Some properties of the M3D-C^1 form of the 3D magnetohydrodynamics equations

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Abstract. We introduce a set of scalar variables and projection operators for the vector momentum and magnetic field evolution equations that have several unique and desirable properties, making them a preferred system for solving the magnetohydrodynamics equations in a torus with a strong toroidal magnetic field. We derive a "weak form" of these equations that explicitly conserves energy and is suitable for a Galerkin finite element formulation provided the basis elements have C^1 continuity. Systems of reduced equations are discussed, along with their energy conservation properties. An implicit time advance is presented that adds diagonally dominant self-adjoint energy terms to the mass matrix to obtain numerical stability.

1. Introduction

Here we are concerned with developing a set of scalar equations that are appropriate for use in finite element computations of three-dimensional global-scale magneto-hydrodynamics (MHD) in a strongly magnetized torus, such as a tokamak. It is well known that there are severe challenges to be overcome in applying the MHD equations to a tokamak. There are multiple time and space scales present, the divergence of the magnetic field must be constrained to vanish, and the treatment must be such as to accurately describe a flow field that avoids compressing the strong externally imposed magnetic field to a large degree [1]. In linear MHD, this latter property has been called the avoidance of spectral pollution [2].

The M3D [3] approach to overcome these difficulties was to introduce a potential and stream function representation for the velocity and the magnetic vector potential and to develop a partially implicit algorithm for integrating the equations in time. In M3D-C^1 we build on that approach by introducing slightly modified forms of the velocity and magnetic field variables that have several desirable properties. They are compatible with appropriate projection operators that lead to energy-conserving weak forms of the equations when the Galerkin method is applied. When the numerical algorithm of differential approximations [4-8] is applied, they also lead to self-adjoint forms for partial-energy terms that are added to the mass matrix to give a stable well-conditioned fully implicit time advance. This technique has been demonstrated in two-dimensional slab [6, 7] and toroidal geometry [8]. The latter reference goes into the two-fluid extensions in some detail. Here we present the full 3D toroidal equations and discuss their properties but restrict our attention in the main text to single fluid resistive MHD for simplicity. The technique presented is readily generalized to include additional terms in the equations, such as gyroviscosity and additional transport terms.

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Another important feature of the formulation presented here is that it leads naturally to several systems of "reduced MHD" equations that are obtained simply by zeroing out one or two of the three scalar variables in the velocity field, and not including the corresponding projection operator for that component with the momentum equation. Similarly, the magnetic field advance can be simplified by zeroing the contribution to the toroidal field by the plasma currents and removing the corresponding projection operator for the magnetic field advance equation. This procedure provides 3D, toroidal, energy-conserving sets of reduced MHD equations that are generalizations of equation sets previously proposed.

We state the resistive MHD equations in Sec. 2 and introduce the forms of the velocity and magnetic vector potential vectors in Sec. 3. Taking the weak form of the projection operators introduced in Sec. 4 and 5, and of the scalar density and pressure equations in Sec. 6 is shown in Sec. 7 to lead to an energy conserving set of evolution equations. Properties of the velocity representation are discussed in Sec. 8, and how this leads to energy conserving subsets of 2-field, Sec. 9, and 4-field, Sec. 10, equations that are energy-conserving toroidal generalizations of equation sets previously proposed. Sec. 11 introduces the implicit time advance and shows the close connection with it and the ideal MHD perturbed energy δW . In sections 12 and 13 we explicitly demonstrate that the partial-energy terms added to the mass matrix for the implicit time advance are self-adjoint. A subset of these same terms is used in the reduced equation sets. The first Appendix covers the two-fluid extensions in the generalized Ohm's law. Appendix B describes the variables and projection operators used in the original M3D code for comparison. Appendix C introduces a self-consistent set of orderings that are used to demonstrate desirable properties of the formulation presented in the main text.

2. The Resistive MHD equations;

Consider the standard resistive MHD equations for the mass density *n*, the fluid velocity **V,** the magnetic field **B**, and the fluid pressure *p* (in Rationalized MKS units):

$$\frac{\partial n}{\partial t} + \nabla\bullet(n\mathbf{V}) = 0 \tag{1a}$$

$$n\left(\frac{\partial \mathbf{V}}{\partial t} + \mathbf{V}\bullet\nabla\mathbf{V}\right) + \nabla p + \nabla\bullet\mathbf{\Pi} = \mathbf{J}\times\mathbf{B} \tag{1b}$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla\times\left(\mathbf{V}\times\mathbf{B} - \eta\mathbf{J}\right) \tag{1c}$$

$$\frac{\partial p}{\partial t} + \mathbf{V}\bullet\nabla p + \gamma p\nabla\bullet\mathbf{V} = \left(\gamma-1\right)\left[\eta\mathbf{J}^2 - \mathbf{\Pi}:\nabla\mathbf{V}\right] \tag{1d}$$

Here, the current density is defined by $\mathbf{J}\equiv\nabla\times\mathbf{B}$ and the condition $\nabla\bullet\mathbf{B}=0$ is implied by Eq. (1c). The viscous stress term is taken to have the form:

$$\begin{aligned}\mathbf{\Pi} &= -\mu\left[\nabla\mathbf{V} + \nabla\mathbf{V}^{\dagger}\right] - 2(\mu_C - \mu)\left(\nabla\bullet\mathbf{V}\right)\mathrm{I}\\ \nabla\bullet\mathbf{\Pi} &= -\mu\nabla^2\mathbf{V} - (2\mu_C - \mu)\nabla\left(\nabla\bullet\mathbf{V}\right)\end{aligned} \tag{1e}$$

We have introduced the adiabatic index, γ (= 5/3 for an ideal gas), the electrical resistivity η, and the isotropic and compressible viscosities μ and μ_C

3. The Form of the Vectors V and B.

We use a physics-motivated decomposition of the vector fields in toroidal geometry. Using a cylindrical coordinate system (R,φ,Z) with $\left|\nabla\varphi\right| = 1/R$, we define the 2D gradient and 2D divergence operators as $\nabla_\perp a \equiv a_R\hat{R} + a_Z\hat{Z}$, $\nabla_\perp \bullet \mathbf{A} \equiv R^{-1}(R\mathbf{A}\bullet\hat{R})_R + (\mathbf{A}\bullet\hat{Z})_Z$. Subscripts denote partial differentiation with respect to *R* and *Z*. We also define the *2D* inner products and Poisson brackets. For any two scalar variables *a* and *b*, we define the inner product:

$$\left(a,b\right) \equiv \nabla_\perp a \bullet \nabla_\perp b = a_R b_R + a_Z b_Z \quad , \tag{2a}$$

and the Poisson bracket

$$\left[a,b\right] \equiv \left[\nabla a \times \nabla b \bullet \nabla\varphi\right] = R^{-1}\left(a_Z b_R - a_R b_Z\right). \tag{2b}$$

The velocity field is represented in terms of the three scalar variables (U,ω,χ) as follows:

$$\mathbf{V} = R^2\nabla U \times \nabla\varphi + \omega R^2 \nabla\varphi + R^{-2}\nabla_\perp \chi \ . \tag{3}$$

This form is chosen so that the in-plane stream function U does not compress the toroidal field (as described in Sec. 8), ω is the toroidal angular frequency, and the in-plane compressibility term χ is orthogonal to the others as described below.

