The Effect of Heat Stress on the Propagation of Sound Waves in a Uniform Magnetized Plasma within the Extended MHD Model

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The tress tensor expresses a linear relationship between the forces acting on a fluid and the applied rate of strain. It is completely analogous to Hooke's law for an elastic solid. In a magnetized plasma the stress tensor can be into three orthogonal components as

$$\Pi = \mathbf{bb}\cdot\Pi + \mathbf{b}\times\mathbf{I}\cdot\Pi + \left(\mathbf{I}-\mathbf{bb}\right)\cdot\Pi \quad ,$$

$$= \Pi_{||} + \Pi_{\wedge} + \Pi_{\perp} \quad . \tag{1}$$

A kinetic theory for a collisional plasma shows that the parallel (||) and perpendicular ($\perp$) components depend on the collision frequency, and are dissipative. However, the traceless "cross" (^) component (called the gyro-viscous stress) is independent of collisions and is not dissipative. A generalized Hooke's law for this component of the stress can be written as

$$\Pi_{\wedge} = \frac{\eta_3}{2}\left[\mathbf{b}\times\mathbf{W}\cdot\left(\mathbf{I}+3\mathbf{bb}\right)+transpose\right] \quad , \tag{2}$$

where

$$\mathbf{W} = \nabla\mathbf{V}_i + \nabla\mathbf{V}_i^T - \frac{2}{3}\mathbf{I}\nabla\cdot\mathbf{V}_i \quad , \tag{3}$$

is the rate of strain tensor, and

$$\eta_3 = \frac{nT_i}{2\Omega} \quad , \tag{4}$$

where $\Omega = eB/m_i$ is the ion gyro-frequency. Inclusion of this form of the gyro-viscosity in the fluid plasma (extended MHD) equations is known to introduce dispersive modifications to the MHD wave spectrum. These require special attention in a numerical solution of these equations.

Subsequent theoretical investigations have suggested that Equation (2) does not contain the correct plasma response to lowest order in the ratio of the ion gyro-radius, $\rho_i = V_{th}/\Omega$, to the macroscopic scale length, and that the gyro-viscous stress should contain an additional term of the form

$$\Pi_{\wedge q} = \frac{5}{2\Omega}\left[\mathbf{b}\times\mathbf{W}_q\cdot\left(\mathbf{I}+3\mathbf{bb}\right)+transpose\right] \quad , \tag{5}$$

where

$$\mathbf{W}_q = \nabla \mathbf{q}_i + \nabla \mathbf{q}_i^T - \frac{2}{3}\mathbf{I}\nabla \cdot \mathbf{q}_i \quad , \tag{6}$$

and $\mathbf{q}_i$ is the *ion* heat flow. We call Equation (5) the *heat stress*. Any closure of the fluid equations that relates $\mathbf{q}_i$ to the (ion) temperature T introduces the possibility of coupling between the heat flow stress in the momentum equation and the compressible ($\sim \nabla \cdot \mathbf{V}$) term in the energy equation to produce modifications of the sound wave dispersion relation. If these modifications are dispersive ($\omega \sim k^2$) they may introduce further complications for numerical algorithms.

In this note we examine the modifications to the sound wave dispersion relation by the heat stress for a homogeneous plasma with a uniform magnetic field. Since we anticipate that the important coupling is between the momentum and (adiabatic) energy equations, only the heat stress contribution to the gyro-viscous stress is considered. We also assume electrostatic perturbations, so there is no need to invoke Ohm's law and Ampere's law, and we use the Braginskii collisional closure for the heat flow. This model contains the essential result. In particular, we find that the perpendicular sound wave indeed has a dispersive modification of order $(\rho_i k_\perp)^2$, but the parallel sound wave is unaffected. Further, the perpendicular propagation is only affected by the "cross" (or gyro) component of the heat flow. The dissipative parallel and perpendicular components have no effect on the waves when the fluid is adiabatic. Further, the wave propagation is unaffected by the electrons (unless one also includes the electron thermal stress in the model), since it is the ion heat flow that enters Equation (6). Thus, more complicated (and accurate) parallel kinetic closures for the electrons can be implemented without concern for their affect on the overall dispersion relation. Since the ion "cross" thermal conductivity is smaller than the electron value by a factor of the mass ratio, the issue can be avoided entirely be excluding the ion "cross" conduction from the model. Further, since the indicated modifications of the momentum equation seem to make no contribution to the dynamics (e.g., sound waves), perhaps the heat stress can ignored completely.

