Ordered Fluid Equations II

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Abstract

The ordered fluid equations of Ref. [1] are re-derived assuming that the velocity is normalized to the ion velocity and the current density to the gradient of the magnetic field. It is found that the principle conclusions of Ref. [1] remain valid in this normalization. High, intermedeiate, and low frequency regimes are again identified. Further, equation consistency in each of these regimes constrains the maximum order (in $\delta = \rho_i / L$) of β that is allowed: $\beta \sim O(\delta^2)$ in the high frequency (Hall) regime, $\beta \sim O(\delta)$ in the intermediate frequency (MHD) regime, and $\beta \sim O(1)$ in the low frequency (drift) regime.

1. Introduction

In a recent note[1], consistent sets of non-dimensional fluid equations were derived systematically by writing the two-fluid equations in non-dimensional form. The resulting equations contain a few non-dimensional parameters that may be considered small. These parameters are the non-dimensional frequency, $\varepsilon = \omega / \Omega_i$, the non-dimensional velocity, $\xi = V_0 / V_{thi}$, the non-dimensional ion gyro-radius, $\delta = \rho_i / L$, and the plasma "beta", $\beta = (V_{thi} / V_A)^2$. (Note that this differs by a factor of 2 from the usual definition of β.) Consistent with the assumptions underlying the derivation of the fluid equations from the kinetic equation, the ion gyro-radius is always considered small, i.e., $\delta << 1$. Fluid models valid in several interesting plasma regimes were then obtained by considering different relative orderings of the remaining parameters with respect to δ, and retaining only low order terms. These models are summarized in Table I.

Table I

Properties of Fluid Models

Model	V_i	ω	β	$\mathbf{J} \times \mathbf{B}$	Whistlers†	KAW††
Hall MHD	V_{thi} / δ	Ω_{ci}	$O(\delta^2)$	$mn \dfrac{d\mathbf{V}_i}{dt} + O(\delta)$	Yes	No
Ideal MHD	V_{thi}	$\delta\Omega_{ci}$	$O(\delta)$	$O(\delta)$	No	No
Drift	δV_{thi}	$\delta^2\Omega_{ci}$	$O(1)$	$\nabla p + O(\delta^2)$	No	Yes

†Whistler waves are high frequency phenomena that disappear as the frequency is ordered successively lower.

††Kinetic Alfvén waves are finite pressure phenomena that appear as β becomes successively larger.

Recently some questions have been raised[2] concerning the choice of normalizations used in this approach, and the effect of these choices on conclusions drawn in Ref. [1]. In particular, in Ref. [1] the velocity was normalized to $V_0 = J_0 / n_0 e$. This effectively measures the velocity in units of the electron velocity. It results in a simple non-dimensional constitutive relationship $\mathbf{J} = n(\mathbf{V}_e - \mathbf{V}_i)$, but non-dimensional parameters then appear explicitly in Ampére's law. A more conventional choice might be to measure the velocity in units of the ion (momentum carrying) velocity, and to normalize the current density to $J_0 = B_0 / \mu_0 L$. Then the non-dimensional Ampére's law takes the simple form $\mathbf{J} = \nabla \times \mathbf{B}$, but the relationship between the current and the velocity explicitly contains non-dimensional parameters. Further, the choice of normalizing velocity used in Ref. [1] results in the relationship $\beta = \delta / \xi$, so that β is not longer an independent parameter. The implied constraints on β for the validity various fluid models is shown in the fourth column of Table I. The generality of these constraints may now be questioned, since they may arise from the particular choice of normalization velocity.

In this short note we reconsider the ordering of the fluid equations when the velocity is measured in units of the ion velocity and the current density in units of $J_0 = B_0 / \mu_0 L$. In this case we again find that the non-dimensional parameters are not independent, but are now related by $\varepsilon = \xi\delta$. Further, we find that consistency of the non-dimensional equations in the different parameter regimes leads to the same constraints on β as found in Ref. [1], and displayed in Table I. Thus the conclusions of Ref. [1] are essentially correct.

