Gravitational Instability as a Test Case for Extended MHD Computations

(or, *Getting into the Sixties in Plasma Physics!*)

D. D. Schnack

August 2005

The extended MHD equations, here meaning resistive MHD with the addition of 2-fluid terms in Ohm's law and the gyro-viscous stress in the equation of motion, present challenges for computational algorithms. Beyond simple dispersion relations for linear waves in uniform media, there are very few known solutions with which computational models can be tested. A non-trivial example is the well-known gravitational instability in 2-dimensional slab geometry, which was studied with extended MHD many years ago by Roberts and Taylor[1]. This simple problem illustrates issues that arise in more general confinement problems, including interchange instability and its stabilization with 2-fluid and finite-Larmor radius (FLR) effects, and so may serve as a candidate for quantitative testing of models and algorithms for extended MHD.

The original motivation for the work of Roberts and Taylor (referred to hereafter as RT) is to correct a calculation published by Lehnert[2], who called into question some results of Rosenbluth, Krall, and Rostoker[3]. (With the lineup of Rosenbluth, Krall, Rostoker, Roberts, and Taylor on one side, and Lehnert on the other, I think you can guess how this comes out.) Rosenbluth, et al[3], had predicted stabilization of the gravitational instability on the basis of kinetic theory. Lehnert[2] used fluid theory to show that the stabilizing effect of Rosenbluth, et al, was "exactly cancelled by another term in the two-fluid equations, leaving only a residual stabilizing effect"[1]. RT showed that the kinetic result of Rosenbluth, et al, can in fact be recovered with a fluid model if "other terms"[1] are used in the ion pressure tensor. Thus, RT showed for the first time that "finite Larmor radius stabilization ... can be obtained from the magnetohydrodynamic [sic] equations"[1], and that it is "not essential to use Vlasov's equation for this type of problem"[1]. The paper has become a classic.

Unfortunately, RT is of classic terseness as well as of importance. The purpose of this note is to work through the analysis of RT in order to elucidate the details of the calculation, and to formulate a relatively simple test problem for benchmarking extended MHD computations. We calculate the wave number above which FLR stabilization occurs (a useful result not given explicitly in RT), which can be tested with computational models. As an aside, we also find that the exact cancellation reported by Lehnert may indeed occur within the equations used by RT if the equilibrium model is inconsistent with a fundamental thermodynamic constraint. Thus, even in this seemingly simple stability problem, careful attention must be paid to the properties of the underlying equilibrium.

We consider the problem of a heavy fluid supported by a light fluid in the presence of a gravitational force. The problem is 2-dimensional in the (x,y) plane, with the gravitational acceleration $\mathbf{g}$ pointing in the negative x-direction, and the density gradient pointing in the positive x-direction. We also assume an exponentially increasing density profile. Thus $\mathbf{G} = \rho\mathbf{g} = -\rho g \mathbf{e}_x$ and $\nabla\rho = \eta\rho\mathbf{e}_x$, where $\eta = 1/L_n$ and L_n is the equilibrium density scale length. (Note that RT take $\mathbf{g}$ to be in the positive x-direction and $\nabla\rho$ in the negative x-direction, so that both g and η have opposite signs from that given here. However, it seems more "natural" to have gravity pointing "down".) The equilibrium magnetic field is in the z-direction, and any velocities are in the (x,y) plane. Including 2-fluid and FLR effects, the continuity and momentum equations, and Ohm's law, are then

$$\frac{\partial\rho}{\partial t} + \nabla\cdot\rho\mathbf{V} = 0 \quad , \tag{1}$$

$$\rho\frac{d\mathbf{V}}{dt} = -\nabla\left(p + \frac{B^2}{2\mu_0}\right) + \rho\mathbf{g} - \nabla\cdot\Pi \quad , \tag{2}$$

