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Viscous Forces Due To Collisional Parallel Stresses For Extended MHD Codes

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Abstract

Various forms of the viscous force caused by collision-induced parallel viscous stresses within a magnetized plasma are presented. A form of it is proposed for initial value extended MHD codes to capture the regular (Braginskii) collisional viscous force effects on short time scales and multi-collisionality regime "residual" viscous forces on collision time scales and longer in axisymmetric toroidal plasmas. Collision-based viscosity coefficients are described in various collisionality regimes: high (Braginskii, Pfirsch-Schlüter), intermediate (plateau) and low (banana). Smoothed formulas for the residual viscous forces induced by electron and ion parallel stresses on collision and longer time scales that encompass all these collisionality regimes are presented. Finally, suggestions are made for implementing and verifying viscous force effects in extended MHD codes.

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I. INTRODUCTION

Extended magnetohydrodynamic (MHD) models used in the CEMM project [1] seek to include all relevant physics needed for simulating fluid-type behavior of magnetically-confined toroidal plasmas. The M3D [2] and NIMROD [3] code projects under the CEMM umbrella have historically focused on using ideal and resistive MHD descriptions of axisymmetric equilibrium toroidal plasmas. Recently, two-fluid effects (e.g., diamagnetic flows, gyroviscosity) have been included. And explorations of plasma flows and their effects have begun.

In order to properly describe the evolution of flows, the anisotropic nature of the viscous force in a magnetized plasma needs to be taken into account. In particular, the viscous forces induced by collision-induced parallel stresses in the plasma need to be taken into account because: 1) they are the largest viscous forces; 2) they collisionally relax the electron and ion flows on their respective time scales; 3) in low collisionality plasmas they lead to important effects in the parallel neoclassical Ohm's law (trapped particle effects on the resistivity and bootstrap current) and the poloidal plasma flow (relaxation to an ion-temperature-gradient-determined value); and 4) it is important for numerical stability and convergence issues to properly treat dissipative effects in the M3D and NIMROD codes.

This report is organized as follows. Section II describes the various components (parallel, cross and perpendicular) of stresses and in particular the collisional viscous stresses in a magnetized plasma. Appendix A describes the collisional (Braginskii) parallel stresses in plasmas containing, as is typical in tokamaks, electrons, hydrogenic ions and impurity ions. The following section (III) presents various forms of parallel stresses and the resultant viscous forces. Thereafter, Section IV proposes specific forms for viscous forces induced by parallel stresses for inclusion in the M3D and NIMROD codes. Asymptotic "residual" parallel viscous force coefficients in high (Braginskii, Pfirsch-Schlüter), intermediate (plateau) and low (banana) collisionality regimes, and smoothed multi-collisionality formulas for both electrons and ions in plasmas with small admixtures of impurity ions are presented in Appendix B. The penultimate section suggests tests to verify the implementation of these viscous forces in the M3D and NIMROD codes. Finally, Section VI summarizes the viscous forces induced by parallel stresses in multi-collisionality regimes for tokamak plasmas that is suggested for use in extended MHD codes. It also discusses some of the possible limitations of the suggested form.

II. COLLISIONAL STRESSES IN A MAGNETIZED PLASMA

The fluid-based viscous force density on a small volume of a plasma species is $-\boldsymbol{\nabla}\cdot\boldsymbol{\pi}$. The fact that this force density is a local differential of the local viscous stress tensor $\boldsymbol{\pi}$ implies that the physical processes that cause it are also local. This is unfortunately not true in low collisionality toroidal plasmas where the collision length $\lambda \equiv v_T/\nu$ is usually many times the toroidal circumference of the experimental device. Nonetheless, the following discussion and analysis will seek to capture the most important low collisionality physics within this local model of viscous forces — to facilitate inclusion of viscous force effects of parallel viscous stresses in the M3D and NIMROD extended-MHD-based initial value codes.