The magnetic vector potential is given in terms of the two scalar variables (f,ψ) and the constant F_0 (proportional to the current in the toroidal field magnets) as:

$$\mathbf{A} = R^2\nabla\varphi \times \nabla f + \psi\nabla\varphi - F_0 \ln R\hat{Z} \ . \tag{4}$$

This form leads to orthogonality between f and ψ in the magnetic energy (below) and to a particularly convenient form for the current density **J**. Note that the gauge condition implied by this form is: $\nabla_\perp \bullet R^{-2}\mathbf{A} = 0$. The magnetic field and current density are calculated in terms of the vector potential variables as:

$$\mathbf{B} = \nabla\times\mathbf{A} = \nabla\psi\times\nabla\varphi - \nabla_\perp f' + F\nabla\varphi \ , \tag{5a}$$

$$= \nabla\psi\times\nabla\varphi - \nabla f' + F^*\nabla\varphi \ , \tag{5b}$$

and

$$\mathbf{J} \equiv \nabla\times\mathbf{B} = \nabla F^* \times \nabla\varphi + R^{-2}\nabla_\perp\psi' - \Delta^*\psi\nabla\varphi \ . \tag{6}$$

The third term in this expression for **J** is the toroidal current density. The second is the poloidal current needed to make the toroidal current divergence-free when it has φ variation. The first term is the remaining poloidal plasma current that contributes to the toroidal field *F*. Here we have defined auxiliary variables: $F \equiv F_0 + R^2\nabla\bullet\nabla_\perp f$, $F^* \equiv F_0 + R^2\nabla^2 f = F + f''$, and the operator: $\Delta^*\psi \equiv R^2\nabla_\perp \bullet R^{-2}\nabla\psi$. Primes denote differentiation with respect to the angle φ. Note that **B** and **J** are manifestly divergence free in these forms.

The forms for the vector fields $\mathbf{V}$ and $\mathbf{B}$ have the property that their corresponding energies do not have any cross-terms when integrated over a volume, as these cross-terms become surface terms that vanish if the normal components of these vectors vanish at the boundaries. Thus, if we define the 2D volume integral (at constant φ) as $d^2R \equiv RdRdZ$, we have:

$$\iint d^2R\, \mathbf{V}^2 = \iint d^2R\left[R^2\left(U,U\right) + R^{-4}\left(\chi,\chi\right) + R^2\omega^2\right] , \tag{7a}$$

$$\iint d^2R\, \mathbf{B}^2 = \iint d^2R\left[R^{-2}\left(\psi,\psi\right) + \left(f',f'\right) + R^{-2}F^2\right] . \tag{7b}$$

The different terms in the velocity and magnetic field vectors, Esq. (3) and (5) are therefore orthogonal in this sense. However, we note that if a non-constant density is present, the kinetic energy will have the additional factor of the density and so the cross term $n\left[U,\chi\right]$ will not vanish, albeit these terms are still approximately orthogonal.

4. The Scalar Momentum Equations

We introduce a set of trial functions with *C^1* continuity, which we denote ν_i. These can either be 2D functions $\nu_i(R,Z)$ or 3D functions $\nu_i(R,\varphi,Z)$. To get scalar forms for the momentum equation, we take the weak form of the following three projection operators applied to Eq. (1b), integrate over the 2D plane *(R,Z)* and perform integration by parts as indicated in Eq. (8):

$$\iint d^2R\, \nu_i \nabla\varphi \bullet \nabla_\perp \times R^2 (1b) \quad \rightarrow \quad \iint d^2R\, R^2 \nabla_\perp \nu_i \times \nabla\varphi \bullet (1b) \tag{8a}$$

$$\iint d^2R\, \nu_i R^2 \nabla\varphi \bullet (1b) \quad \rightarrow \quad \iint d^2R\, \nu_i R^2 \nabla\varphi \bullet (1b) \tag{8b}$$

$$-\iint d^2R\, \nu_i \nabla_\perp \bullet R^{-2} (1b) \quad \rightarrow \quad \iint d^2R\, R^{-2} \nabla_\perp \nu_i \bullet (1b) \tag{8c}$$

The boundary terms from the integration by parts are assumed to vanish here, but will be the subject of a future publication. By comparing the integrands on the right in Eq. (8) with the form of the velocity in Eq. (3), we see that after the integration by parts, these projection operators are equivalent to taking the inner product of the momentum equation (1b) separately with each of the three terms in the velocity field, but with the trial function ν_i replacing each of the three scalars (U,ω,χ). We show in Sec. 7 that this property leads to an energy-conserving set of discrete equations, to two energy-conserving subsets of reduced equations, and in Sec. 11 we show that this leads to self-adjoint energy terms being introduced in an implicit time advance. After substituting for the magnetic field and velocity from Eqns. (3) and (5), Eqns. (8a-c) give the following integrands:

$$
\begin{aligned}
&nR^2\left(\nu_i,\dot{U}\right)-n\left[\nu_i,\dot{\chi}\right] \\
&=-n\left\{\begin{aligned}&\left(\nabla_\perp^2 U-\frac{2}{R^4}\chi_Z\right)\left(R^4\left[\nu_i,U\right]+\left(\nu_i,\chi\ \right)\right)-\omega\left[\nu_i,\chi'\right]+R^2\left(\nu_i,U'\right)\\ &+R^2\omega^2\nu_{iZ}-\frac{1}{2}R^2\left[\nu_i,R^2\left(U,U\right)+\frac{1}{R^4}\left(\chi,\chi\right)\right]+R^2\left[\nu_i,\left[U,\chi\right]\right]\end{aligned}\right\} \\
&\Delta^*\psi\left[\left[\nu_i,\psi\right]-\left(\nu_i,f'\right)\right]+\frac{F}{R^2}\left(\nu_i,\psi'\right)+F\left[\nu_i,f''\right]-R^2\left[p,\nu_i\right] \\
&-\mu\left[R^2\left(\nabla_\perp^2\nu_i\right)\left(\nabla_\perp^2 U\right)-\left(\nu_i,U''\right)\right]\quad ,
\end{aligned}
\tag{9a}
$$