and

$$\mathbf{E} = -\mathbf{V}\times\mathbf{B} - \frac{M}{\rho e}\left[\nabla\left(\frac{B^2}{2\mu_0}\right) + \nabla p_e\right] \quad , \tag{3}$$

where p is the total fluid pressure, p_e is the electron pressure, M is the ion mass, and Π is the gyro-viscous stress tensor, which in this case is given by the Braginskii[4] expression

$$\Pi_{xx} = -\Pi_{yy} = -\rho\nu\left(\frac{\partial V_y}{\partial x} + \frac{\partial V_x}{\partial y}\right) \quad , \tag{4}$$

and

$$\Pi_{xy} = \Pi_{yx} = \rho\nu\left(\frac{\partial V_x}{\partial x} - \frac{\partial V_y}{\partial y}\right) \quad . \tag{5}$$

Following RT, we have written $\rho\nu \equiv \eta_3 = p/2\Omega$, so that $\nu = a^2\Omega/2$, where $a^2 = V_{th}^2/\Omega^2$ is the square of the ion Larmor radius, $V_{th}^2 = T/M$ is the square of the ion thermal speed, and $\Omega = eB/M$ is the ion gyro-frequency. (Note that our definition of ν differs by a factor of 2 from RT.) As noted in RT, "ν has the dimensions (but not the exact physical significance) of a kinematic viscosity"; indeed, the gyro-viscous force is not dissipative.

The stationary equilibrium is given by Equation (2) as

$$\frac{d}{dx}\left(p_0 + \frac{B_0^2}{2\mu_0}\right) = -\rho_0 g \quad . \tag{6}$$

Since only the density profile is specified (as $d\rho_0/dx = \eta\rho_0$), there is some arbitrariness in the pressure and magnetic field, as long as they do not violate any physical principles. One natural choice is $B_0 = constant$ and

$$\frac{dp_0}{dx} = -\rho_0 g \quad , \tag{7}$$

so that pressure is a *decreasing* function of *x*, as in hydrostatic equilibrium. However, it is a fundamental law of thermodynamics that $(\partial p/\partial \rho)_S > 0$, so that the pressure must be a monotonic function of the density. This condition is violated by Equation (7), since the resulting pressure decreases as the density increases. We thus require that the pressure be an increasing function of the x-coordinate. If we assume an equation of state of the form $p = p(\rho)$ (a *barotropic* fluid, which encompasses isothermal and adiabatic fluids as special cases), then

$$\frac{\partial p_0}{\partial x} = \left(\frac{\partial p}{\partial \rho}\right)_0 \frac{\partial \rho_0}{\partial x} = C_s^2 \eta \rho_0 \quad , \tag{8}$$

where C_s is the sound speed. The magnetic field must then vary as

$$\frac{d}{dx}\left(\frac{B_0^2}{2\mu_0}\right) = -\left(g + \eta C_s^2\right)\rho_0(x) \quad , \tag{9}$$

which depends on the specific form of $p(\rho)$. (RT is mute on the subject of equilibrium force balance, except to state that "the magnetic field is in the *z* direction and essentially uniform". This would seem to imply Equation (7), except that this choice is both unphysical and inconsistent with their results, as we shall see. Perhaps a more accurate statement would be that the equilibrium magnetic pressure varies no rapidly than the equilibrium fluid pressure.)

We have not yet addressed the evolution of the magnetic field. Using Equation (2), Equation (3) can be written as

$$\mathbf{E} = -\mathbf{V} \times \mathbf{B} + \frac{M}{\rho e}\left[\rho \frac{d\mathbf{V}}{dt} + \nabla p_i - \rho \mathbf{g} + \nabla \cdot \Pi\right] \quad , \tag{9}$$

where p_i is the ion pressure, so that the magnetic field only explicitly enters the dynamics through the total pressure

