**On Pfirsch-Schlüter flows in MHD codes (Rev-1)** SCJ 11/07/06

The Pfirsch-Schlüter flows in toroidal geometry can be derived from the 3 equations:

$$\nabla p = \vec{J} \times \vec{B} \tag{1}$$

$$\vec{E} + \vec{V} \times \vec{B} = \eta \vec{J} \tag{2}$$

$$\nabla \times \vec{E} = 0 \tag{3}$$

It is easiest to work in the axisymmetric flux coordinate system $(\psi, \theta, \varphi)$ so that the magnetic field and Jacobian can be written as:

$$\vec{B} = \nabla\varphi \times \nabla\psi + g(\psi)\nabla\varphi \tag{4}$$

and

$$J \equiv \left[\nabla\psi \times \nabla\theta \bullet \nabla\varphi\right]^{-1} \tag{5}$$

Note that $|\nabla\varphi|^2 = 1/R^2$ . Combine (3) and (2) to rewrite (2) in the form:

$$\nabla\Phi + \frac{1}{2\pi} V_L \nabla\varphi + \vec{V} \times \vec{B} = \eta \vec{J} \tag{2'}$$

The first term in (2)' is the gradient of a single valued scalar potential, and the second term represents the electric field due to an applied loop voltage (it is necessary in order to make $\Phi$ single valued).

Equation (1) gives the perpendicular part of the current density, but it is not divergence free. The condition that the current be divergence-free and consistent with (2)' gives:

$$\vec{J} = \frac{\vec{B} \times \nabla p}{B^2} + \left[\frac{V_L \left\langle \vec{B} \bullet \nabla\varphi \right\rangle}{2\pi\eta \left\langle B^2 \right\rangle} + g p' \left( \frac{1}{\left\langle B^2 \right\rangle} - \frac{1}{B^2} \right)\right] \vec{B} \tag{6}$$

Note that if we take the dot product of (6) and then surface average, it is consistent with taking the dot product of (2)' and surface averaging, using the fact that $\left\langle B \bullet \nabla\Phi \right\rangle = 0$ .

Now, we can use Equations (2)' and (6) to solve for the perpendicular velocity. Take both $\nabla\psi \times \vec{B} \bullet$ and $\vec{B} \bullet$ of Equation (2)', perform appropriate surface averages, eliminate the first term (scalar potential), and solve for the cross field particle flux:

Thus,

$$\nabla\psi\times\vec{B}\bullet\left[\nabla\Phi+\frac{1}{2\pi}V_L\nabla\varphi+\vec{V}\times\vec{B}=\eta\vec{J}\right]$$

$$\nabla\psi\bullet\left[\vec{B}\times\nabla\Phi+\frac{1}{2\pi}V_L\vec{B}\times\nabla\varphi+\vec{B}\times\vec{V}\times\vec{B}=\eta\vec{B}\times\vec{J}\right]$$

$$\frac{g}{J}\Phi_\theta+\frac{1}{2\pi}V_L\frac{|\nabla\psi|^2}{R^2}+B^2\vec{V}\bullet\nabla\psi=-\eta p'|\nabla\psi|^2$$

$$J\vec{V}\bullet\nabla\psi=-\eta p'\frac{J|\nabla\psi|^2}{B^2}-\frac{1}{2\pi}V_L\frac{J|\nabla\psi|^2}{B^2R^2}-\frac{g}{B^2}\Phi_\theta \qquad (6.1)$$

Now we need to eliminate the last term. This is done by taking another projection of (2)':

$$\vec{B}\bullet\left[\nabla\Phi+\frac{1}{2\pi}V_L\nabla\varphi+\vec{V}\times\vec{B}=\eta\vec{J}\right]$$

$$\Phi_\theta+\frac{1}{2\pi}V_LJ\vec{B}\bullet\nabla\varphi=\eta\left[\frac{V_L\left\langle\vec{B}\bullet\nabla\varphi\right\rangle JB^2}{2\pi\eta\left\langle B^2\right\rangle}+gp'J\left(\frac{B^2}{\left\langle B^2\right\rangle}-1\right)\right] \qquad (6.2)$$

Now use (6.2) to eliminate the scalar electrostatic potential from (6.1) and surface average:

$$\left\langle\vec{V}\bullet\nabla\psi\right\rangle=-\eta p'\left\langle\frac{|\nabla\psi|^2}{B^2}\right\rangle\left[1+g^2\left\langle\frac{|\nabla\psi|^2}{B^2}\right\rangle^{-1}\left(\left\langle B^{-2}\right\rangle-\left\langle B^2\right\rangle^{-1}\right)\right]-\frac{1}{2\pi}V_L\left[1-\left\langle\frac{B_T^2}{B^2}\right\rangle\right], \qquad (7)$$

where $B_T^2\equiv g^2/R^2$ . The first term in brackets in (7) is the Pfirsch-Schlüter diffusion term, and the second is the classical pinch. Note that we can define:

$$q_*^2\equiv g^2\left\langle\frac{|\nabla\psi|^2}{B^2}\right\rangle^{-1}\left(\left\langle B^{-2}\right\rangle-\left\langle B^2\right\rangle^{-1}\right) \qquad (7.1)$$

In the large aspect ratio, circular limit, q* reduces to the safety factor. Then (7) becomes:

$$\left\langle\vec{V}\bullet\nabla\psi\right\rangle=-\eta p'\left\langle\frac{|\nabla\psi|^2}{B^2}\right\rangle\left[1+2q_*^2\right]-\frac{1}{2\pi}V_L\left[1-\left\langle\frac{B_T^2}{B^2}\right\rangle\right] \qquad (7.2)$$

The only way to make this particle flux zero would be to replace (2) with $\vec{E}+\vec{V}\times\vec{B}=\eta(\vec{J}-\vec{J}_0)$ and set $V_L$=0, where $\vec{J}_0$ is EXACTLY given by (6). It is unlikely that any physical current drive source will be of this exact form.