The detailed calculation is straightforward. The minimal set of relevant equations is

$$\rho_0 \frac{\partial \mathbf{V}}{\partial t} = -\nabla p - \nabla \cdot \Pi_{\wedge q} \quad , \tag{7}$$

$$\frac{\partial p}{\partial t} = -\gamma p_0 \nabla \cdot \mathbf{V} \quad , \tag{8}$$

where $\Pi_{\wedge q}$ is given by Equations (5) and (6), and the heat flow is taken to be

$$\mathbf{q} = -\kappa_{\|}\nabla_{\|}T - \kappa_\perp \nabla_\perp T - \kappa_\wedge \mathbf{b} \times \nabla_\perp T \quad . \tag{9}$$

For a uniform plasma with a constant magnetic field in the z-direction, Equations (7-9) become, after assuming $\exp(i\omega t + ik_\perp y + ik_{\|}z)$ dependence,

$$i\omega\rho_0 V_x = \frac{ik^3}{5\Omega n_0} F_1(\theta)\, p \quad , \tag{10}$$

$$i\omega\rho_0 V_y = \left(-ik\sin\theta - \frac{ik^3}{5\Omega n_0}F_2(\theta)\right)p \quad , \tag{11}$$

$$i\omega\rho_0 V_z = \left(-ik\cos\theta + \frac{ik^3}{5\Omega n_0}F_3(\theta)\right)p \quad , \tag{12}$$

$$i\omega p = -i\gamma p_0 k\left(V_y\sin\theta + V_z\cos\theta\right) \quad , \tag{13}$$

where

$$F_1(\theta) = \left(2\sin^2\theta + \cos^2\theta\right)\left(\kappa_\perp\sin\theta - \kappa_\wedge\cos\theta\right) - \kappa_{||}\cos^2\theta\sin\theta \quad , \tag{14}$$

$$F_2(\theta) = \kappa_\wedge\sin\theta\left(2\sin^2\theta + \cos^2\theta\right) \quad , \tag{15}$$

$$F_3(\theta) = \kappa_\wedge\sin^2\theta\cos\theta \quad , \tag{16}$$

and we have defined $k_{||} = k\cos\theta$ and $k_\perp = k\sin\theta$.

Setting the determinant of Equations (10-13) to zero yields the dispersion relation

$$\omega^2 = C_s^2 k^2\left[1 + \frac{\kappa_\wedge}{5n_0\Omega}f(\theta)k^2\right] \quad , \tag{17}$$

where $C_s^2 = \gamma p_0/\rho_0$ is the sound speed. The angular dependence is given by

$$f(\theta) = \frac{1}{2}\sin\theta\left(2\sin^2\theta + \cos^2\theta - \sin\theta\cos\theta\right) \quad , \tag{18}$$

which has the property that $f(0) = 0$ and $f(\pi/2) = 1$.

Equation (17) exhibits a dispersive modification to the sound wave for oblique to normal propagation $(0 < \theta \le \pi/2)$. However, the dispersion of the parallel sound wave $(\theta = 0)$ is unaffected. Note that the dispersive modification depends only on the "cross" component of the ion thermal conductivity. Using the expression $\kappa_\wedge = 5n_0T/2m_i\Omega$, Equation (17) can be written as

$$\omega^2 = C_s^2 k^2\left[1 + f(\theta)\left(\rho_i k\right)^2\right] \quad , \tag{19}$$

which exhibits the FLR nature of the correction explicitly.

We remark that the expressions for the heat flow and coefficients of thermal conductivity appearing above refer to the *ions*. Only the *ion* "cross" thermal conductivity need be considered. Since this is smaller than the electron thermal conductivity be a factor of the mass ratio, perhaps this effect can be neglected. More sophisticated non-local closure expressions for the parallel electron heat flow will not affect the propagation of sound waves.