Gravitational Instability as a Test Case for Extended MHD Computations

(or, *Getting into the Sixties in Plasma Physics!*)

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1.0 Introduction

The extended MHD equations, here meaning resistive MHD with the addition of 2-fluid terms in Ohm's law and the gyro-viscous stress in the equation of motion, present challenges for computational algorithms. Beyond simple dispersion relations for linear waves in uniform media, there are very few known solutions with which computational models can be tested. A non-trivial example is the well-known gravitational instability in 2-dimensional slab geometry, which was studied with extended MHD many years ago by Roberts and Taylor[1]. This simple problem illustrates issues that arise in more general confinement problems, including interchange instability and its stabilization with 2-fluid and finite-Larmor radius (FLR) effects, and so may serve as a candidate for quantitative testing of models and algorithms for extended MHD.

The original motivation for the work of Roberts and Taylor (referred to hereafter as RT) is to correct a calculation published by Lehnert[2], who called into question some results of Rosenbluth, Krall, and Rostoker[3]. (With the lineup of Rosenbluth, Krall, Rostoker, Roberts, and Taylor on one side, and Lehnert on the other, I think you can guess how this comes out.) Rosenbluth, et al[3], had predicted stabilization of the gravitational instability on the basis of kinetic theory. Lehnert[2] used fluid theory to show that the stabilizing effect of Rosenbluth, et al, was "exactly cancelled by another term in the two-fluid equations, leaving only a residual stabilizing effect"[1]. RT showed that the kinetic result of Rosenbluth, et al, can in fact be recovered with a fluid model if "other terms"[1] are used in the ion pressure tensor. Thus, RT showed for the first time that "finite Larmor radius stabilization ... can be obtained from the magnetohydrodynamic [sic] equations"[1], and that it is "not essential to use Vlasov's equation for this type of problem"[1]. The paper has become a classic.

Unfortunately, RT is of classic terseness as well as of importance. The purpose of this note is to work through the analysis of RT in order to elucidate the details of the calculation, and to formulate a relatively simple test problem for benchmarking extended MHD computations. We calculate the wave number k_0 above which FLR stabilization occurs (a useful result not given explicitly in RT), which can be tested with computational models. As an aside, we also find that the exact cancellation reported by Lehnert may indeed occur within the equations used by RT if the equilibrium model is inconsistent with a fundamental thermodynamic constraint. Thus, even in this seemingly simple stability problem, careful attention must be paid to the properties of the underlying equilibrium. We extend the analysis to include the collisionless ion heat flow in the gyro-viscous stress tensor[5-9], and find that it has negligible effect on the stability results. We also derive the extended MHD dispersion relation directly from the separate equations for ion and electron fluids. This reveals the role played by compressibility, along with the stabilizing mechanism for the mode. Finally, we estimate the value of the toroidal mode number n at which stabilization should occur for a tokamak edge plasma. For the parameters used in the benchmark case, we estimate that stabilization should occur at $m \sim 160$ and $n = m/q \sim 40$. At these scales $(k_0 a)^2 << 1$, so that the fluid theory remains valid.

2.0 The Gravitational Instability in Extended MHD

We consider the problem of a heavy fluid supported by a light fluid in the presence of a gravitational force. The problem is 2-dimensional in the (x,y) plane, with the gravitational acceleration $\mathbf{g}$ pointing in the negative x-direction, and the density gradient pointing in the positive x-direction. We also assume an exponentially increasing density profile. Thus $\mathbf{G} = \rho\mathbf{g} = -\rho g \mathbf{e}_x$ and $\nabla\rho = \eta\rho\mathbf{e}_x$, where $\eta = 1/L_n$ and L_n is the equilibrium density scale length. (Note that RT take $\mathbf{g}$ to be in the positive x-direction and $\nabla\rho$ in the negative x-direction, so that both g and η have opposite signs from that given here. However, it seems more "natural" to have gravity pointing "down".) The equilibrium magnetic field is in the z-direction, and any velocities are in the (x,y) plane. Including 2-fluid and FLR effects, the continuity and momentum equations, and Ohm's law, are then

$$\frac{\partial\rho}{\partial t} + \nabla\cdot\rho\mathbf{V} = 0 \quad , \tag{1}$$

$$\rho\frac{d\mathbf{V}}{dt} = -\nabla\left(p + \frac{B^2}{2\mu_0}\right) + \rho\mathbf{g} - \nabla\cdot\Pi \quad , \tag{2}$$