$$
\begin{aligned}
nR^2\nu_i\dot{\omega}=-n\nu_i&\left\{R^2\left[R^2\omega,U\right]+\frac{1}{R^2}\left(R^2\omega,\chi\right)+R^2\omega\omega'\right\} \\
&-\frac{\nu_i}{R^2}\left(\psi,\psi'\right)+\nu_i\left[F^*,\psi\right]+\nu_i\left[f',\psi'\right]-\nu_i\left(f',F^*\right)-\nu_i p' \\
&+\left[\nu_i\mu\Delta^*(R^2\omega)+2\mu_C\nu_i\omega''\right]\quad ,
\end{aligned}
\tag{9b}
$$

$$
\begin{aligned}
&\frac{n}{R^4}\left(\nu_i,\dot{\chi}\right)+n\left[\nu_i,\dot{U}\right] \\
&=n\left\{\begin{aligned}&\left(\nabla_\perp^2 U-\frac{2}{R^4}\chi_Z\right)\left(\left(\nu_i,U\right)-\frac{1}{R^2}\left[\nu_i,\chi\right]\right)-\frac{\omega}{R^4}\left(\nu_i,\chi'\right)-\omega\left[\nu_i,U'\right]\\ &+\frac{1}{R}\omega^2\nu_{iR}-\frac{1}{2R^2}\left(\nu_i,R^2\left(U,U\right)+\frac{1}{R^4}\left(\chi,\chi\right)\right)-\frac{1}{R^2}\left(\nu_i,\left[\chi,U\right]\right)\end{aligned}\right\} \\
&-\Delta^*\psi\left[\frac{1}{R^4}\left(\psi,\nu_i\right)+\frac{1}{R^2}\left[\nu_i,f'\right]\right]-\frac{F}{R^4}\left[\left(\nu_i,F^*\right)-\left[\nu_i,\psi'\right]\right]-\frac{1}{R^2}\left(\nu_i,p\right) \\
&-2\mu_C\frac{1}{R^2}\Delta^\dagger\nu_i\Delta^\dagger\chi\quad .
\end{aligned}
\tag{9c}
$$

Here and elsewhere we are denoting time derivatives as $\dot{U}\equiv\partial U/\partial t$, etc. We again note that the terms $-n\left[\nu_i,\dot{\chi}\right]$ and $n\left[\nu_i,\dot{U}\right]$ on the left sides of Eqns. (9a) and (9c) would vanish for constant density, n = const.

5. The Magnetic Field Evolution Equations

In a manner analogous to the momentum equation, we obtain the scalar weak form of the magnetic field evolution equations by applying the following two projection operators to Eq. (1c), integrating over the 2D plane, and performing integration by parts:

$$
\iint d^2R\ \nu_i\nabla\varphi\bullet\nabla_\perp\times(1c)\quad\rightarrow\quad\iint d^2R\ \nabla_\perp\nu_i\times\nabla\varphi\bullet(1c)\quad , \tag{10a}
$$

$$\iint d^2R\, \nu_i \nabla\varphi \bullet (1c) \quad \rightarrow \quad \iint d^2R\, \nu_i \nabla\varphi \bullet (1c) \quad . \tag{10b}$$

As in the discussion following Eq. (8), if we compare the integrands on the right in Eq. (10) with the form of the magnetic field in Eq. (5b), we see that these projection operators are equivalent to taking the inner product of the magnetic field evolution equation (1c) with the first and third terms in the magnetic field, but with the trial function ν_i replacing the scalar quantities ψ and *F**. In this case, there is no need to take the third projection, which would be

$$-\iint d^2R\, \nu_i \nabla \bullet (1c) \quad \rightarrow \quad \iint d^2R\, \nabla \nu_i \bullet (1c) \ , \tag{10c}$$

since the divergence constraint on the magnetic field assures that this is satisfied. After substituting for the magnetic field and velocity from Eqns. (3) and (5), Eqns. (10a,b) give the following integrands:

$$\begin{aligned}
&\frac{1}{R^2}(\nu_i,\dot{\psi}) = -[U,\psi]\Delta^*\nu_i - \frac{1}{R^2}\left(\nu_i, R^2\left(f',U\right)\right) + \left(\frac{F'}{R^2}\right)(\nu_i,U) \\
&+\left(\frac{F}{R^2}\right)(\nu_i,U') - \omega'\left(\frac{1}{R^2}(\nu_i,\psi) + [\nu_i, f']\right) - \omega\left(\frac{1}{R^2}(\nu_i,\psi') + [\nu_i, f'']\right) \\
&-\frac{1}{R^2}\left(\nu_i, R^{-2}(\psi,\chi)\right) + \frac{1}{R^2}\left(\nu_i,[f',\chi]\right) - \frac{1}{R^4}F'[\nu_i,\chi] - \frac{1}{R^4}F[\nu_i,\chi'] \\
&-\frac{\eta}{R^2}\left[\Delta^*\nu_i\Delta^*\psi - \frac{1}{R^2}(\nu_i,\psi'') - [\nu_i, f'''] - [\nu_i, F']\right] \quad ,
\end{aligned} \tag{11a}$$

and

$$\begin{aligned}
-\left(\nu_i, \dot{f}\right) = \frac{\nu_i}{R^2}\dot{F} &= F[\nu_i,U] + \frac{1}{R^4}F(\nu_i,\chi) - \omega[\nu_i,\psi] + \omega(\nu_i, f') \\
&-\frac{\eta}{R^2}(\nu_i,F) - \frac{\eta}{R^2}(\nu_i, f'') + \frac{\eta}{R^2}[\nu_i,\psi'] \quad .
\end{aligned} \tag{11b}$$

Eq. (11b) is seen to be an evolution equation both for *f* and for *F*.

6. Density and pressure evolution:

The density and pressure equations follow directly from Eqns. (1a) and (1d) after substituting for the magnetic field and velocity from Eqns. (3) and (5),

$$\dot{n} + \left[nR^2,U\right] + (n\omega)' + \nabla_\perp \bullet \left[\frac{n}{R^2}\nabla_\perp\chi\right] = 0 \, , \tag{12}$$

$$\dot{p}+\left[pR^2,U\right]+\left(p\omega\right)'+\nabla_{\perp}\bullet\left[\frac{p}{R^2}\nabla_{\perp}\chi\right]=-\left(\gamma-1\right)p\left(-2U_z+\omega'+\frac{1}{R^2}\Delta^*\chi\right)$$

$$+\left(\gamma-1\right)\left\{\begin{array}{l}\mu\left[R^2\left(\nabla_{\perp}^2U\right)\left(\nabla_{\perp}^2U\right)+\left(U',U'\right)\right]+\frac{\mu}{R^2}\left(R^2\omega,R^2\omega\right)+2\mu_C\omega'\omega'+2\mu_C\frac{1}{R^2}\Delta^{\dagger}\chi\Delta^{\dagger}\chi \\ +\frac{\eta}{R^2}\Delta^*\psi\Delta^*\psi+\frac{\eta}{R^4}\left(\psi',\psi'\right)+\frac{\eta}{R^2}\left[\psi',F^*\right]+\frac{\eta}{R^2}\left(F^*,F^*\right)\end{array}\right\} \cdot \tag{13}$$