$$p_T = p + \frac{B^2}{2\mu_0} \quad . \tag{10}$$

Therefore, as far as the dynamics are concerned, perturbations to the magnetic field can be ignored, and all perturbed pressure forces can be viewed as entering through the fluid pressure p. It is then a significant, and consistent, simplification to assume that all perturbations are electrostatic, so that $\nabla \times \mathbf{E} = 0$. [RT calls this the low beta approximation, meaning that, for a given change in the total pressure p_T, the required relative change in fluid pressure is much larger than the required change in the magnetic field (by a factor of B_0/μ_0), and the latter can therefore be ignored.] Setting the curl of Equation (9) to zero, and assuming that the ions are barotropic, yields

$$-\nabla \times (\mathbf{V} \times \mathbf{B}) + \frac{M}{e}\nabla \times \left[\frac{d\mathbf{V}}{dt} + \frac{C_s^2}{\rho}\nabla \rho + \frac{1}{\rho}\nabla \cdot \Pi\right] = 0 \quad , \tag{11}$$

since $\nabla \times \mathbf{g} = 0$. In the present case, with $\mathbf{B} = B\mathbf{e}_z$ and $\mathbf{V}$ in the (*x*,*y*) plane, this becomes

$$\nabla \cdot \mathbf{V} + \frac{1}{\Omega}\nabla \times \left[\frac{d\mathbf{V}}{dt} + \frac{C_s^2}{\rho}\nabla\rho + \frac{1}{\rho}\nabla \cdot \Pi \right] = 0 \quad . \tag{12}$$

Given a relationship for $p(\rho)$, Equations (1), (2), and (12) are four equations in the 4 unknowns ρ, $\mathbf{V}$ (2 components), and $p_T = p + B^2/2\mu_0$. Equation (12) (or, equivalently, $\nabla \times \mathbf{E} = 0$) serves as an "equation of state" to close the system and determine p_T. This is analogous to the common assumption of incompressibility in hydrodynamics, except that the fluid is not longer strictly incompressible. (It is interesting that non-solenoidal velocity fields can lead to no change in the magnetic field, but such is extended MHD.) If, as in RT, we further assume that the fluid is isothermal ("we assume ... that temperature variations can be ignored"[1]), then $C_s^2 = constant$, and Equation (12) becomes

$$\nabla \cdot \mathbf{V} + \frac{1}{\Omega}\nabla \times \left[\frac{d\mathbf{V}}{dt} - \frac{1}{\rho^2}\nabla\rho \times \nabla \cdot \Pi \right] = 0 \quad , \tag{13}$$

since Π is a symmetric tensor. Equations (1), (2) and (13) are the equations of the model.

Following RT, we linearize about the equilibrium state, assuming variations of the form $\exp(i\omega t + iky)$. We ignore explicit variations of the coefficients in the *x*-direction, which requires $\eta L_x << 1$, where L_x is the maximum value of *x*. With $L_x \sim \lambda_y = 2\pi/k$, this implies $\eta << |k|$. The linearized components of the gyro-viscous stress are

$$(\nabla \cdot \Pi)_x = -(\rho_0 \nu_0)' ikV_x + \rho_0 \nu_0 k^2 V_y \quad , \tag{14}$$

and

$$(\nabla \cdot \Pi)_y = -(\rho_0 \nu_0)' ikV_y - \rho_0 \nu_0 k^2 V_x \quad , \tag{15}$$

where $(..)'$ indicates differentiation with respect to *x*. Now, with RT, we let the entire variation of the gyro-viscous coefficient enter through the equilibrium density ρ_0, so that $\nu_0 = constant$. (This implies constant temperature and ignores the variation of Ω_0 with *x*, but is consistent with assumption that B_0 is "essentially uniform".) Equations (14) and (15) are then

$$(\nabla \cdot \Pi)_x = -\nu_0 \eta \rho_0 ikV_x + \rho_0 \nu_0 k^2 V_y \quad , \tag{16}$$

and

$$(\nabla \cdot \Pi)_y = -\nu_0 \eta \rho_0 ikV_y - \rho_0 \nu_0 k^2 V_x \quad . \tag{17}$$