2. Ordered Equations

Following Ref. [2], we choose a normalization in which length is measured in units of L, time in units of ω^{-1}, velocity in units of $V_0 = \omega / L$, magnetic field in units of B_0, current density in units of $J_0 = B_0 / \mu_0 L$, electric field in units of $E_0 = V_0 B_0$, density in units of n_0, and pressure in units of $p_{i0} = n_0 m_i V_{th_i}^2 = 2 n_0 T_i$. (Note that the factor of 2 differs from the usual definition.) Then, neglecting the electron mass, the non-dimensional 2-fluid equations become:

Continuity:

$$\frac{\partial n}{\partial t} = -\nabla \cdot n\mathbf{V}_i \quad , \tag{1}$$

Center of Mass Equation of Motion:

$$n\varepsilon^2 \frac{d\mathbf{V}_i}{dt} = \delta^2 \left(-\nabla p + \frac{1}{\beta} \mathbf{J} \times \mathbf{B} - \varepsilon \nabla \cdot \Pi_i^{gv} \right) \quad , \tag{2}$$

Electron Equation of Motion:

$$\varepsilon \mathbf{E} = -\delta^2 \left(\frac{1}{\beta} \mathbf{V}_e \times \mathbf{B} + \frac{T_e}{T_i} \frac{\nabla p_e}{n} \right) \quad , \tag{3}$$

Constitutive Relation:

$$\delta^2 \mathbf{J} = n \left(\beta \varepsilon \mathbf{V}_i - \delta^2 \mathbf{V}_e \right) \quad . \tag{4}$$

As in Ref. [1], we do not consider the energy equation. The total non-dimensional pressure (measured in units of p_{i0}) appearing in Equation (2) is $p = p_i + (T_e / T_i) p_e$. The non-dimensional parameters $\varepsilon = \omega / \Omega_i$, $\delta = \rho_i / L$, and $\beta = \left(V_{th_i} / V_A\right)^2$ are the same as in Ref. [1]. An additional parameter $\xi = V_0 / V_{th_i}$, which appears in Ref. [1], is no longer independent, but is related to the others by $\varepsilon = \xi\delta$. All normalized variables that appear in Equations (1-4) are to be considered $O(1)$. The ion gyro-radius is always considered to be small ($\delta << 1$). The relative magnitudes of the terms in the equations are to be determined by the magnitudes of ε and β relative to δ.

Before proceeding, we note that, since the first and third terms in Equation (4) are $O(\delta^2)$, consistency requires that $\beta\varepsilon \sim \delta^2$. Thus $\beta \sim \delta^2 / \varepsilon = \delta / \xi$ (since $\varepsilon = \xi\delta$; see above). *This is precisely the constraint on β that was found in Ref. [1].* Therefore, in this approach only the frequency ε can be specified independently. (The temperature ratio T_e / T_i can also be specified.)

As in Ref. [1] we will consider three regimes: a fast ordering, $\varepsilon \sim 1$; an intermediate ordering, $\varepsilon \sim \delta$, and a slow ordering, $\varepsilon \sim \delta^2$. These correspond to Hall MHD, ideal MHD, and drift MHD, respectively.

3. Fast Ordering: $\varepsilon \sim 1$

We consider the high frequency case $\varepsilon \sim 1$, or $\omega \sim \Omega_i$. This implies that $\xi \equiv V_0 / V_{th_i} = \varepsilon / \delta \sim O(1/\delta)$, so that the characteristic flows can be much larger than the ion sound speed, and $\beta \sim O(\delta^2)$. Then Equations (2-4) become

$$n \frac{d\mathbf{V}_i}{dt} = \delta^2 \left(-\nabla p + \frac{1}{\beta} \mathbf{J} \times \mathbf{B} - \nabla \cdot \Pi_i^{gv} \right) \quad , \tag{5}$$

$$\mathbf{E} = -\delta^2 \left(\frac{1}{\beta} \mathbf{V}_e \times \mathbf{B} + \frac{T_e}{T_i} \frac{\nabla p_e}{n} \right) \quad , \tag{6}$$

and

$$\delta^2 \mathbf{J} = n \left(\beta \mathbf{V}_i - \delta^2 \mathbf{V}_e \right) \quad . \tag{7}$$

Using Equation (7) in Equation (6) to eliminate the electron velocity, we find the generalized Ohm's law to be

$$\mathbf{E} = -\mathbf{V}_i \times \mathbf{B} + \delta^2 \frac{1}{n} \left(\frac{1}{\beta} \mathbf{J} \times \mathbf{B} - \frac{T_e}{T_i} \frac{\nabla p_e}{n} \right) \quad . \tag{8}$$

With the constraint $\beta \sim O(\delta^2)$, these equations are identical (except for the temperature ratio) to the Hall MHD equations of Ref. [1], Section 4, and the comments of Ref. [1] apply. The model requires the retention of both the gyro-viscous stress and the Hall term. (The electron pressure term can be neglected unless the electrons are very much hotter than the ions, i.e., $T_e / T_i \sim O(1/\delta^2)$.)