7. Energy Conservation:

The well known demonstration of energy conservation by Eqns. (1) depends on the equivalent of the following vector identities being satisfied by the scalar equations:

$$n\mathbf{V}\bullet\left(\mathbf{V}\bullet\nabla\mathbf{V}\right)=n\mathbf{V}\bullet\nabla\left(\frac{1}{2}V^2\right) \tag{14a}$$

$$\mathbf{V}\bullet\nabla\bullet\mathbf{\Pi}=-\mathbf{\Pi}:\nabla\mathbf{V}+\text{ ST} \tag{14b}$$

$$\mathbf{B}\bullet\left[\left(\nabla\times\mathbf{V}\right)\times\mathbf{B}\right]=-\mathbf{V}\bullet\left(\nabla\times\mathbf{B}\right)\times\mathbf{B}+\text{ ST} \tag{14c}$$

$$\mathbf{B}\bullet\left[\nabla\times\left(\eta\nabla\times\mathbf{B}\right)\right]=-\eta\left(\nabla\times\mathbf{B}\right)^2+\text{ ST} \tag{14d}$$

(Here we denote by ST those surface terms that vanish at the boundary if the normal velocities and Poynting fluxes vanish there). It is seen that these vector identities are satisfied exactly in the present formulation because of the equivalence of the weak forms of the projection operators that we use to taking the inner product of Eq. (1b) with each of the vector components of **V** and of Eq. (1c) with each of the vector components of **B**. Since the weak form of the equations (9) and (10) holds true for any function ν_i in the admissible function space, to explicitly demonstrate energy conservation, we make the particular choices in Table 1.

Table 1: Trial functions used to show energy conservation

Equation	ν_i
9a	U
9b	ω
9c	χ
10a	ψ
10b	F^*

With the functions of Table 1 inserted as indicated, we sum equations (9a-c) and (10a-c) together with 1/2$\mathbf{V}^2$ times Eq. (12) and $(\gamma\text{-}1)^{-1}$ times Eq. (13), and integrate over the volume to obtain the explicit demonstration of energy conservation. After many cancellations and integrations by parts, this yields

$$\frac{d}{dt}\iiint dV\left\{\begin{matrix}\frac{1}{2}n\left[R^2\left(U,U\right)+R^2\omega^2+\frac{1}{R^4}\left(\chi,\chi\right)+2\left[\chi,U\right]\right] \\ +\frac{1}{2}\left[\frac{1}{R^2}\left(\psi,\psi\right)+\frac{F^2}{R^2}+\left(f',f'\right)\right]+\frac{1}{(\gamma-1)}p\end{matrix}\right\} = 0 + \text{ST}.$$

As discussed earlier, the final term in the kinetic energy term, $n\left[\chi,U\right]$, would integrate to zero if the density n were constant.

8. Properties of the Velocity Field:

The toroidal magnetic field in a standard tokamak is much greater in strength than the poloidal field. The dominant contribution to the toroidal field, denoted by the constant F_0 in Eq. (4) and those that follow, is produced by the large toroidal field coils. It follows that the leading order plasma motion will be such as to convect the strong externally imposed toroidal field but not to compress it [9]. The desire to represent this dominant plasma flow field with a single scalar variable led to the particular form of the velocity field vector introduced in Eq. (3).

One sees from the toroidal field evolution equation, Eq. (11b) that the only place that U enters in this integrand is in the first term on the right. This term can be written as

$$\frac{\nu_i}{R^2}\dot{F} = F\left[\nu_i,U\right]+\cdots = -\nu_i\left[F,U\right]+\cdots = -\nu_i\nabla_\perp F\bullet\nabla_\perp U\times\nabla\varphi+\cdots \quad . \tag{15}$$

We see that since $\nabla F_0 = 0$, U only convects the part of F that is produced by the plasma currents, but does not compress the toroidal field. We therefore expect that to a very good approximation, the plasma motion in the *(R,Z)* plane will be represented by the first term in the velocity equation.

To understand better how this comes about, suppose the first term in the velocity field were given by

$$\mathbf{V} = R^m\nabla U\times\nabla\varphi + . \quad . \quad .$$

(In Eq. (3) we have *m=2*). The $\nabla\varphi$ component of the magnetic field evolution equation, Eq. (1c) becomes

$$\begin{aligned}\frac{1}{R^2}\dot{F} &= -\nabla\bullet\left[\mathbf{V}\frac{F}{R^2}\right]+\cdot \quad \cdot \quad \cdot \\ &= -\nabla\bullet\left[R^{m-2}\nabla U\times\nabla\varphi F\right]+\cdot \quad \cdot \\ &= -R^{m-2}\nabla U\times\nabla\varphi\bullet\nabla F - F\nabla U\times\nabla\varphi\bullet\nabla R^{m-2}\end{aligned} \tag{15a}$$

Since the second term contains F_0 (if $m\neq 2$) but the first term does not (since $\nabla F_0 = 0$), it will dominate in this equation and hence restrict the velocity field associated with the U variable as there is a large energy penalty associated with compressing the toroidal field. This is the physical basis for choosing the form of the velocity field in Eq. (3), and for the reduced MHD model presented below which follows from it.

9. Generalized two-field equations

Since the entire system of equations conserves energy, we can also obtain energy-conserving subsets by setting one or two of the velocity variables (ω,χ) to zero, and either keeping or setting to zero the vector potential variable *f*. Thus, by eliminating Eqns. (9b), (9c), and (11b) and setting $\omega = \chi = f = 0$ in the remaining equations, we obtain the following two integrands and two equations:

$$\begin{aligned} nR^2\left(\nu_i,\dot{U}\right) = -n\left\{\left(\nabla_\perp^2 U\right)R^4\left[\nu_i,U\right] + R^2\left(\nu_i,U'\right) - \frac{1}{2}R^2\left[\nu_i,R^2\left(U,U\right)\right]\right\} \\ + \Delta^*\psi\left[\nu_i,\psi\right] + \frac{F_0}{R^2}\left(\nu_i,\psi'\right) - R^2\left[p,\nu_i\right] - \mu\left[R^2\left(\nabla_\perp^2\nu_i\right)\left(\nabla_\perp^2 U\right) - \left(\nu_i,U''\right)\right] \quad , \end{aligned} \tag{16a}$$

$$\frac{1}{R^2}\left(\nu_i,\dot{\psi}\right) = -\left[U,\psi\right]\Delta^*\nu_i + \left(\frac{F_0}{R^2}\right)\left(\nu_i,U'\right) - \frac{\eta}{R^2}\left[\Delta^*\nu_i\Delta^*\psi - \frac{1}{R^2}\left(\nu_i,\psi''\right)\right] \quad , \tag{16b}$$

$$\dot{n} + \left[nR^2,U\right] = 0 \tag{16c}$$

$$\dot{p} + \left[pR^2,U\right] = \left(\gamma-1\right)\left\{\begin{array}{l} 2U_z p + \mu\left[R^2\left(\nabla_\perp^2 U\right)\left(\nabla_\perp^2 U\right) + \left(U',U'\right)\right] \\ + \dfrac{\eta}{R^2}\Delta^*\psi\Delta^*\psi + \dfrac{\eta}{R^4}\left(\psi',\psi'\right) \end{array}\right\} \tag{16d}$$