and

$$\mathbf{E} = -\mathbf{V}\times\mathbf{B} - \frac{M}{\rho e}\left[\nabla\left(\frac{B^2}{2\mu_0}\right) + \nabla p_e\right] \quad , \tag{3}$$

where p is the total fluid pressure, p_e is the electron pressure, M is the ion mass, and Π is the gyro-viscous stress tensor, which in this case is given by the Braginskii[4] expression

$$\Pi_{xx} = -\Pi_{yy} = -\rho\nu\left(\frac{\partial V_y}{\partial x} + \frac{\partial V_x}{\partial y}\right) \quad , \tag{4}$$

and

$$\Pi_{xy} = \Pi_{yx} = \rho\nu\left(\frac{\partial V_x}{\partial x} - \frac{\partial V_y}{\partial y}\right) \quad . \tag{5}$$

Following RT, we have written $\rho\nu \equiv \eta_3 = p/2\Omega$, so that $\nu = a^2\Omega/2$, where $a^2 = V_{th}^2/\Omega^2$ is the square of the ion Larmor radius, $V_{th}^2 = T/M$ is the square of the ion thermal speed, and $\Omega = eB/M$ is the ion gyro-frequency. (Note that our definition of ν differs by a factor of 2 from RT.) As noted in RT, "ν has the dimensions (but not the exact physical significance) of a kinematic viscosity"; indeed, the gyro-viscous force is not dissipative.

The stationary equilibrium is given by Equation (2) as

$$\frac{d}{dx}\left(p_0+\frac{B_0^2}{2\mu_0}\right)=-\rho_0 g \quad . \tag{6}$$

Since only the density profile is specified (as $d\rho_0/dx=\eta\rho_0$), there is some arbitrariness in the pressure and magnetic field, as long as they do not violate any physical principles. One natural choice is $B_0 = constant$ and

$$\frac{dp_0}{dx}=-\rho_0 g \quad , \tag{7}$$

so that pressure is a *decreasing* function of *x*, as in hydrostatic equilibrium. However, it is a fundamental law of thermodynamics that $(\partial p/\partial\rho)_S>0$, so that the pressure must be a monotonic function of the density. This condition is violated by Equation (7), since the resulting pressure decreases as the density increases. We thus require that the pressure be an increasing function of the x-coordinate. If we assume an equation of state of the form $p=p(\rho)$ (a *barotropic* fluid, which encompasses isothermal and adiabatic fluids as special cases), then

$$\frac{\partial p_0}{\partial x}=\left(\frac{\partial p}{\partial\rho}\right)_0\frac{\partial\rho_0}{\partial x}=C_s^2\eta\rho_0 \quad , \tag{8}$$

where C_s is the sound speed. The magnetic field must then vary as

$$\frac{d}{dx}\left(\frac{B_0^2}{2\mu_0}\right)=-\left(g+\eta C_s^2\right)\rho_0(x) \quad , \tag{9}$$

which depends on the specific form of $p(\rho)$. (RT is mute on the subject of equilibrium force balance, except to state that "the magnetic field is in the *z* direction and essentially uniform". This would seem to imply Equation (7), except that this choice is both unphysical and inconsistent with their results, as we shall see. Perhaps a more accurate statement would be that the equilibrium magnetic pressure varies no more rapidly than the equilibrium fluid pressure.)

We have not yet addressed the evolution of the magnetic field. Using Equation (2), Equation (3) can be written as

$$\mathbf{E}=-\mathbf{V}\times\mathbf{B}+\frac{M}{\rho e}\left[\rho\frac{d\mathbf{V}}{dt}+\nabla p_i-\rho\mathbf{g}+\nabla\cdot\Pi\right] \quad , \tag{9}$$

where p_i is the ion pressure, so that the magnetic field only explicitly enters the dynamics through the total pressure

$$p_T = p+\frac{B^2}{2\mu_0} \quad . \tag{10}$$

Therefore, as far as the dynamics are concerned, perturbations to the magnetic field can be ignored, and all perturbed pressure forces can be viewed as entering through the fluid pressure p. It is then a significant, and consistent, simplification to assume that all perturbations are electrostatic, so that $\nabla \times \mathbf{E} = 0$. [RT calls this the low beta approximation, meaning that, for a given change in the total pressure p_T, the required relative change in fluid pressure is much larger than the required change in the magnetic field (by a factor of B_0 / μ_0), and the latter can therefore be ignored.] Setting the curl of Equation (9) to zero, and assuming that the ions are barotropic, yields

