SOME DISPERSION PROPERTIES OF THE GYRO-VISCOUS FLUID EQUATIONS IN A UNIFORM MEDIUM

D. D. Schnack

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1. Introduction

Extensions to the usual resistive MHD equations are required to capture two-fluid and finite-Larmor radius (FLR) effects. These extra terms include the Hall term and the electron pressure gradient in Ohm's law, and the divergence of the gyro-viscous stress in the fluid momentum equation. While considerable attention has been paid to the physical and computational consequences of the modifications to Ohm's law, the role of the gyro-viscous force does not appear to have been as thoroughly studied. The purpose of this note is to examine the linear dispersion properties of the full set of equations including both the generalized Ohm's law and the gyro-viscous force, to identify characteristics of this dispersion that may lead to computational difficulties, and to envision paths to an efficient time stepping algorithm when all terms are taken into account.

In a previous note[1], several possible orderings for the two-fluid equations were identified. In all cases the ratio of the ion Larmor radius to the macroscopic scale length, $\delta = \rho_i / L$, was considered to be a small parameter, i.e., it was assumed that $\delta << 1$. Three distinct sets of equations were identified depending on the assumed ordering of other parameters with respect to δ. These are: Hall MHD, which is valid at very low plasma β and accommodates very vast flows; resistive MHD, which is valid at relatively low β and accommodates fast, but not supersonic, flows; and, drift MHD, which allows for high β but is limited to very slow flows. These models are summarized in Table I.

Table I

Properties of Fluid Models

Model	V_i	ω	β	$\mathbf{J} \times \mathbf{B}$	Whistlers†	KAW††
Hall MHD	V_{thi} / δ	Ω_{ci}	$O(\delta^2)$	$mn \dfrac{d\mathbf{V}_i}{dt} + O(\delta)$	Yes	No
Ideal MHD	V_{thi}	$\delta\Omega_{ci}$	$O(\delta)$	$O(\delta)$	No	No
Drift	δV_{thi}	$\delta^2 \Omega_{ci}$	$O(1)$	$\nabla p + O(\delta^2)$	No	Yes

†Whistler waves are high frequency phenomena that disappear as the frequency is ordered successively lower.

††Kinetic Alfvén waves are finite pressure phenomena that appear as β becomes successively larger.

Hall and drift MHD represent the lowest order (in δ) FLR and two-fluid corrections to the resistive MHD model. In both cases the gyro-viscous force enters at the same order in δ as the pressure gradient; it is inconsistent to include one without also including the other. Only resistive MHD escapes this ordering, and the gyro-viscosity is then ignored. As indicated in Table I, the Hall and drift models introduce new normal modes (whistlers and kinetic Alfvén waves) that are dispersive, i.e., they lead to dispersion relations of the form $\omega^2 \sim k^4$. An explicit time advance of these terms will therefore have a CFL restriction of the form $\Delta t < \Delta x^2$, which is unacceptable for high resolution, long time scale simulations. The appropriate semi-implicit treatment of these modes has been the topic of several recent notes[2,3].

The general form of the gyro-viscous force is $-\nabla \cdot \Pi_{gv}$, where

$$\Pi_{gv} = \frac{\eta_3}{2}\left[\mathbf{b} \times \mathbf{W} \cdot (\mathbf{I} + 3\mathbf{bb}) + transpose\right] \quad , \tag{1}$$

is the gyro-viscous stress tensor, $\mathbf{W}$ is the rate of strain tensor,

$$\mathbf{W} = \nabla \mathbf{V}_i + \nabla \mathbf{V}_i^T - \frac{2}{3}\mathrm{I}\nabla \cdot \mathbf{V}_i \quad , \tag{2}$$

with

$$\eta_3 = \frac{nT_i}{2\Omega} \quad , \tag{3}$$

and $\Omega = eB / m_i$ is the ion gyro-frequency. The gyro-viscous force is not dissipative because it is not caused particle collisions (η_3 is independent of the collision frequency ν); it represents a momentum transport inherent in the gyro-motion of the particles, which is reversible. (It can be shown explicitly[4] from Equation (1) that the work done by the gyro-viscous force vanishes, i.e., $\nabla \mathbf{v} : \Pi_{gv} = 0$.) From the form of Equation (2), the gyro-viscous force appears diffusive. Indeed, η_3 / mn as the dimensions of a diffusion coefficient (m^2/sec). However, the aforementioned dissipationless property of this term requires that the gyro-viscous momentum equation must remain hyperbolic, i.e., it must produce waves. Since equations of similar form (e.g., the Hall term in Ohm’s law) produce dispersive waves, we anticipate that this will also be the case with the gyro-viscosity.