It is shown in Appendix C that the force terms neglected in the evolution equation for *U* in going from Eq. (9a) to (16a), which all contain *f*, are all higher order in a systematic ordering and this is why these simplified equations are of interest. This set of equations is seen to be an energy-conserving toroidal generalization of the “Strauss Equations”[10]. A further reduction is possible by eliminating the last two equations and setting the density *n* to a constant n_0 and the pressure *p* to zero. This set of equations, called the two-field reduced equations, consists of just (16a) and (16b) with $n=n_0$ and $p=0$. This reduced equation set no longer has the exact energy conservation property but is nevertheless useful for many applications.

10. Generalized 4-field Equations:

Another set of energy-conserving reduced equations is obtained by keeping the full magnetic field advance equations for ψ and *f* but just eliminating the third velocity equation Eq. (9c) and setting $\chi = 0$ in the remaining equations. This leads to a set of toroidal equations where the component of the velocity field in the (*R,Z*) plane does not compress the toroidal field. These equations are toroidal energy-conserving generalizations of those presented in [11,12] which can be extended to two-fluid MHD as discussed in the Appendix.

11. The Implicit Time Advance Equations:

It has been shown by several authors [4-8] that a stable implicit numerical time stepping algorithm, now known as the method of differential approximation, can be obtained by replacing Eq. (1b) by the following equation, and then by applying centered time differences and either centered space differences or finite elements:

$$\left\{n-\theta^2(\delta t)^2 L\right\}\frac{\partial \mathbf{V}}{\partial t}+n\mathbf{V}\bullet\nabla\mathbf{V}+\nabla p+\nabla\bullet\mathbf{\Pi}=\mathbf{J}\times\mathbf{B} \quad . \tag{17}$$

This is followed by an implicit time advance for the magnetic field, pressure, and density using the advanced time velocity. Here we have introduced the implicit parameter θ, where $1/2 \le \theta \le 1$ for numerical stability, δt is the time-step, and L is the linear ideal MHD operator [13]:

$$L\{\mathbf{V}\}=\left\{\nabla\times\left[\nabla\times\left(\mathbf{V}\times\mathbf{B}\right)\right]\right\}\times\mathbf{B}+\left(\nabla\times\mathbf{B}\right)\times\left[\nabla\times\left(\mathbf{V}\times\mathbf{B}\right)\right]+\nabla\left(\mathbf{V}\bullet\nabla p+\gamma p\nabla\bullet\mathbf{V}\right) \tag{18}$$

When the operator projections in Eq. (8) are applied to Eq. (17), new terms, called partial energy terms, appear on the left sides of Eqns. (9abc) but the right sides remain unchanged. The left sides of these integrands then become:

$$nR^2\left(\nu_i,\dot{U}\right)-n\left[\nu_i,\dot{\chi}\right]-\theta^2(\delta t)^2\left\{\begin{array}{l} \delta W_{11}^{(0)}(\nu_i,\dot{U})+\delta W_{11}^{(1)}(\nu_i,\dot{U}')+\delta W_{11}^{(2)}(\nu_i,\dot{U}'') \\ +\delta W_{12}^{(1)}(\nu_i,\dot{\omega}')+\delta W_{12}^{(2)}(\nu_i,\dot{\omega}'') \\ +\delta W_{13}^{(0)}(\nu_i,\dot{\chi})+\delta W_{13}^{(1)}(\nu_i,\dot{\chi}')+\delta W_{13}^{(2)}(\nu_i,\dot{\chi}'') \end{array}\right\}=\cdots \tag{19a}$$

$$nR^2\nu_i\dot{\omega}-\theta^2(\delta t)^2\left\{\begin{array}{l} \delta W_{21}^{(1)}(\nu_i,\dot{U}')+\delta W_{21}^{(2)}(\nu_i,\dot{U}'') \\ +\delta W_{22}^{(0)}(\nu_i,\dot{\omega})+\delta W_{22}^{(2)}(\nu_i,\dot{\omega}'') \\ +\delta W_{23}^{(0)}(\nu_i,\dot{\chi})+\delta W_{23}^{(1)}(\nu_i,\dot{\chi}')+\delta W_{23}^{(2)}(\nu_i,\dot{\chi}'') \end{array}\right\}=\cdots \tag{19b}$$

$$\frac{n}{R^4}\left(\nu_i,\dot{\chi}\right)+n\left[\nu_i,\dot{U}\right]-\theta^2(\delta t)^2\left\{\begin{array}{l} \delta W_{31}^{(0)}(\nu_i,\dot{U})+\delta W_{31}^{(1)}(\nu_i,\dot{U}')+\delta W_{31}^{(2)}(\nu_i,\dot{U}'') \\ +\delta W_{32}^{(0)}(\nu_i,\dot{\omega})+\delta W_{32}^{(1)}(\nu_i,\dot{\omega}')+\delta W_{32}^{(2)}(\nu_i,\dot{\omega}'') \\ +\delta W_{33}^{(0)}(\nu_i,\dot{\chi})+\delta W_{33}^{(1)}(\nu_i,\dot{\chi}')+\delta W_{33}^{(2)}(\nu_i,\dot{\chi}'') \end{array}\right\}=\cdots \tag{19c}$$

Properties of these partial energy terms as applied to an axisymmetric equilibrium with zero equilibrium flow are discussed in the next section.