$$-\nabla \times (\mathbf{V} \times \mathbf{B}) + \frac{M}{e} \nabla \times \left[\frac{d\mathbf{V}}{dt} + \frac{C_s^2}{\rho} \nabla \rho + \frac{1}{\rho} \nabla \cdot \Pi \right] = 0 \quad , \tag{11}$$

since $\nabla \times \mathbf{g} = 0$. In the present case, with $\mathbf{B} = B\mathbf{e}_z$ and $\mathbf{V}$ in the (x,y) plane, this becomes

$$\nabla \cdot \mathbf{V} + \frac{1}{\Omega} \nabla \times \left[\frac{d\mathbf{V}}{dt} + \frac{C_s^2}{\rho} \nabla \rho + \frac{1}{\rho} \nabla \cdot \Pi \right] = 0 \quad . \tag{12}$$

Given a relationship for $p(\rho)$, Equations (1), (2), and (12) are four equations in the 4 unknowns ρ, $\mathbf{V}$ (2 components), and $p_T = p + B^2 / 2\mu_0$. Equation (12) (or, equivalently, $\nabla \times \mathbf{E} = 0$) serves as an "equation of state" to close the system and determine p_T. This is analogous to the common assumption of incompressibility in hydrodynamics, except that the fluid is not longer strictly incompressible. (It is interesting that non-solenoidal velocity fields can lead to no change in the magnetic field, but such is extended MHD.) If, as in RT, we further assume that the fluid is isothermal ("we assume ... that temperature variations can be ignored"[1]), then $C_s^2 = constant$, and Equation (12) becomes

$$\nabla \cdot \mathbf{V} + \frac{1}{\Omega} \nabla \times \frac{d\mathbf{V}}{dt} - \frac{1}{\Omega \rho^2} \nabla \rho \times \nabla \cdot \Pi = 0 \quad , \tag{13}$$

since Π is a symmetric tensor. Equations (1), (2) and (13) are the equations of the model.

Following RT, we linearize about the equilibrium state, assuming variations of the form $\exp(i\omega t + iky)$. We ignore explicit variations of the coefficients in the x-direction, which requires $\eta L_x << 1$, where L_x is the maximum value of x. With $L_x \sim \lambda_y = 2\pi / k$, this implies $\eta << |k|$. The linearized components of the gyro-viscous stress are

$$(\nabla \cdot \Pi)_x = -(\rho_0 \nu_0)' ikV_x + \rho_0 \nu_0 k^2 V_y \quad , \tag{14}$$

and

$$(\nabla \cdot \Pi)_y = -(\rho_0 \nu_0)' ikV_y - \rho_0 \nu_0 k^2 V_x \quad , \tag{15}$$

where $(..)'$ indicates differentiation with respect to x. Now, with RT, we let the entire variation of the gyro-viscous coefficient enter through the equilibrium density ρ_0, so that $\nu_0 = constant$. (This implies constant temperature and ignores the variation of Ω_0 with x, but is consistent with assumption that B_0 is "essentially uniform".) Equations (14) and (15) are then

$$\left(\nabla \cdot \Pi\right)_x = -\nu_0 \eta \rho_0 ik V_x + \rho_0 \nu_0 k^2 V_y \quad , \tag{16}$$

and

$$\left(\nabla \cdot \Pi\right)_y = -\nu_0 \eta \rho_0 ik V_y - \rho_0 \nu_0 k^2 V_x \quad . \tag{17}$$

With this, the final set of linearized equations is

$$i\omega\rho \quad + \quad \eta\rho_0 V_x \quad + \quad ik\rho_0 V_y \quad = 0 \ , \tag{18}$$

$$\frac{g}{\rho_0}\rho \quad + \quad \left(i\omega - \zeta\eta\nu_0 ik\right)V_x \quad + \quad \zeta\nu_0 k^2 V_y \quad = 0 \ , \tag{19}$$

$$-\zeta\nu_0 k^2 V_x \quad + \quad (i\omega - \zeta\eta\nu_0 ik)V_y \quad + \quad \frac{ik}{\rho_0} p_T = 0 \ , \tag{20}$$

$$\left(\xi\frac{\omega k}{\Omega_0} + \xi\zeta\frac{\nu_0\eta}{\Omega_0}k^2\right)V_x \quad + \quad \left(1 + \xi\zeta\frac{\eta^2\nu_0}{\Omega_0}\right)ikV_y \quad = 0 \ . \tag{21}$$