In the following sections we investigate the dispersion properties of the gyro-viscous/Hall equations for the special case of an infinite uniform plasma imbedded in a straight, uniform magnetic field. We find that the presence of the gyro-viscous force introduces new normal modes that are indeed dispersive; like the whistler and kinetic Alfvén waves, they have $\omega^2 \sim k^4$. For the case of parallel propagation they appear simply as corrections to the whistler branch. (However, they persist in the absence of the Hall term.) For the case of perpendicular propagation they appear as corrections to the compressional (fast) magnetosonic mode. Neither the whistler or kinetic Alfvén waves propagate in the perpendicular direction. The perpendicular dispersive waves thus represent a new set of normal modes of the system that must be accounted for when time advance algorithms are considered.

1. The Gyro-viscous Fluid Equations

Neglecting electron inertia and assuming quasi-neutrality and an adiabatic equation of state, the equations for a gyro-viscous fluid are

$$\frac{\partial n}{\partial t} = -\nabla \cdot n\mathbf{V} \quad , \tag{4a}$$

$$mn\frac{d\mathbf{V}}{dt} = -\nabla\left(p_e + p_i\right) + \mathbf{J} \times \mathbf{B} - \nabla \cdot \Pi_{gv} \quad , \tag{4b}$$

$$\mathbf{J} = ne(\mathbf{V} - \mathbf{V}_e) = \frac{1}{\mu_0}\nabla \times \mathbf{B} \quad , \tag{4c}$$

$$\mathbf{E} = -\mathbf{V}_e \times \mathbf{B} - \frac{1}{ne}\nabla p_e = -\mathbf{V} \times \mathbf{B} + \frac{1}{ne}\left[\mathbf{J} \times \mathbf{B} - \nabla p_e\right] \quad , \tag{4d}$$

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E} \quad , \tag{4e}$$

$$\frac{\partial p_i}{\partial t} = -\gamma p_i \nabla \cdot \mathbf{V} \quad , \tag{4f}$$

and

$$\frac{\partial p_e}{\partial t} = -\gamma p_e \nabla \cdot \mathbf{V}_e = -\gamma p_e \nabla \cdot \left(\mathbf{V} - \frac{1}{ne}\mathbf{J}\right) \quad . \tag{4g}$$

Here $\mathbf{V}$ is the total bulk velocity of the momentum carrying component of the plasma, $\mathbf{V} = \mathbf{V}_i$; it contains all the relevant drift motions. In particular, when the Hall term ($\mathbf{J} \times \mathbf{B}$) is included in Ohm's law, Equation (4d), the $\mathbf{E} \times \mathbf{B}$, diamagnetic, and polarization drifts are all contained in $\mathbf{V}$.

We assume a uniform plasma with straight field lines and no zero-order gradients or current. We choose a Cartesian coordinate system (x, y, z) with the z-axis positively aligned with the mean magnetic field $\mathbf{B}_0$. In this coordinate system, and under the assumption that $\delta << 1$, the components of the gyro-viscous stress tensor, Equation (1), are written[5]

$$\Pi_{xx} = -\Pi_{yy} = -\eta_3 W_{xy} \quad , \tag{5a,b}$$

$$\Pi_{xy} = \Pi_{yx} = \frac{1}{2}\eta_3\left(W_{xx} - W_{yy}\right) \quad , \tag{5c,d}$$

$$\Pi_{xz} = \Pi_{zx} = -\eta_4 W_{yz} \quad , \tag{5e,f}$$

$$\Pi_{yz} = \Pi_{zy} = \eta_4 W_{xz} \quad , \tag{5g,h}$$

$$\Pi_{zz} = 0 \quad , \tag{5i}$$

where $\eta_4 = 2\eta_3$, and the components of the rate of strain tensor, Equation (2), are given by

$$W_{ij} = \frac{\partial V_j}{\partial x_i} + \frac{\partial V_i}{\partial x_j} - \frac{2}{3}\delta_{ij}\nabla \cdot \mathbf{V} \quad . \tag{6}$$

