EFFECT OF COMPRESSIBILITY ON THE TWO-FLUID GRAVITATIONAL INTERCHANGE INSTABILITY

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In a recent note, Barnes[1] investigated the effect of compressibility on the linear stability of the MHD gravitational interchange mode in slab geometry, finding a destabilizing effect. In particular, an unstable mode was shown to persist even in the limit of infinite density gradient scale length (zero density gradient). The modifications were shown to cause a significant increase in the MHD growth rate for the parameters being considered in NIMROD benchmark cases. The additional mode is analogous to a buoyancy instability: downward moving dense fluid is compressed, reinforcing the displacement.

We extend this work by considering the effect of compressibility on the gravitational instability within the two-fluid model. The original treatment of this problem by Roberts and Taylor[2] assumed the perturbations to be electrostatic. In the MHD regime this assumption implies incompressibility. The two-fluid regime requires finite compressibility to maintain charge neutrality, but this can be accommodated consistently within the electrostatic model[3]. Here we remove the electrostatic constraint and allow compressibility to affect changes in the pressure, density, and magnetic field. Not surprisingly, we find that the primary effect of compressibility is to increase the MHD growth rate, thus driving the stabilizing wave number to larger values (shorter wave lengths). There are also finite-β corrections to the drift frequency, but these are of lesser importance for strongly magnetized plasmas.

The two-fluid equations are

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \mathbf{V} = 0 \quad , \tag{1}$$

$$\rho \frac{d\mathbf{V}}{dt} = -\nabla \left(p + \frac{B^2}{2\mu_0} \right) + \rho \mathbf{g} \quad , \tag{2}$$

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E} \quad , \tag{3}$$

$$\mathbf{E} = -\mathbf{V} \times \mathbf{B} + \frac{M}{\rho e} \left(\frac{d\mathbf{V}}{dt} - \nabla p_i - \rho \mathbf{g} \right) \quad , \tag{4}$$

$$\frac{\partial p}{\partial t} = -\mathbf{V} \cdot \nabla p - \Gamma p \nabla \mathbf{V} \quad , \tag{5}$$

where $p = p_i + p_e$ is the total pressure, and the hoop stress term has been removed from Equation (2). This differs from the MHD model by the term in parentheses on the right hand side of Equation (4).

Following Roberts and Taylor[2], we assume slab geometry with a uniform field $\mathbf{B}_0 = B_0\mathbf{e}_z$, $\rho_0(x) = \rho_0 \exp(x/L)$, and $\mathbf{g} = -g\mathbf{e}_x$. The equilibrium condition is then

$$\frac{dp_0}{dx} = -\rho_0 g \quad . \tag{6}$$

We linearize Equations (1-5) about this state, assume a perturbation with variation $\exp(i\omega t + iky)$, and ignore all x-variation. With perturbed velocity $\mathbf{V} = u\mathbf{e}_x + v\mathbf{e}_y$, we have

$$i\omega\rho + \frac{\rho_0}{L}u + ik\rho_0 v = 0 \quad , \tag{7}$$

$$i\omega\rho_0 u = -\rho g \quad , \tag{8}$$

$$i\omega\rho_0 v = -ik\left(p + \frac{B_0}{\mu_0}B\right) \quad , \tag{9}$$

$$i\omega B = -ikB_0 v - \frac{M}{e}\omega k u \quad , \tag{11}$$

$$i\omega p = -ik\Gamma p_0 v + \rho_0 g u \quad . \tag{12}$$

The last term in Equation (11) is the two-fluid correction to MHD. Equations (11) and (12) can be combined to yield

$$i\omega\left(p + \frac{B_0}{\mu_0}B\right) = \left(\rho_0 g - \frac{MB_0}{\mu_0 e}\omega k\right)u - \rho_0 C^2 ikv \quad , \tag{13}$$

where

$$C^2 = \frac{\Gamma p_0}{\rho_0} + \frac{B_0^2}{\mu_0\rho_0} = C_s^2 + V_A^2 = V_A^2\left(1 + \frac{\Gamma\beta}{2}\right) \tag{14}$$

is the magneto-acoustic speed. Then Equation (9) becomes

$$i\omega v = -\frac{k}{\omega}\left(\rho_0 g - \frac{MB_0}{\mu_0 e}\omega k\right)u + \frac{ik^2}{\omega}\rho_0 C^2 v \quad . \tag{15}$$

Equations (7), (8), and (15) determine the linear properties of the system. Setting their determinant to zero gives the dispersion relation

$$\rho_0 C^2 \underbrace{\left(\omega^2 + \frac{g}{L}\right)}_{\substack{\textit{MHD instability} \\ \textit{Incompressible}}} + \underbrace{\frac{MB_0}{\mu_0 e}\omega k}_{2-\textit{fluid effects}} - \underbrace{\rho_0 g}_{\substack{\textit{Compressible correction} \\ \textit{to MHD instability}}} = 0 \quad . \tag{16}$$

At low frequency ($\omega^2 / C^2k^2 << 1$), this can be written as

$$\omega^2 - \omega_*\omega + \gamma_g^2 = 0 \quad , \tag{17}$$

where

$$\omega_* = \frac{gk/\Omega}{1+\frac{\Gamma\beta}{2}} \tag{18}$$

is the drift frequency, and

$$\gamma_g^2 = \frac{g}{L} + \frac{g^2}{C^2} \tag{19}$$

is the MHD growth rate as given by Barnes[1]. The last term is the compressible correction to the growth rate. Note that instability persists as $L \to \infty$. There is also a finite-β correction to the drift frequency.

Equation (16) has stable solutions provided $\omega_* > 2\gamma_g$. This defines the stabilizing wave number within the two-fluid model. Finite-β effects decrease ω_* and compressible effects increase γ_g, so both tend to push the stabilizing wave number to larger values (smaller wave lengths).

These predictions are being tested with the NIMROD code. Results of these studies will be provided in future revisions of this note.

References

1. D. C. Barnes, "Another Calculation of the MHD Growth Rate for the g-mode", unpublished note, Feb. 8, 2006.

2. K. V. Roberts and J. B. Taylor, Phys. Rev. Letters **8**, 197 (1962).

3. D. D. Schnack, "Gravitational Instability as a Test Case for Extended MHD Computations", unpublished note, Sept. 12, 2005.