With this, the final set of linearized equations is

$$i\omega\rho \quad + \quad \eta\rho_0 V_x \quad + \quad ik\rho_0 V_y \quad = 0 \ , \tag{18}$$

$$\frac{g}{\rho_0}\rho \quad + \quad (i\omega - \zeta\eta\nu_0 ik)V_x \quad + \quad \zeta\nu_0 k^2 V_y \qquad = 0 \ , \quad (19)$$

$$-\zeta\nu_0 k^2 V_x \quad + \quad (i\omega - \zeta\eta\nu_0 ik)V_y \quad + \quad \frac{ik}{\rho_0}p_T = 0 \ , \quad (20)$$

$$\left(\xi\frac{\omega k}{\Omega_0} + \xi\zeta\frac{\nu_0\eta}{\Omega_0}k^2\right)V_x \quad + \quad \left(1 + \xi\zeta\frac{\eta^2\nu_0}{\Omega_0}\right)ikV_y \qquad = 0 \ . \quad (21)$$

We have introduced the parameters ξ and ζ so that $\xi = 0,\ \zeta = 0$ indicates ideal MHD, $\xi = 1,\ \zeta = 0$ indicates extended Ohm's law but no gyro-viscosity, $\xi = 0,\ \zeta = 1$ indicates gyro-viscosity but no extended Ohm's law, and $\xi = 1,\ \zeta = 1$ indicates both extended Ohm's law (2-fluid) and gyro-viscous (FLR) effects.

Accordingly, the dispersion relation can be found in the corresponding regimes. For *ideal MHD* ($\xi = 0,\ \zeta = 0$) we have

$$\omega^2 + g\eta = 0 \quad , \qquad (22)$$

so that there is instability with growth rate $\gamma = \sqrt{g\eta}$ independent of the wave number k. With *gyro-viscosity only* ($\xi = 0,\ \zeta = 1$) we have

$$\omega^2 - \nu_0\eta k\omega + g\eta = 0 \quad . \qquad (23)$$

The solution is

$$2\omega = \nu_0\eta k \pm \sqrt{(\nu_0\eta k)^2 - 4g\eta} \quad . \qquad (24)$$

There are two real (stable) roots if

$$k^2 > k_{GV}^2 = \frac{4g}{\nu_0^2\eta} \quad . \qquad (25)$$

With *two-fluid effects only* ($\xi = 1,\ \zeta = 0$) we have

$$\omega^2 - \frac{gk}{\Omega_0}\omega + g\eta = 0 \quad . \qquad (26)$$

The solution is

$$2\omega = \frac{gk}{\Omega_0} \pm \sqrt{\left(\frac{gk}{\Omega_0}\right)^2 - 4g\eta} \quad . \qquad (27)$$

There are two real (stable) roots if

$$k^2 > k_{2F}^2 = \frac{4\eta\Omega_0^2}{g} \quad . \qquad (28)$$

Finally, with *full extended MHD (two-fluid + gyro-viscous)* ($\xi = 1,\ \zeta = 1$) we have

$$\left[1+\frac{\nu_0}{\Omega_0}\left(\eta^2+k^2\right)\right]\omega^2 - \left\{\frac{gk}{\Omega_0}+\nu_0\eta k\left[1+\frac{\nu_0}{\Omega_0}\left(\eta^2-k^2\right)\right]\right\}\omega + g\eta\left[1+\frac{\nu_0}{\Omega_0}\left(\eta^2-k^2\right)\right]=0 \quad . \tag{29}$$

Taking note that $\nu_0 k^2/\Omega_0 = (ka)^2/2 << 1$, and $\eta^2 << k^2$, we find to lowest order in small quantities that

$$\omega^2 - \left(\frac{gk}{\Omega_0}+\nu_0\eta k\right)\omega + g\eta = 0 \quad . \tag{30}$$