The (negative) components of the gyro-viscous force are then

$$\left(\nabla\cdot\Pi\right)_x = -\eta_3\nabla_\perp^2 V_y - \eta_4\frac{\partial}{\partial z}\left(\frac{\partial V_y}{\partial z}+\frac{\partial V_z}{\partial y}\right) \quad , \tag{7a}$$

$$\left(\nabla\cdot\Pi\right)_y = +\eta_3\nabla_\perp^2 V_x + \eta_4\frac{\partial}{\partial z}\left(\frac{\partial V_x}{\partial z}+\frac{\partial V_z}{\partial x}\right) \quad , \tag{7b}$$

and

$$\left(\nabla\cdot\Pi\right)_z = -\eta_4\frac{\partial}{\partial z}\left(\frac{\partial V_x}{\partial y}-\frac{\partial V_y}{\partial x}\right) \quad , \tag{7c}$$

where

$$\nabla_\perp^2 \equiv \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2} \quad . \tag{8}$$

We note that the perpendicular and parallel components of the (negative) gyro-viscous force can be written in vector notation as

$$\left(\nabla\cdot\Pi\right)_\perp = \left[\eta_3\nabla_\perp^2 + \eta_4(\mathbf{b}\cdot\nabla)^2\right](\mathbf{b}\times\mathbf{V}) + \eta_4(\mathbf{b}\cdot\nabla)\left[\mathbf{b}\cdot\nabla\times\left(V_\parallel\mathbf{I}\right)\right] \quad , \tag{9}$$

and

$$\mathbf{b}\cdot\left(\nabla\cdot\Pi\right) = -\eta_4(\mathbf{b}\cdot\nabla)(\mathbf{b}\cdot\nabla\times\mathbf{V}) \quad . \tag{10}$$

The expressions given by Equations (7-10) assume that η_3 and η_4 are constant.

The only assumptions made in obtaining Equations (4a-g) are: 1) quasi-neutrality; 2) ignoring electron inertia; and, 3) the form of the closure for the stress tensor (which assumes $\delta << 1$).

We now linearize Equations (4a-g) about the uniform background state and assume plane wave solutions of the form $\exp(i\omega t + i\mathbf{k}\cdot\mathbf{x})$. We note that the last term in Equation (4d), the $-\nabla p_e / n$ contribution to Ohm's law, is annihilated by the curl operator even in the presence of density fluctuations because of the assumed uniformity of the background electron pressure. Similar comments apply to the last term in Equation (4g), the $\nabla\cdot(\mathbf{J}/n)$ contribution to the electron pressure. We can therefore omit density evolution from the stability analysis; the perturbed density can be determined *a posteriori* directly from Equation (4a). Equations (4f-g) can then be combined into the single equation

$$\frac{\partial p}{\partial t} = -\gamma p\nabla\cdot\mathbf{V} \quad , \tag{11}$$

where $p = p_i + p_e$.

The linearized gyro-viscous equations then become

$$i\omega mnV_x = -ik_x p + \frac{iB_0}{\mu_0}\left(k_z B_x - k_x B_z\right) - \eta_3 k_\perp^2 V_y - \eta_4 k_z^2 V_y - \eta_4 k_y k_z V_z \quad , \tag{12a}$$

$$i\omega mnV_y = -ik_y p + \frac{iB_0}{\mu_0}\left(k_z B_y - k_y B_z\right) + \eta_3 k_\perp^2 V_x + \eta_4 k_z^2 V_x - \eta_4 k_x k_z V_z \quad , \tag{12b}$$

$$i\omega mnV_z = -ik_z p - \eta_4 k_y k_z V_x + \eta_4 k_x k_z V_y \quad , \tag{12c}$$

$$i\omega B_x = ik_z B_0 V_x + \frac{B_0}{\mu_0 ne} k_z \left(k_y B_z - k_z B_y\right) \quad , \tag{12d}$$

$$i\omega B_y = ik_z B_0 V_y + \frac{B_0}{\mu_0 ne} k_z \left(k_z B_x - k_x B_z\right) \quad , \tag{12e}$$

$$i\omega B_z = -ik_y B_0 V_y - ik_x B_0 V_x + \frac{B_0}{\mu_0 ne} k_z \left(k_x B_y - k_y B_x\right) \quad , \tag{12f}$$

and

$$i\omega p = -i\gamma p_0 k_z V_z \quad , \tag{12g}$$

where $k_\perp^2 = k_x^2 + k_y^2$.