We have introduced the parameters ξ and ζ so that $\xi = 0,\ \zeta = 0$ indicates ideal MHD, $\xi = 1,\ \zeta = 0$ indicates extended Ohm's law but no gyro-viscosity, $\xi = 0,\ \zeta = 1$ indicates gyro-viscosity but no extended Ohm's law, and $\xi = 1,\ \zeta = 1$ indicates both extended Ohm's law (2-fluid) and gyro-viscous (FLR) effects, i.e., extended MHD.

Accordingly, the dispersion relation can be found in the corresponding regimes. For *ideal MHD* ($\xi = 0,\ \zeta = 0$) we have

$$\omega^2 + g\eta = 0 \quad , \tag{22}$$

so that there is instability with growth rate $\gamma = \sqrt{g\eta}$ independent of the wave number k. With *gyro-viscosity only* ($\xi = 0,\ \zeta = 1$) we have

$$\omega^2 - \nu_0\eta k\omega + g\eta = 0 \quad . \tag{23}$$

The solution is

$$2\omega = \nu_0\eta k \pm \sqrt{\left(\nu_0\eta k\right)^2 - 4g\eta} \quad . \tag{24}$$

There are two real (stable) roots if

$$k^2 > k_{GV}^2 = \frac{4g}{\nu_0^2\eta} \quad . \tag{25}$$

With *two-fluid effects only* ($\xi = 1,\ \zeta = 0$) we have

$$\omega^2 - \frac{gk}{\Omega_0}\omega + g\eta = 0 \quad . \tag{26}$$

The solution is

$$2\omega = \frac{gk}{\Omega_0} \pm \sqrt{\left(\frac{gk}{\Omega_0}\right)^2 - 4g\eta} \quad . \tag{27}$$

There are two real (stable) roots if

$$k^2 > k_{2F}^2 = \frac{4\eta\Omega_0^2}{g} \quad . \tag{28}$$

Finally, with *full extended MHD (two-fluid + gyro-viscous)* ($\xi = 1$, $\zeta = 1$) we have

$$\begin{aligned} &\left[1 + \frac{\nu_0}{\Omega_0}\left(\eta^2 + k^2\right)\right]\omega^2 \\ &\qquad - \left\{\frac{gk}{\Omega_0} + \nu_0\eta k\left[1 + \frac{\nu_0}{\Omega_0}\left(\eta^2 - k^2\right)\right]\right\}\omega \\ &\qquad\qquad + g\eta\left[1 + \frac{\nu_0}{\Omega_0}\left(\eta^2 - k^2\right)\right] = 0 \quad . \end{aligned} \tag{29}$$

Taking note that $\nu_0 k^2 / \Omega_0 = (ka)^2 / 2 << 1$, and $\eta^2 << k^2$, we find to lowest order in small quantities that

$$\omega^2 - \left(\frac{gk}{\Omega_0} + \nu_0\eta k\right)\omega + g\eta = 0 \quad . \tag{30}$$

The solution is

$$2\omega = \frac{gk}{\Omega_0} + \nu_0\eta k \pm \sqrt{\left(\frac{gk}{\Omega_0} + \nu_0\eta k\right)^2 - 4g\eta} \quad . \tag{31}$$

There are two real (stable) roots if

$$k^2 > k_{EMHD}^2 = \frac{4g\eta}{\left(\frac{g}{\Omega_0} + \nu_0\eta\right)^2} \quad . \tag{32}$$

(The coefficients of ω differ by a sign, and in the case of the factor $\nu_0\eta k$ by a factor of 2, from the results of RT. This is because of the differences in signs of both g and η, and the factor of 2 in the definition of ν_0, as stated previously. The predictions of stabilization thresholds remain unchanged.)

With regard to the stabilization thresholds, we note that Equation (25) is equivalent to $\omega_\nu^2 > 4\omega_g^2$, Equation (28) is equivalent to $\omega_k^2 > 4\omega_g^2$, and Equation (32) is equivalent to $\omega_\nu^2 + \omega_k^2 > 4\omega_g^2$, where $\omega_\nu = \nu_0\eta k$, $\omega_k = gk / \Omega_0$, and $\omega_g^2 = g\eta$.

