{u}_{i\perp} - \mu\nabla_{\perp}^{2}\vec{u}_{i\perp} = -2Tk_B\nabla_{\perp}n_1 + \vec{J}_1\times\vec{B}_0 \tag{A6: 1a}$$
$$-\nabla_{\perp}\phi_1 + \vec{u}_{1e\perp}\times\vec{B} = -\frac{k_BT}{ne}\nabla_{\perp}n_1 \tag{A6:1b}$$
$$-ik_{\parallel}(n_1k_BT - en^0\phi_1) + \frac{\nu_{ei}m}{e}J_{\parallel} = 0 \tag{A6:1c}$$

$$i\omega n_1 + u_1^x n_x^0 - ik_y u_i^0 n_1 = 0 \qquad \text{(A6:1d)}$$

$$ik_x u_1^x + ik_y u_1^y = 0 \qquad \text{(A6:1e)}$$

From Eq. (A6:1e)

$$\vec{u}_{i\perp} = \nabla\Omega \times \hat{z}; \quad u_1^x = ik_y\Omega\,,\; u_1^y = -ik_x\Omega \qquad \text{(A6:2)}$$

Take the $\hat{z}$ component of the curl of (1a):

$$\left[i(\omega - k_y u_i^0) n^0 M\left(k_x^2 + k_y^2\right) + \mu\left(k_x^2 + k_y^2\right)^2\right]\Omega = -B_0 ik_\| J_\| \qquad \text{(A6: 3a)}$$

From (A6:1b), using $\nabla\bullet\vec{J} = \nabla\bullet\left[ne\left(\vec{u}_i - \vec{u}_e\right)\right] = 0$, we have the identities:

$$\vec{u}_{e\perp} = \vec{u}_{e0\perp} + \frac{\vec{B}\times\nabla\phi}{B^2} - \frac{k_B T}{ne}\frac{\vec{B}\times\nabla n_1}{B^2} = \vec{u}_{i\perp} - \frac{1}{ne}\vec{J}_\perp$$

$$\vec{J} = ne\left(\vec{u}_{i0} + \nabla\Omega\times\hat{z} - \frac{\vec{B}\times\nabla\phi_1}{B^2} + \frac{k_B T}{ne}\frac{\vec{B}\times\nabla(n_0+n_1)}{B^2}\right) + \vec{J}_\| \qquad \text{(A6:4)}$$

$$\frac{1}{e}\nabla\bullet\vec{J}_1 = \nabla n\bullet\left(\vec{u}_{i0} + \nabla\Omega\times\hat{z} - \frac{\vec{B}\times\nabla\phi}{B^2}\right) + \frac{ik_\|}{e}J_\|$$

Thus,

$$0 = ik_y n_x^0\Omega_1 + \frac{n_x^0}{B_0}ik_y\phi_1 - ik_y u_i^0 n_1 + \frac{ik_\|}{e}J_\| \qquad \text{(A6:3b)}$$

We can write Equations (A6:3a, 3b,1c, and 1d as a matrix equation:

$$\begin{bmatrix} \dfrac{-ib^2 M\Omega_i^2 n^0 M}{k_B T}\left[(\omega - k_y u_i^0) - i\dfrac{\nu_{ii} b^2}{4h_\pi^2}\right] & 0 & 0 & iB_0 k_\| \\ k_y n_x^0 & \dfrac{n_x^0}{B_0}k_y & -k_y u_i^0 & \dfrac{k_\|}{e} \\ 0 & ik_\| e n^0 & -ik_\| k_B T & \dfrac{\nu_{ei} m}{e} \\ k_y n_x^0 & 0 & \omega - k_y u_i^0 & 0 \end{bmatrix} \bullet \begin{bmatrix}\Omega \\ \phi_1 \\ n_1 \\ J_\|\end{bmatrix} = 0$$

$$\begin{bmatrix} -ib^2\left[(\omega - k_y u_i^0) - i\dfrac{\nu_{ii} b^2}{4h_\pi^2}\right] & 0 & 0 & i \\ 0 & -u_i^0 k_y & -\omega & 1 \\ 0 & -1 & 1 & \dfrac{i\nu_{ei} m}{k_\|^2 k_B T} \\ -u_i^0 k_y & 0 & \left(\omega - k_y u_i^0\right) & 0 \end{bmatrix} \bullet \begin{bmatrix}\Omega \\ \phi_1 \\ n_1 \\ J_\|\end{bmatrix} = 0$$

$$-ib^2\left[(\omega-k_y u_i^0)-i\frac{\nu_{ii}b^2}{4h_\pi^2}\right]\begin{bmatrix}-u_i^0 k_y & -\omega & 1\\ -1 & 1 & \dfrac{i\nu_{ei}m}{k_\parallel^2 k_B T}\\ 0 & \left(\omega-k_y u_i^0\right) & 0\end{bmatrix}$$

$$+u_i^0 k_y\begin{bmatrix}0 & 0 & i\\ -u_i^0 k_y & -\omega & 1\\ -1 & 1 & \dfrac{i\nu_{ei}m}{k_\parallel^2 k_B T}\end{bmatrix}=0$$

$$ib^2\left[(\omega-k_y u_i^0)-i\frac{\nu_{ii}b^2}{4h_\pi^2}\right]\left(\omega-k_y u_i^0\right)\begin{bmatrix}-u_i^0 k_y & 1\\ -1 & \dfrac{i\nu_{ei}m}{k_\parallel^2 k_B T}\end{bmatrix}+iu_i^0 k_y\begin{bmatrix}-u_i^0 k_y & -\omega\\ -1 & 1\end{bmatrix}=0$$

$$b^2\left(\omega-k_y u_i^0\right)\left[(\omega-k_y u_i^0)-i\frac{\nu_{ii}b^2}{4h_\pi^2}\right]\left[1+\frac{ik_\parallel^2 k_B T}{\nu_{ei}m u_i^0 k_y}\right]=i\left(\omega+k_y u_i^0\right)\frac{k_\parallel^2 k_B T}{\nu_{ei}m}$$