12. Direct Evaluation of Energy Terms.

The partial energy terms, $\delta W_{kj}(a_k,b_j)$ are obtained by taking the 2D integral of the inner product of the k^{th} velocity component with the operator L operating on the j^{th} velocity component. For example: $\delta W_{13}(U,\chi)\equiv R^2\nabla U\times\nabla\varphi\bullet L\left\{R^{-2}\nabla_\perp\chi\right\}$. In implementing the Galerkin method, each finite element basis function is used in place of the first variable and the second variable is expanded in basis functions. Integration by parts is done so that no more than two derivatives appear on any scalar, consistent with restrictions on C^1 elements. We use the equilibrium equation, $\Delta^*\psi=-R^2dp/d\psi-FdF/d\psi$, to simplify the energy terms and put them in a manifestly self-adjoint form. Note that "prime" means $\partial/\partial\varphi$ (*e.g.:* $U' \equiv \partial U/\partial\varphi$). We also make use of the identities (obtained by integration by parts) that for any scalars *a*, *b*, and *c*:

$$c(a,b) = -\frac{a}{R^2}\left(cR^2, b\right) - ac\Delta^* b$$
$$a[b,c] = -c[b,a]$$

The terms that appear in the integrands of Eq. (19) are as follows:

$$\delta W_{11}^{(0)}(\nu_i,U) = -\frac{1}{R^2}\left(R^2[U,\psi], R^2[\nu_i,\psi]\right) + R^2\frac{d}{d\psi}\left(F\frac{dF}{d\psi}\right)[U,\psi][\nu_i,\psi] - 4\gamma p U_z \nu_{iz}$$

$$\delta W_{11}^{(1)}(\nu_i,U') = -\frac{F}{R^2}\left(U', R^2[\nu_i,\psi]\right) + \frac{F}{R^2}\left(\nu_i, R^2[U',\psi]\right) + \Delta^*\psi F[\nu_i,U']$$

$$\delta W_{11}^{(2)}(\nu_i,U'') = \frac{F^2}{R^2}\left(U'',\nu_i\right)$$

$$\delta W_{12}^{(1)}(\nu_i,\omega') = \frac{\omega'}{R^2}\left(\psi, R^2[\nu_i,\psi]\right) + \omega' F[\nu_i,F] + \omega' R^2[\nu_i,p] + 2\gamma p\omega'\nu_{iz}$$

$$\delta W_{12}^{(2)}(\nu_i,\omega'') = -\omega''\frac{F}{R^2}\left(\psi,\nu_i\right)$$

$$\delta W_{13}^{(0)}(\nu_i,\chi) = \frac{1}{R^2}\left(R^{-2}(\chi,\psi), R^2[\nu_i,\psi]\right) - \frac{d}{d\psi}\left(F\frac{dF}{d\psi}\right)[\nu_i,\psi](\chi,\psi) - \frac{d^2p}{d\psi^2}[\nu_i,\psi](\chi,\psi) + \frac{2\gamma}{R^2}p\nu_{iZ}\Delta^*\chi$$

$$\delta W_{13}^{(1)}(\nu_i,\chi') = +\frac{F}{R^4}\left[R^2[\nu_i,\psi],\chi'\right] - \frac{F}{R^2}\left(\nu_i, R^{-2}(\chi',\psi)\right) + \Delta^*\psi\frac{F}{R^4}\left(\chi',\nu_i\right)$$

$$\delta W_{13}^{(2)}(\nu_i,\chi'') = \frac{F^2}{R^4}[\chi'',\nu_i]$$

$$\delta W_{21}^{(1)}(\nu_i,U') = -\frac{\nu_i}{R^2}\left(\psi, R^2[U',\psi]\right) - \nu_i F[U',F] - \nu_i R^2[U',p] - 2\gamma p\nu_i U_Z'$$

$$\delta W_{21}^{(2)}(\nu_i,U'') = -\nu_i\frac{F}{R^2}\left(\psi,U''\right)$$

$$\delta W_{22}^{(0)}(\nu_i,\omega) = -R^2[\nu_i,\psi][\omega,\psi]$$

$$\delta W_{22}^{(2)}(\nu_i,\omega'') = \nu_i\frac{\omega''}{R^2}\left(\psi,\psi\right) + \gamma\nu_i p\omega''$$

$$\delta W_{23}^{(0)}(\nu_i,\chi) = F[\nu_i,\psi]\left[R^{-2}\Delta^*\chi + \left(R^{-2},\chi\right)\right]$$

$$\delta W_{23}^{(1)}(\nu_i,\chi') = +\frac{\nu_i}{R^2}\left(\psi, R^{-2}(\chi',\psi)\right) + F\frac{\nu_i}{R^4}\left(F,\chi'\right) + \frac{\nu_i}{R^2}\left(p,\chi'\right) + \gamma p\frac{\nu_i}{R^2}\Delta^*\chi'$$

$$\delta W_{23}^{(2)}(\nu_i,\chi'') = -\frac{\nu_i}{R^4}F[\chi'',\psi]$$

$$\delta W_{31}^{(0)}(\nu_i,U) = +\frac{1}{R^2}\left(R^{-2}(\nu_i,\psi), R^2[U,\psi]\right) - \frac{1}{R^2}\frac{d}{d\psi}\left(F\frac{dF}{d\psi}\right)[U,\psi](\nu_i,\psi) - \frac{d^2p}{d\psi^2}[U,\psi](\nu_i,\psi) + \frac{2\gamma}{R^2}pU_Z\Delta^*\nu_i$$

$$\delta W_{31}^{(0)}(\nu_i,U') = -\frac{F}{R^4}\left[R^2[U',\psi],\nu_i\right] + \frac{F}{R^2}\left(U', R^{-2}(\nu_i,\psi)\right) - \Delta^*\psi\frac{F}{R^4}\left(\nu_i,U'\right)$$

$$\delta W_{31}^{(2)}(\nu_i,U'') = \frac{F^2}{R^4}[\nu_i,U'']$$

$$\delta W_{32}^{(0)}(\nu_i,\omega) = F\left[\omega,\psi\right]\left[R^{-2}\Delta^*\nu_i + \left(R^{-2},\nu_i\right)\right]$$
$$\delta W_{32}^{(1)}(\nu_i,\omega') = -\frac{\omega'}{R^2}\left(\psi,R^{-2}\left(\nu_i,\psi\right)\right) - F\frac{\omega'}{R^4}\left(F,\nu_i\right) - \frac{\omega'}{R^2}\left(p,\nu_i\right) - \gamma p\frac{\omega'}{R^2}\Delta^*\nu_i$$
$$\delta W_{32}^{(2)}(\nu_i,\omega'') = -\frac{\omega''}{R^4}F\left[\nu_i,\psi\right]$$
$$\delta W_{33}^{(0)}(\nu_i,\chi) = -\frac{1}{R^2}\left(R^{-2}\left(\chi,\psi\right),R^{-2}\left(\nu_i,\psi\right)\right) - F^2R^2\left[\nabla\bullet\frac{1}{R^4}\nabla\nu_i\right]\left[\nabla\bullet\frac{1}{R^4}\nabla\chi\right] + \frac{1}{R^6}\frac{d}{d\psi}\left(F\frac{dF}{d\psi}\right)\left(\chi,\psi\right)\left(\nu_i,\psi\right)$$
$$\qquad -\frac{\gamma p}{R^4}\Delta^*\chi\Delta^*\nu_i + \frac{1}{R^4}\frac{d^2p}{d\psi^2}\left(\chi,\psi\right)\left(\nu_i,\psi\right)$$
$$\delta W_{33}^{(1)}(\nu_i,\chi') = -\frac{F}{R^4}\left[\nu_i,R^{-2}\left(\chi',\psi\right)\right] + \frac{F}{R^4}\left[\chi',R^{-2}\left(\nu_i,\psi\right)\right] - \frac{1}{R^6}\Delta^*\psi F\left[\chi',\nu_i\right]$$
$$\delta W_{33}^{(2)}(\nu_i,\chi'') = +\frac{F^2}{R^8}\left(\chi'',\nu_i\right)$$