Equations (25), (28), and (32) predict stabilization of the gravitational instability for sufficiently large wave number. Stabilization occurs due to both 2-fluid and gryo-viscous effects, and is more effective (occurs at lower k) with both 2-fluid and gyro-viscous terms. These quantitative predictions should be testable with computational models of extended MHD. The gryo-viscous stabilization occurs because of the x-variation of the equilibrium gyro-viscous $\rho_0\nu_0$, so it is important to retain this effect in the computations.

3.0 The Effect of Ion Heat Stress

Several papers[5-9] have shown that the expression for the gyro-viscous stress given by Equations (4) and (5) is formally incomplete, in that it should be supplemented by an additional term of the same form but with the velocity replaced by the ion heat flux, and the coefficient replaced by 2/5Ω. Implementation of these terms requires a closure expression for the ion heat flux. It has recently been suggested[10] that, using the Braginskii expression[4] for collisional heat flux, these new terms may have negligible effect on the fluid dispersion relations. Here we examine the effect of these terms on the gravitational stability and show that the effect is similarly negligible.

When the ion heat stress is included in the gyro-viscous force, and ignoring the collisonal heat flux[10], Equations (18-21) become

$$i\omega\rho + \eta\rho_0 V_x + ik\rho_0 V_y = 0 , \quad (33)$$

$$\frac{g}{\rho_0}\rho + (i\omega - \eta\nu_0 ik)V_x + \nu_0 k^2 V_y = 0 , \quad (34)$$

$$-\nu_0 k^2 V_x + (i\omega - \eta\nu_0 ik)V_y + \frac{ik}{\rho_0}\left(1 + \frac{2\kappa_\wedge k^2}{5n_0\Omega}\right)p_T = 0 , \quad (35)$$

$$\left(\frac{\omega k}{\Omega_0} + \frac{\nu_0\eta}{\Omega_0}k^2\right)V_x + \left(1 + \frac{\eta^2\nu_0}{\Omega_0}\right)ikV_y - \frac{ik}{\rho_0}\frac{2\kappa_\wedge k^2}{5n_0\Omega}p_T = 0 . \quad (36)$$

There is now additional coupling between Equations (35) and (36). (We consider here only the extended MHD case, with $\xi = \zeta = 1$, and assumed an isothermal model.) Using the relations $\kappa_\wedge = 5n_0 T / 2\Omega$ and $\nu_0 = a^2\Omega/2$, Equations (33-36) can be expressed in terms of the small parameters $\delta = ka$ and $\varepsilon = \eta a$. Since $k^2 >> \eta^2$, we can ignore terms that are $O(\varepsilon^2)$, but we will retain terms that are $O(\delta^2)$ and $O(\delta\varepsilon)$. After some algebra, the resulting dispersion relation is, to order consistent with Equation (30),

$$2\omega = \frac{gk}{\Omega}\left(1 + \frac{\eta^2}{k^2}\right) + \nu_0\eta k \pm \sqrt{\left[\frac{gk}{\Omega}\left(1 + \frac{\eta^2}{k^2}\right) + \nu_0\eta k\right]^2 - 4g\eta} \quad . \quad (37)$$

This is identical with the result of RT (Equation (31)) except for terms that are $O(\eta^2 / k^2) << 1$. The heat stress therefore makes a negligible contribution to the gravitational interchange instability. (We note that the term $\nu_0\eta k = (ka)^2\eta\Omega / 2k$ must be retained, as in RT, since $\eta\Omega / k$ may be large.)

5.0 Full Two-Fluid Description

The results presented in Section 3 arise from setting the determinant of Equations (18-21) to zero. The solution procedure offers little insight into the physics of the two-fluid and FLR stabilization. Some progress can be made in this regard by considering the two-fluid form to the fluid equations for ions and electrons. For simplicity, only two-fluid effects are included. Ignoring terms that are $O(m/M)$, where m is the electron mass, the equations are

$$\frac{\partial n_i}{\partial t} = -\nabla \cdot n_i \mathbf{V}_i \quad , \tag{38}$$

$$\frac{\partial n_e}{\partial t} = -\nabla \cdot n_i \mathbf{V}_e \quad , \tag{39}$$