The collisional Braginskii [4] viscous stresses are given by

$$\boldsymbol{\pi} = \boldsymbol{\pi}_{\parallel} + \boldsymbol{\pi}_{\wedge} + \boldsymbol{\pi}_{\perp}, \quad \text{parallel, cross (gyroviscous) and perpendicular stresses.} \tag{1}$$

Here, the subscripts $\parallel, \wedge, \perp$ indicate parallel, cross and perpendicular directions relative to the local magnetic field $\mathbf{B}(\mathbf{x})$. A strongly magnetized plasma species is defined as one that has a small collision frequency ν compared to the gyrofrequency ω_c and small gyroradius $\varrho \equiv v_T/\omega_c$ compared to cross and perpendicular gradient scale lengths of plasma properties and electromagnetic fields. For strongly magnetized toroidal plasmas of fusion interest a small gyroradius expansion is usually appropriate:

$$\delta \equiv \varrho/a \ll 1, \quad \text{small gyroradius expansion.} \tag{2}$$

Here, a is a characteristic macroscopic plasma dimension, typically the plasma minor radius. For arbitrary flow velocity magnitudes and properties, the characteristic scalings of the parallel, cross and perpendicular stresses can be written for $R_0 q \gtrsim \lambda \gtrsim a$ as

$$\boldsymbol{\pi}_{\parallel} \sim \nu\lambda^2\,\boldsymbol{\nabla}_{\parallel}\mathbf{V}, \quad \boldsymbol{\pi}_{\wedge} \sim \nu\varrho\lambda\,\mathbf{B}\times\boldsymbol{\nabla}\mathbf{V}/B \sim \delta\,\boldsymbol{\pi}_{\parallel}, \quad \boldsymbol{\pi}_{\perp} \sim \nu\varrho^2\,\boldsymbol{\nabla}_{\perp}\mathbf{V} \sim \delta^2\,\boldsymbol{\pi}_{\parallel}, \quad \text{scalings.} \tag{3}$$

Thus, the parallel viscous stress $\boldsymbol{\pi}_{\parallel}$ is usually dominant in small gyroradius, magnetized toroidal plasmas. The remainder of this report will concentrate on it. [For plasma perturbations extended long distances along field lines (i.e., $|\boldsymbol{\nabla}_{\parallel}| \lesssim 1/a$) but radially localized to a small fraction of the minor radius (i.e., $|\boldsymbol{\nabla}_{\perp}| \sim k_{\perp} \gg 1/a$), the cross and perpendicular stress effects can become larger than this scaling indicates by factors of $k_{\perp}a$ and $(k_{\perp}a)^2$; for such cases the cross (gyroviscous) force can exceed the parallel stress force — as is the case for many perturbed diamagnetic flow effects on MHD-type instabilities.]

The parallel viscous stresses for electrons and ions were originally written by Braginskii [4] for each species in the form (here, z is the local coordinate along $\mathbf{B}$)

$$\boldsymbol{\pi}_{\parallel} = -\,\eta_0\, W_{zz}, \quad W_{zz} \equiv 2\,\frac{\partial V_z}{\partial z} - \frac{2}{3}\,(\boldsymbol{\nabla}\cdot\mathbf{V}). \tag{4}$$

For an electron-ion plasma with a hydrogenic ion species (i.e., $Z_i = 1$), the collisional viscosity coefficients for electrons and ions are [4] ($\tau \equiv 1/\nu$, $\lambda \equiv v_T/\nu$ and $v_T \equiv \sqrt{2T/m}$)

$$\eta_0^i = 0.96\, n_i T_i\, \tau_i = 0.48\, m_i n_i\, \nu_i \lambda_i^2, \quad \eta_0^e = 0.73\, n_e T_e\, \tau_e = 0.365\, m_e n_e\, \nu_e \lambda_e^2. \tag{5}$$