13. Symmetry Properties of Energy Terms:

The forms of the energies in Section 12 have the following symmetries that follow from the self-adjointness property.

$$\delta W_{11}^{(0)}(\nu_i,U) = \delta W_{11}^{(0)}(U,\nu_i), \quad \delta W_{11}^{(1)}(\nu_i,U') = -\delta W_{11}^{(1)}(U',\nu_i), \quad \delta W_{11}^{(2)}(\nu_i,U'') = \delta W_{11}^{(2)}(U'',\nu_i)$$
$$\delta W_{22}^{(0)}(\nu_i,\omega) = \delta W_{22}^{(0)}(\omega,\nu_i), \quad \delta W_{22}^{(2)}(\nu_i,\omega'') = \delta W_{22}^{(2)}(\omega'',\nu_i)$$
$$\delta W_{33}^{(0)}(\nu_i,\chi) = \delta W_{33}^{(0)}(\chi,\nu_i), \quad \delta W_{33}^{(1)}(\nu_i,\chi') = -\delta W_{33}^{(1)}(\chi',\nu_i), \quad \delta W_{33}^{(2)}(\nu_i,\chi'') = \delta W_{33}^{(2)}(\chi'',\nu_i)$$
$$\delta W_{12}^{(1)}(\nu_i,\omega') = -\delta W_{21}^{(1)}(\omega',\nu_i), \quad \delta W_{12}^{(2)}(\nu_i,\omega'') = \delta W_{21}^{(2)}(\omega'',\nu_i)$$
$$\delta W_{13}^{(0)}(\nu_i,\chi) = \delta W_{31}^{(0)}(\chi,\nu_i), \quad \delta W_{13}^{(1)}(\nu_i,\chi') = -\delta W_{31}^{(1)}(\chi',\nu_i), \quad \delta W_{13}^{(2)}(\nu_i,\chi'') = \delta W_{31}^{(2)}(\chi'',\nu_i)$$
$$\delta W_{23}^{(0)}(\nu_i,\chi) = \delta W_{32}^{(0)}(\chi,\nu_i), \quad \delta W_{23}^{(1)}(\nu_i,\chi') = -\delta W_{32}^{(1)}(\chi',\nu_i), \quad \delta W_{23}^{(2)}(\nu_i,\chi'') = \delta W_{32}^{(2)}(\chi'',\nu_i)$$

If we define $V_1 = U, V_2 = \omega, V_3 = \chi$ and denote phi derivatives as: $V_1^{(1)} = U', V_1^{(2)} = U''$, etc, we can write these compactly as:

$$\delta W_{jk}^{(l)}(\nu_i,V_k^{(l)}) = (-1)^l\,\delta W_{kj}^{(l)}(V_j^{(l)},\nu_i) \tag{20}$$

14. Summary

Equations (9abc) for the velocity and (11ab) for the magnetic field variables are the weak forms of the 2D integrands for the momentum and magnetic field advance for a resistive MHD plasma. They are similar to the equations used by the M3D code, given in Appendix B, but are different in some important ways. They are preferable for use in a 3D MHD finite element code if the elements posses C^1 continuity with respect to *(R,Z)*. These forms satisfy the equivalent of the vector identities that are required to obtain energy conservation. These are supplemented by the density and pressure evolution equations in Eq. (12) and (13). These equations manifestly preserve the divergence-free condition on the magnetic field and will accurately describe plasma motions that minimize the compression of the strong toroidal field. Because the formalism of this paper only involved 2D integrals over *(R,Z)* at constant toroidal angle φ, it is generally applicable to any kind of discrimination in the toroidal direction: spectral, finite difference, or finite element.

To construct a well behaved numerically stable implicit time advance, called the differential approximation, we add the partial energy terms given in Eq. (19abc) to the inertial terms on the left of Eqns. (9abc) as indicated. These terms are shown to be self-adjoint and thus their addition to the mass matrix should lead to a well conditioned matrix equation for the new time velocities. This clean result, and that of energy conservation for the discrete forms, stem from the equivalence of the projection operators that we use to taking the inner product of the momentum equation with each term in the velocity vector, and the magnetic field evolution equation with each term in the magnetic field vector.

It is shown in Sections 9 and 10 how to use this formalism to construct well-behaved reduced systems of equations that also conserve energy and whose matrix equation for the new time advance is also well conditioned.

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Appendix A: 2-Fluid Extensions

The primary extension of the resistive MHD model to the two-fluid model is to change Eqns. (1c) and (1d) to [6]:

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times \left(\mathbf{V} \times \mathbf{B} - \eta \mathbf{J} - \frac{1}{ne}\left[\mathbf{J} \times \mathbf{B} - \nabla p_e\right]\right) \tag{A-1}$$

$$\frac{\partial p}{\partial t} + \mathbf{V}\bullet\nabla p + \gamma p \nabla\bullet\mathbf{V} = (\gamma - 1)\left[\eta \mathbf{J}^2 - \mathbf{\Pi} : \nabla \mathbf{V}\right] + \frac{n^\gamma}{ne}\mathbf{J}\bullet\nabla\left(\frac{p_e}{n^\gamma}\right) \tag{A-2}$$

It is also necessary to add an additional evolution equation for the electron pressure, p_e, although it does not directly enter into the energy balance. As in the discussion in Sec. 7, the energy associated with the new terms will vanish identically up to a surface term because we are effectively taking the inner product of Eq. (A-1) through the weak form of the projection operators that we use. Thus, the vector identities needed,

$$\begin{aligned}
\mathbf{B}\bullet\nabla \times \left(-\frac{1}{ne}\left[\mathbf{J} \times \mathbf{B} - \nabla p_e\right]\right) &= \mathbf{J}\bullet\left(-\frac{1}{ne}\left[\mathbf{J} \times \mathbf{B} - \nabla p_e\right]\right) + ST, \\
&= \frac{1}{ne}\mathbf{J}\bullet\nabla p_e + ST, \\
&= \frac{p_e}{n^2 e}\mathbf{J}\bullet\nabla n + ST',
\end{aligned} \tag{A-3}$$

are satisfied exactly by the weak form. This last term then combines with the new term in Eq. (A-2) [when multiplied by $(\gamma\text{-}1)^{-1}$] to give only a surface term.