$$M n_i \left(\frac{\partial \mathbf{V}_i}{\partial t} + \mathbf{V}_i \cdot \nabla \mathbf{V}_i \right) = e n_i (\mathbf{E} + \mathbf{V}_i \times \mathbf{B}) - M C_{Si}^2 \nabla n_i + M n_i \mathbf{g} \quad , \tag{40}$$

$$0 = -e n_e (\mathbf{E} + \mathbf{V}_e \times \mathbf{B}) \quad , \tag{41}$$

$$\nabla \times \mathbf{E} = 0 \quad , \tag{42}$$

$$\nabla \cdot \mathbf{E} = c^2 \mu_0 \rho_q \quad , \tag{43}$$

$$\rho_q = e(n_i - n_e) \quad . \tag{44}$$

Here $\mathbf{V}_i$ and $\mathbf{V}_e$ are the velocities relative to a stationary reference frame, and we have assumed a barotropic fluid with $C_{Si}^2 = (\partial p_i / \partial \rho)_S$ as the square of the ion sound speed. We linearize about a quasi-neutral equilibrium, but do not as yet assume that the perturbation is quasi-neutral, i.e., we take $n_{i0} = n_{e0} = n_0$, but do not assume that n_e is equal to n_i. (Here and in what follows equilibrium quantities are denoted by the subscript 0 while perturbed quantities are unsubscripted.) We take the equilibrium quantities to depend only on *x*, and, as in RT, ignore the *x*-dependence of the perturbation quantities. Then Equations (38-44) yield the equilibrium conditions

$$V_{xi0} = 0 \quad , \tag{45}$$

$$V_{yi0} = \frac{g}{\Omega} + \frac{C_{Si}^2}{\Omega L_n} - \frac{E_{x0}}{B} \quad , \tag{46}$$

$$V_{xe0} = 0 \quad , \tag{47}$$

$$V_{ye0} = -\frac{E_{x0}}{B} \quad , \tag{48}$$

$$E_{x0} = constant \quad , \tag{49}$$

$$E_{y0} = 0 \quad . \tag{50}$$

The ions and electrons experience their common $\mathbf{E} \times \mathbf{B}$ in the *y*-direction in response to the zero order electric field. The ions experience an additional gravitational drift in the *y*-direction

$$\mathbf{V}_{gi} = -\frac{1}{\Omega}\mathbf{b}\times\mathbf{g} \quad . \tag{51}$$

(The electrons experience a similar drift in the opposite direction of the ions, but it is $O(m/M)$ with respect to the ion drift, and has been ignored.) The equilibrium electric field is arbitrary, and represents a coordinate transformation to a reference frame with relative y-velocity $V_{y0} = E_{x0}/B$.

Using this equilibrium, the linearized two-fluid equations are

$$i(\omega + kV_{yi0})n_i = -\frac{n_0}{L_n}(V_{xi} + ikL_nV_{yi}) \quad , \tag{52}$$

$$i(\omega + kV_{ye0})n_e = -\frac{n_0}{L_n}V_{ye} \quad , \tag{53}$$

$$i(\omega + kV_{yi0})\frac{n_0}{L_n}V_{xi} = \Omega\frac{n_0}{L_n}V_{yi} + \frac{C_{Si}^2}{\Omega L_n}n_i \quad , \tag{54}$$

$$i(\omega + kV_{yi0})V_{yi} = -\Omega V_{xi} + \Omega\frac{E_y}{B} - ik\frac{C_{Si}^2}{L_n}n_i \quad , \tag{55}$$

$$V_{xe} = \frac{E_y}{B} \quad , \tag{56}$$

$$ikE_y = c^2\mu_0 e(n_i - n_e) \quad . \tag{57}$$

(Here, $L_n = 1/\eta$ is the density scale length.)

These are six equations in the 6 unknowns n_i, n_e, V_{xi}, V_{yi}, V_{xe} and E_y, and by themselves are no more revealing than Equations (18-21). However, it is interesting to compare them with the MHD case, which is obtained in the limit $\Omega \to \infty$. Then the equilibrium conditions (46) and (48), along with the choice $E_{x0} = 0$ (a stationary reference frame) dictate no ion or electron drifts. Under the same conditions, Equation (54) implies that $V_{yi} = 0$, and Equations (55) and (56) yield $V_{xi} \sim E_y/\omega B = V_{xe}$, so that to an excellent approximation the ions and electrons move together with common velocity $\mathbf{V} = \mathbf{E}\times\mathbf{B}/B^2$ (which in turn implies that $\mathbf{E} + \mathbf{V}\times\mathbf{B} = 0$). There is no displacement parallel to the wave vector $\mathbf{k} = k\mathbf{e}_y$. This is equivalent to the constraint $\nabla\cdot\mathbf{V} = 0$, which is implied by Equation (13) in this limit. Using Equation (52) in Equation (54) immediately provides the MHD result $\omega^2 = -g/L_n$ (see Equation (22)).