4. Intermediate Ordering: $\varepsilon \sim \delta$

In this case $\xi \sim 1$ and $\beta \sim \delta$, and the equations are

$$n\frac{d\mathbf{V}_i}{dt} = -\nabla p + \frac{1}{\beta}\mathbf{J}\times\mathbf{B} - \delta\nabla\cdot\Pi_i^{gv} \quad , \tag{9}$$

$$\mathbf{E} = -\delta\left(\frac{1}{\beta}\mathbf{V}_e\times\mathbf{B} + \frac{T_e}{T_i}\frac{\nabla p_e}{n}\right) \quad , \tag{10}$$

and

$$\delta\mathbf{J} = n\left(\beta\mathbf{V}_i - \delta\mathbf{V}_e\right) \quad . \tag{11}$$

The generalized Ohm's law is

$$\mathbf{E} = -\mathbf{V}_i\times\mathbf{B} + \delta\frac{1}{n}\left(\frac{1}{\beta}\mathbf{J}\times\mathbf{B} - \frac{T_e}{T_i}\frac{\nabla p_e}{n}\right) \quad . \tag{12}$$

To lowest order in δ, this is the ideal MHD model discussed in Ref. [1], Section 5. Neither the gyro-viscosity, the Hall term, nor the electron pressure need be retained.

5. Slow Ordering: $\varepsilon \sim \delta^2$

In this case $\xi \sim \delta$ and $\beta \sim 1$, and the equations are

$$n\delta^2\left(\frac{d\mathbf{V}_i}{dt} + \nabla\cdot\Pi_i^{gv}\right) = -\nabla p + \frac{1}{\beta}\mathbf{J}\times\mathbf{B} \quad , \tag{13}$$

$$\mathbf{E} = -\frac{1}{\beta}\mathbf{V}_e\times\mathbf{B} - \frac{T_e}{T_i}\frac{\nabla p_e}{n} \quad , \tag{14}$$

and

$$\mathbf{J} = n\left(\beta\mathbf{V}_i - \mathbf{V}_e\right) \quad . \tag{15}$$

The generalized Ohm's law is

$$\mathbf{E} = -\mathbf{V}_i\times\mathbf{B} + \frac{1}{n\beta}\mathbf{J}\times\mathbf{B} - \frac{T_e}{T_i}\frac{\nabla p_e}{n} \quad . \tag{16}$$

These are identical to the equations derived in Ref. [1], Section 6. The drift model is obtained by making the transformation $\mathbf{V}_i = \mathbf{V}_E + \mathbf{V}_{*i}$, where

$$\mathbf{V}_E = \frac{\mathbf{E}\times\mathbf{B}}{B^2} \tag{17}$$

is the "MHD" velocity, and

$$\mathbf{V}_{*i} = \frac{1}{nB^2}\mathbf{B}\times\nabla p_i \tag{18}$$

is the non-dimensional ion diamagnetic drift velocity. Then Equation (13) simplifies by means of the gyro-viscous cancellation, as described in Ref. [1], while the generalized Ohm's law becomes

$$\mathbf{E} = -\mathbf{V}_E \times \mathbf{B} + \underbrace{\frac{1}{n}\left(-\nabla_{\perp} p + \frac{1}{\beta}\mathbf{J} \times \mathbf{B}\right)}_{O(\delta^2)} - \frac{T_e}{T_i}\frac{\nabla_{\|} p_e}{n} \quad , \tag{19}$$

$$= -\mathbf{V}_E \times \mathbf{B} - \frac{T_e}{T_i}\frac{\nabla_{\|} p_e}{n} + O(\delta^2) \quad . \tag{20}$$

As discussed in Ref. [1], the drift model eliminates whistler waves due to force balance, but retains the dispersive kinetic Alfvén branch.

5. Discussion

We have shown that the results of Ref. [1] survive with the suggested[2] velocity normalization. In particular, the restrictions on the magnitude of β in the fluid models remain valid. In Ref. [1] these constraints arise directly from the definition of the non-dimensional variables and the choice of normalization. Here they are a result of the requirement to balance the magnitude of the terms in the constitutive relation between the current density and the species velocities. Each approach leads to identical conclusions, as it should since the physics content of the equations is independent of the units used to measure the individual variables.

6. Acknowledgement

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References

1. D. D. Schnack, Ordered Fluid Equations, August 2003 (unpublished note).

2. S. Jardin and J. Ramos, private communication (2004).