The new terms that replacing Eq. (1c) by Eq. (A-1) are as follows. To the right side of Eq. (11a) we add:

$$= \quad \cdot \quad -\frac{1}{neR^2}\left[\begin{array}{l}
\left[\frac{1}{R^2}(\psi,\psi') + [\psi', f'] + \left[\psi, F^*\right] + \left(f', F^*\right)\right]\Delta^*\nu_i \\
+\Delta^*\psi'\left[\frac{1}{R^2}(\nu_i,\psi) + [\nu_i, f']\right] + \Delta^*\psi\left[\frac{1}{R^2}(\nu_i,\psi') + [\nu_i, f'']\right] \\
+\frac{F'}{R^2}\left[\left(\nu_i, F^*\right) - [\nu_i,\psi']\right] + \frac{F}{R^2}\left[\left(\nu_i, F'^*\right) - [\nu_i,\psi'']\right] - \frac{1}{n}p_e'(\nu_i, n) \\
-\frac{n'}{n}\left[\Delta^*\psi\left[\frac{1}{R^2}(\nu_i,\psi) + [\nu_i, f']\right] + \frac{F}{R^2}\left[\left(\nu_i, F^*\right) - [\nu_i,\psi']\right] - (\nu_i, p_e)\right]
\end{array}\right]. \tag{A-4}$$

To the right side of Eq. (11b) we add:

$$= \quad \cdot + \frac{1}{ne}\left[\frac{\Delta^*\psi}{R^2}\left[(\nu_i, f') - [\nu_i,\psi]\right] - \left(\frac{F}{R^2}\right)\left\{\left[\nu_i, F^*\right] + \frac{1}{R^2}(\nu_i,\psi')\right\} - [\nu_i, p_e]\right]. \tag{A-5}$$

Appendix B: The M3D Potentials and Projection Operators

In the original C^0 M3D code [3], the velocity is expressed in terms of a different set of scalars U, χ, and V_φ as

$$\mathbf{V} = R^2\nabla U \times \nabla\varphi + RV_\varphi\nabla\varphi + \nabla_\perp\chi \tag{B.1}$$

The magnetic vector potential, and magnetic field are given by:

$$\mathbf{A} = \psi\nabla\varphi + R\nabla f \times \nabla\varphi - F_0 \ln R\hat{z} \tag{B.2a}$$

$$\begin{aligned}\mathbf{B} &= \nabla\psi \times \nabla\varphi + R^{-1}\nabla_\perp f' + F\nabla\varphi \\ &= \nabla\psi \times \nabla\varphi + R^{-1}\nabla f' + (F - \frac{1}{R}f'')\nabla\varphi\end{aligned}, \tag{B.2b}$$

where the auxiliary variable F is here defined by $F = F_0 - R^2\nabla\bullet\frac{1}{R}\nabla_\perp f$.

Projection operators for momentum are

$$-R\nabla\varphi\cdot\nabla\times \tag{B.3}$$

to extract the U component;

$$\hat{R}\cdot \text{ and } \hat{\mathrm{Z}}\cdot \tag{B.4}$$

for $\nabla_\perp\chi$, which leads to two equations that still contain terms with time derivatives of U, and requires an elliptic solve for χ itself; and

$$R\nabla\varphi\cdot \tag{B.5}$$

for the toroidal component. The evolution of poloidal flux ψ is found by applying the projection operator

$$R^2\nabla\varphi\cdot \tag{B.6}$$

to the time derivative of the magnetic vector potential, Eq. (B.2a). This derivative is determined by Ohm's and Faraday's laws only to within the gradient of an auxiliary scalar Φ which will appear in the ψ equation. The choice of gauge $\nabla_\perp\cdot\frac{1}{R}\mathbf{A} = 0$ implies that this is the electrostatic potential (to within a sign), leading to the auxiliary equation

$$\nabla\bullet\frac{1}{R}\nabla_\perp\Phi = \nabla_\perp\frac{1}{R}\cdot\mathbf{E} \tag{B.7}$$

where **E** is the electric field given by Ohm's law. Finally, the toroidal field evolution may be projected out of (B.2b) with the same projection operator (B.6).

Appendix C: Orderings

Let R be a typical major radius, and a be a typical minor radius. Define the ordering small parameter $\varepsilon = a/R$. Gradients in the *(R,Z)* plane are assumed ordered as: $\nabla_\perp \sim 1/a$, however in the toroidal direction we have: $\nabla\varphi \sim 1/R$. It then follows that we can order the quantities that appear in the magnetic field and pressure as:

$$
\begin{aligned}
f &\sim R\varepsilon^4 \tilde{f} \\
\psi &\sim R^2\varepsilon^2\tilde{\psi} \\
p &\sim \varepsilon^2 \tilde{p} \\
F &\sim R \\
\nabla F &\sim \varepsilon \tilde{f} \\
\nabla R &\sim 1
\end{aligned}
$$

Here $\tilde{f}$, $\tilde{\psi}$ and $\tilde{p}$ are dimensionless quantities of order unity. This gives for the relative magnitude of the 3 terms in the magnetic field:

$$
\begin{array}{ccccc}
\nabla\psi\times\nabla\varphi & : & \nabla_\perp f & : & F\nabla\varphi \\
\varepsilon & : & \varepsilon^3 & : & 1+\varepsilon^2
\end{array},
$$

and for the 3 term in the current density:

$$
\begin{array}{ccccc}
\nabla F^*\times\nabla\varphi & : & R^{-2}\nabla_\perp\psi' & : & \Delta^*\psi\nabla\varphi \\
\varepsilon^2 & : & \varepsilon^2 & : & \varepsilon
\end{array}
$$

This corresponds to the standard low-beta tokamak ordering. Applying this ordering to the energy terms in Sections 11-12 gives the following relations between the diagonal terms and their corresponding kinetic energies:

$$
\frac{\delta W_{11}(U,U)}{R^2\left(U,U\right)} \sim \varepsilon^2 , \qquad \frac{\delta W_{22}(\omega,\omega)}{R^2\omega^2} \sim \varepsilon^2 , \qquad \frac{\delta W_{33}(\chi,\chi)}{\frac{1}{R^4}\left(\chi,\chi\right)} \sim 1 .
$$

This implies that a plasma motion that is seeking to minimize the perturbed energy must not have a large χ component in the velocity field. Also, we see that the off-diagonal terms satisfy the following ordering,

$$
\frac{\delta W_{12}(U,\omega)\delta W_{21}(\omega,U)}{\delta W_{11}(U,U)\delta W_{22}(\omega,\omega)} \sim \varepsilon^2 , \qquad \frac{\delta W_{13}(U,\chi)\delta W_{31}(\chi,U)}{\delta W_{11}(U,U)\delta W_{33}(\chi,\chi)} \sim \varepsilon^2 ,
$$

so that the partial energy matrix is diagonally dominant in this sense.