It is instructive to rewrite the perturbed two-fluid equations in non-dimensional form. Defining $N_\alpha = n_\alpha/n_0$, $v_\alpha = V_\alpha/V_0$, $V_0 = g/\Omega$, $W = \omega/\Omega$, $W_{\alpha 0} = kV_{y\alpha 0}/\Omega$, $V_E = E_y/B$, $\omega_A = V_A/L_n$, and $\alpha = kL_n$ (not to be confused with the species subscript), we have

$$i(W + W_{i0})N_i = -\frac{\omega_A}{\Omega}(v_{xi} + i\alpha v_{yi}) \quad , \tag{58}$$

$$i\left(W + W_{e0}\right)N_e = -\frac{\omega_A}{\Omega}V_E \quad , \tag{59}$$

$$i\left(W + W_{i0}\right)v_{xi} = v_{yi} + \frac{\gamma\beta}{2}\frac{\omega_A}{\Omega}N_i \quad , \tag{60}$$

$$i\left(W + W_{i0}\right)v_{yi} = -v_{xi} + V_E - i\alpha\frac{\gamma\beta}{2}\frac{\omega_A}{\Omega}N_i \quad , \tag{61}$$

$$i\alpha\frac{V_A^2}{c^2}\frac{\omega_A}{\Omega}V_E = N_i - N_e \quad . \tag{62}$$

We note that, in contrast with MHD, the gravitational acceleration $\mathbf{g}$ enters only through the individual particle drifts $W_{\alpha 0}$, and does not appear explicitly in the dynamics of the perturbed quantities. Also, Equation (64) implies that the normalized charge separation is $O(V_A^2/c^2) << 1$, as is required for the neglect of relativistic effects. We can therefore assume quasi-neutrality. However, that does not imply that the perturbed electric field vanishes. It is given by Equation (56) as the electron $\mathbf{E}\times\mathbf{B}$ drift, and is related to the other perturbed quantities through quasi-neutrality and Equations (58) and (59), i.e.,

$$V_E = \frac{W + W_{e0}}{W + W_{i0}}\left(v_{xi} + i\alpha v_{yi}\right) \quad . \tag{63}$$

This does *not* imply that $\nabla\cdot\mathbf{E} = 0$; rather, any finite value of $\nabla\cdot\mathbf{E}$ can only result in an insignificant charge separation. We again assume low-β, so that the last two terms in Equations (60) and (61) can be dropped. This eliminates sound waves that do not play a fundamental role in the instability.

Under these assumptions, the dispersion relation obtained from Equations (59), (60) and (63) is

$$\left(W + W_{i0}\right)\left[\left(W + W_{i0}\right)^2 - \alpha\left(W + W_{e0}\right)\right] + W_{e0} - W_{i0} = 0 \quad , \tag{64}$$

which is a cubic equation in W. However, if we assume low frequency (or large enough α), we have

$$\alpha\left(W + W_{i0}\right)\left(W + W_{e0}\right) + W_{i0} - W_{e0} = 0 \quad , \tag{65}$$

or, in terms of the dimensional variables,

$$\omega^2 + k\left(\frac{g}{\Omega} - 2\frac{E_{x0}}{B}\right)\omega - k^2\frac{E_{x0}}{B}\left(\frac{g}{\Omega} - \frac{E_{x0}}{B}\right) + \frac{g}{L_n} = 0 \quad . \tag{66}$$

The coefficients containing $g/\Omega - E_{x0}/B$ are related to the choice of reference frame through Equations (46) and (48) (with $\beta = 0$). In a frame with stationary electrons and drifting ions, we have $E_{x0} = 0$ and Equation (66) becomes

$$\omega^2 + \frac{gk}{\Omega}\omega + \frac{g}{L_n} = 0 \quad , \tag{67}$$

while in a frame with stationary ions and drifting electrons, we have

$$\omega^2 - \frac{gk}{\Omega}\omega + \frac{g}{L_n} = 0 \quad , \tag{68}$$

which is identical with the two-fluid result given by Equation (26). The choice of reference frame thus accounts for the sign discrepancy between Equation (26) and the RT result. The impact of this choice is also discussed in RT.