Solutions of Equations (12a-g) for the special cases of parallel and perpendicular propagation are discussed in the next sections.

3. Parallel Propagation

For the case of parallel propagation we set $k_x = k_y = 0$ in Equations (12a-g). The result is

$$i\omega V_x = \frac{iB_0 k_z}{\mu_0 mn} B_x - \frac{\eta_4 k_z^2}{nm} V_y \quad , \tag{13a}$$

$$i\omega V_y = \frac{iB_0 k_z}{\mu_0 mn} B_y + \frac{\eta_4 k_z^2}{nm} V_x \quad , \tag{13b}$$

$$i\omega B_x = iB_0 k_z V_x - \frac{B_0 k_z^2}{\mu_0 ne} B_y \quad , \tag{13c}$$

$$i\omega B_y = iB_0 k_z V_y + \frac{B_0 k_z^2}{\mu_0 ne} B_x \quad , \tag{13d}$$

$$i\omega V_z = -\frac{ik_z}{mn} p \quad , \tag{13e}$$

and

$$i\omega p = -i\gamma p_0 k_z V_z \quad . \tag{13f}$$

Equations (13e-f) describe parallel sound waves, and provide no new modes. The remaining analysis is simplified by defining new variables $V_\pm = V_x \pm iV_y$ and $B_\pm = B_x \pm iB_y$. Then adding and subtracting Equations (13a-b) and (13c-d), respectively, we find

$$i\omega V_\pm = \frac{iB_0 k_z}{\mu_0 ne} B_\pm \pm \frac{i\eta_4 k_z^2}{nm} V_\pm \quad , \tag{14a}$$

$$i\omega B_{\pm} = iB_0 k_z V_{\pm} \pm \frac{iB_0 k_z^2}{\mu_0 ne} B_{\pm} \quad . \tag{14b}$$

Equations (14a-b) are two sets of coupled equations, one for the pair (V_+, B_+), and the other for the pair (V_-, B_-). They represent right (-) and left (+) polarized waves. Defining

$$\omega_A = \frac{B_0 |k_z|}{\sqrt{\mu_0 mn}} = V_A |k_z| \quad , \tag{15}$$

$$\omega_4 = \frac{\eta_4 k_z^2}{mn} = \frac{V_{th_i}^2}{2\Omega} k_z^2 = \frac{1}{2} (\rho_i k_z)^2 \Omega \quad , \tag{16}$$

and

$$\omega_W = \frac{B_0 k_z^2}{\mu_0 ne} = \left(\frac{\omega_A}{\Omega} \right)^2 \Omega = \frac{1}{\beta} (\rho_i k_z)^2 \Omega \quad , \tag{17}$$

the parallel dispersion relation is obtained as

$$(\omega \pm \omega_4)(\omega \pm \omega_W) = \omega_A^2 \quad , \tag{18}$$

where the + (-) sign refers to the solution corresponding to V_- and B_- (V_+ and B_+), and $\rho_i = V_{th_i} / \Omega$ is the ion Larmor radius. There are thus four independent solutions, two for the left polarized wave, and two for the right.

Assuming $\rho_i k_z << 1$, as is required for the gyro-viscous closure, the solutions for the left and right polarized waves can be written as

$$\omega_{L\pm} = V_A k_z \left[\pm 1 + \frac{1+\beta}{2\sqrt{\beta}} (\rho_i k_z) \right] \quad , \tag{19a,b}$$

and

$$\omega_{R\pm} = V_A k_z \left[\pm 1 - \frac{1+\beta}{2\sqrt{\beta}} (\rho_i k_z) \right] \quad . \tag{20a,b}$$

The whistler result is obtained by setting $1 + \beta \to 1$ in the numerator of the last terms. The gyro-viscosity thus gives an $O(\beta)$ correction to the parallel whistler branch.