The solution is

$$2\omega = \frac{gk}{\Omega_0}+\nu_0\eta k \pm \sqrt{\left(\frac{gk}{\Omega_0}+\nu_0\eta k\right)^2 - 4g\eta} \quad . \tag{31}$$

There are two real (stable) roots if

$$k^2 > k_{EMHD}^2 = \frac{4g\eta}{\left(\frac{g}{\Omega_0}+\nu_0\eta\right)^2} \quad . \tag{32}$$

(The coefficients of ω differ by a sign, and in the case of the factor $\nu_0\eta k$ by a factor of 2, from the results of RT. This is because of the differences in signs of both g and η, and the factor of 2 in the definition of ν_0, as stated previously. The predictions of stabilization thresholds remain unchanged.)

With regard to the stabilization thresholds, we note that Equation (25) is equivalent to $\omega_\nu^2 > 4\omega_g^2$, Equation (28) is equivalent to $\omega_k^2 > 4\omega_g^2$, and Equation (32) is equivalent to $\omega_\nu^2 + \omega_k^2 > 4\omega_g^2$, where $\omega_\nu = \nu_0\eta k$, $\omega_k = gk/\Omega_0$, and $\omega_g^2 = g\eta$.

Equations (25), (28), and (32) predict stabilization of the gravitational instability for sufficiently large wave number. Stabilization occurs due to both 2-fluid and gryo-viscous effects, and is more effective (occurs at lower k) with both 2-fluid and gyro-viscous terms. These quantitative predictions should be testable with computational models of extended MHD. The gryo-viscous stabilization occurs because of the x-variation of the equilibrium gyro-viscous $\rho_0\nu_0$, so it is important to retain this effect in the computations.

It is of interest to see what effect the choice of equilibrium has on the solution. For, example, suppose we had chosen initially $B_0 = constant$, so that force balance is given by Equation (7). Then, along with Equation (8), we have

$$\frac{dp_0}{dx} = \left(\frac{\partial p}{\partial\rho}\right)_0 \frac{d\rho_0}{dx} = C_s^2\eta\rho_0 = -\rho_0 g \quad , \tag{33}$$

which implies that g and η are related by $g = -C_s^2 \eta$. Then the coefficient of ω in Equation (15) becomes

$$\begin{aligned} \frac{gk}{\Omega_o} + v_0 \eta k &= -\frac{C_s^2 \eta k}{\Omega_0} + v_0 \eta k = -\frac{C_s^2 \eta k \Omega_0}{\Omega_0^2} + v_0 \eta k \\ &= -a^2 \Omega_0 \eta k + v_0 \eta k = -2 v_0 \eta k + v_0 \eta k = -v_0 \eta k \quad , \end{aligned} \tag{34}$$

so that the stabilizing effect is significantly modified. Perhaps this is related to the "almost exact" cancellation and "residual stabilization" found by Lehnert[2]. In any case, it is unphysical and Equation (30) is the proper result as given by both RT and Rosenbluth, et al.[3].

It should be relatively straight forward to design a test case that fits within the parameters of this problem that will demonstrate and benchmark 2-fluid and FLR stabilization at large enough wave number. It would also be interesting to estimate the sizes of the relevant terms for the ELM benchmark problem to see whether or not extended MHD stabilization can be expected.

References

1. K. V. Roberts and J. B. Taylor, Phys. Rev. Letters **8**, 197 (1962).

2. B. Lehnert, Phys. Rev. Letters **7**, 442 (1961).

3. M. N. Rosenbluth, N. A. Krall, and N. Rostoker, (IAEA, Salzburg, September, 1961, Paper CN-10/170), Nucl. Fusion Suppl. Pt. **1**, 143 (1962).

4. S. I. Braginskii, Rev. Plasma Phys. **1**, 205 (Consultants Bureau, New York, 1965).