The primary difference between the MHD and two-fluid models is the role of compressibility. In MHD, the electrostatic assumption and the form of Ohm's law *require* $\nabla \cdot \mathbf{V} = 0$. A non-solenoidal velocity field would result in compression of the magnetic field. As a consequence, the electron and ion motions are tied together, moving with their common $\mathbf{E} \times \mathbf{B}$ drift in the *x*-direction. The two-fluid treatment allows the decoupling of the electron and ion motions because of their separate gravitational drifts. Now consider an element of the ion fluid moving into a region of lower (or higher) electron density. If the ion fluid were incompressible, there would be charge imbalance. The ion fluid element must therefore be compressed (or expanded) just enough to maintain charge balance. Non-solenoidal ion flow is thus a *requirement of quasi-neutrality*; it is enabled by the perturbed ion drifts, as given by Equation (13). This results in a *y*-component of the perturbed ion flow, which is related to the electron $\mathbf{E} \times \mathbf{B}$ flow by means of Equation (63). (This is *exactly* the amount of compressibility needed to maintain charge neutrality.) This perturbed ion flow does work on the perturbed electric field. Since E_y increases with k, this implies a wave number at which there is insufficient free energy to drive the instability.

The gyro-viscosity leads to an additional ion drift that can also interact with the perturbed electric field, and modify the stability threshhold (see Equation (13)).

5.0 Estimates of Stabilization in the Tokamak Edge

We now apply these results to the case of the tokamak edge. We take *B* to be the toroidal field, *y* to be the poloidal direction, and **g** to point in the negative *R*-direction. Defining the characteristic velocities $V_g^2 = g/\eta$ and $V_{th}^2 = a^2\Omega^2$, the expression for the stabilizing wave number, Equation (32), can be written as

$$k_0^2 = \frac{4\left(V_g / V_{th}\right)^2}{a^2\left[1 + \left(V_g / V_{th}\right)^2\right]^2} \quad . \tag{64}$$

The effective gravitational acceleration is can be expressed approximately as $g = \left|\mathbf{B}_0 \cdot \nabla \mathbf{B}_0\right| / \mu_0 \rho_0 \sim V_A^2 \kappa$, where κ is the normal curvature of the field lines and V_A is the Alfvén speed. Then $\left(V_g / V_{th}\right)^2 = \kappa L_n / \beta$, where $L_n = 1/\eta$ is the density scale length and $\beta = \left(V_{th} / V_A\right)^2$ is (approximately) the plasma beta. Either the poloidal or toroidal field and curvature can be used for the estimate. However, since $\kappa_{pol} \sim \kappa_{tor} / \varepsilon$ and $\beta_{pol} \sim \left(q/\varepsilon\right)^2 \beta_{tor}$,

$$\left(\frac{V_g}{V_{th}}\right)^2_{pol} = \frac{\kappa_{pol} L_n}{\beta_{pol}} = \frac{\varepsilon}{q^2} \frac{\kappa_{tor} L_n}{\beta_{tor}} << \left(\frac{V_g}{V_{th}}\right)^2_{tor} \quad , \tag{65}$$

so the poloidal field will cause stabilization at a smaller wave number than will the toroidal field. Using Equation (39) with values of $\kappa_{tor} \sim 0.3$ m^{-1}, $\beta_{tor} \sim 10^{-2}$, $\varepsilon \sim 0.3$, $q = 4$, and $L_n \sim 10^{-2}$ m, we have $(V_g / V_{th})^2 \sim 6 \times 10^{-3}$. The toroidal field is relevant for estimating the gyro-radius; with $B \sim 1$ T, we get $a \sim 10^{-3}$ m. Finally, making the association $k_0 \sim m/r$, we expect to find stabilization for $m \sim 160$, or $n = m/q \sim 40$. The MHD growth rate (valid for long wavelengths) is $\gamma = \sqrt{g\eta} \sim 10^6$ sec^{-1}. Note that $(k_0 a)^2 = 3 \times 10^{-2} << 1$, so that the theory remains valid at the stabilizing wave number.

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