4. Perpendicular Propagation

For the case of perpendicular propagation we set $k_y = k_z = 0$ in Equations (12a-g). We now obtain

$$i\omega mnV_x = -ik_x p - \frac{iB_0 k_x}{\mu_0} B_z - \eta_3 k_x^2 V_y \quad , \tag{21a}$$

$$i\omega mnV_y = \eta_3 k_x^2 V_x \quad , \tag{21b}$$

$$i\omega p = -i\gamma p_0 k_x V_x \quad , \tag{21c}$$

and

$$i\omega B_z = -iB_0 k_x V_x \quad . \tag{21d}$$

The dispersion relation is easily found to be

$$\frac{\omega^2}{\omega_s^2 + \omega_A^2} = 1 + \frac{\omega_3^2}{\omega_s^2 + \omega_A^2} \quad , \tag{22}$$

where $\omega_s^2 = C_s^2 k_x^2$, C_s is the sound speed, and $\omega_3 = \omega_4 / 2$. This is clearly a modification of the fast magnetosonic wave. The frequency can be expressed as

$$\omega^2 = V_A^2 k_x^2 \left[1 + \frac{\gamma\beta}{2} + \frac{\beta}{16} (\rho_i k_x)^2 \right] \quad , \tag{23}$$

so that the mode is now dispersive. From Equations (21b) and (23) we see that, to lowest order in $\rho_i k_x$,

$$\frac{iV_y}{V_x} = \pm \frac{1}{4} \frac{\sqrt{\beta}}{1 + \gamma\beta / 2} \rho_i k_x \quad . \tag{24}$$

The mode is therefore elliptically polarized in the plane perpendicular to **B**, with the major axis in the direction of **k** and the minor axis perpendicular to both **k** and **B**. The eccentricity of the ellipse is proportional to $\rho_i k_x$.

5. Discussion

The gyro-viscous stress introduces new normal modes to the magneto-plasma system. The form of Equation (19), (20), and (23) suggests that these modes are related to interactions between the electromagnetic field of the wave and the cyclotron gyration of the ions. It is likely that a more complete kinetic treatment would indicate resonances near the ion cyclotron frequency. The expressions reported here are the lowest order (in ρ_i / λ) manifestations of these kinetic modes. They appear at the same order as both the whistler and kinetic Alfvén waves, and so cannot be ignored in a two-fluid model.

For the case of parallel propagation the new modes appear as $O(\beta)$ corrections to the whistler branch. This does not lead to any new dispersion. However, for the case of perpendicular propagation new modes appear as dispersive modifications to the fast magnetosonic wave.

Time step limitations due to dispersive modes must be dealt with (or at least recognized) in any time advance algorithm. Semi-implicit formulations for the whistler and kinetic Alfvén branches are presently being implemented and tested[2,3]. These treatments may be sufficient to stabilize also the gyro-viscous correction to the parallel whistler wave. However, since the whistler and kinetic Alfvén modes do not propagate perpendicular to the magnetic field, it seems unlikely that the semi-implicit formulations presently under consideration will be sufficient to stabilize the gyro-viscous dispersive correction to the magnetosonic wave. A new semi-implicit operator must be sought.

One key to an accurate semi-implicit treatment is to construct the operator based on knowledge of the underlying equations that are to be advanced. This suggests that semi-implicit operator based on Equations (9) and (10) may be useful. Note that purely parallel propagation results from the second term in Equation (9) ($\eta_4 (\mathbf{b} \cdot \nabla)^2$), while purely perpendicular propagation arises from the first term ($\eta_3 \nabla_\perp^2$). Since the parallel waves may be stabilized by existing semi-implicit operators[2,3], perhaps a new semi-implicit method for stabilizing the dispersive magnetosonic wave could be based entirely on the first term, i.e., $\eta_3 \nabla_\perp^2 (\mathbf{b} \times \mathbf{V})$. (The last term in Equation (9) and Equation (10) affect oblique propagation, which has not been considered here. It is possible (likely?) that this will complicate the relatively simple approach that has been suggested.)

We remark again that, in this formulation, the bulk velocity $\mathbf{V}$ contains all the ion drifts. In drift MHD one transforms to a frame of reference moving with just the common $\mathbf{E} \times \mathbf{B}$ (MHD) drift of the ions and electrons. In that case the gyro-viscous cancellation eliminates almost all of the terms in the gyro-viscous force and results in an apparently greatly simplified form of the equations. It is not clear (at least to me) how the gyro-viscous modes derived here survive this transformation. However, if the gyro-viscous cancellation is valid, one would think that it is implicitly contained in Equations (4a-g), and that these new modes are fundamental to both models. Future work will investigate these modes in the context of the drift MHD model.

Finally we note that the gyro-viscous modifications to the usual MHD equations were used by Roberts and Taylor[6] in 1962 to describe the FLR stabilization of the gravitational interchange mode. They point out that consistent (and correct!) results require the simultaneous inclusion of both the Hall term in Ohm's law and the gyro-viscous force in the momentum equation.

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