The parallel stresses can be written in general (for arbitrary collisionality in an inhomogeneous magnetized plasma) in the Chew-Goldberger-Low form as [5]

$$\boldsymbol{\pi}_{\parallel} \equiv \pi_{\parallel}\,(\hat{\mathbf{b}}\hat{\mathbf{b}} - \mathsf{I}/3), \quad \hat{\mathbf{b}}\cdot\boldsymbol{\pi}_{\parallel}\cdot\hat{\mathbf{b}} = (2/3)\,\pi_{\parallel}, \quad \text{parallel stress tensor.} \tag{6}$$

Here, $\pi_{\parallel}(\mathbf{x}, t) \equiv p_{\parallel} - p_{\perp}$ is the pressure anisotropy, which is a scalar function of space and time. Also, $\hat{\mathbf{b}} \equiv \mathbf{B}/B$ is a unit vector along the local magnetic field $\mathbf{B}$ and I is the identity tensor (dyad). In the Braginskii high collisionality regime $\pi_{\parallel}$ is given for each species by

$$\boxed{\pi_{\parallel} \equiv -\,(3/2)\,\eta_0\,\hat{\mathbf{b}}\cdot\mathsf{W}_V\cdot\hat{\mathbf{b}},} \quad \text{collision-induced pressure anisotropy,} \tag{7}$$

in which the rate of strain in the plasma species induced by the flow velocity $\mathbf{V}$ is

$$\mathsf{W}_V \equiv \boldsymbol{\nabla}\mathbf{V} + (\boldsymbol{\nabla}\mathbf{V})^{\mathsf{T}} - (2/3)\,\mathsf{I}\,(\boldsymbol{\nabla}\cdot\mathbf{V}), \quad \text{rate of strain induced by } \mathbf{V}. \tag{8}$$

The superscript T is the transpose of that tensor (dyad); thus, W_V is a symmetric tensor.

The Braginskii [4] closures for the parallel viscous stress $\boldsymbol{\pi}_{\parallel}$ were developed for MHD-type applications where the flow velocity $\mathbf{V}$ is assumed to be large compared to the heat flow velocity $\mathbf{V}_q \equiv -\,2\mathbf{q}/5nT$ and higher order flow-type moments (energy-weighted heat flow etc.) — but still gyroradius small compared to thermal speeds: $|\mathbf{V}|/v_T \sim \delta \ll 1$. However, in two-fluid treatments which include diamagnetic flows, the diamagnetic-type heat flows $\mathbf{V}_{sq}$ are comparable to the diamagnetic flows $\mathbf{V}_{s*} \equiv \mathbf{B}\times\boldsymbol{\nabla}p_s/(n_s q_s B^2)$ and cannot be neglected. Then, the rate of strain tensor is modified [6]: $\mathsf{W}_V \to \mathsf{W}_V + \mathsf{W}_q$, where the rate of strain tensor for heat flows is

$$\mathsf{W}_q \equiv (-2/5nT)\,[\,\boldsymbol{\nabla}\mathbf{q} + (\boldsymbol{\nabla}\mathbf{q})^{\mathsf{T}} - (2/3)\,\mathsf{I}\,(\boldsymbol{\nabla}\cdot\mathbf{q})\,]. \tag{9}$$

Similarly, the stress tensor gets modified: $\boldsymbol{\pi} \to \boldsymbol{\pi}_V + \boldsymbol{\pi}_q$, in which $\boldsymbol{\pi}_q$ represents parallel heat stresses. Viscosity coefficients including these heat flow effects and allowing for impure plasmas (i.e., for $Z_{\text{eff}} \equiv \sum_i n_i Z_i^2/n_e > 1$) are discussed in Appendix A.

III. VISCOUS FORCE INDUCED BY COLLISIONAL PARALLEL STRESSES

Next, various geometric forms of the pressure anisotropy and the viscous force they induce will be explored. Using various vector and tensor identities and the definition of the local curvature of the magnetic field, $\boldsymbol{\kappa} \equiv (\hat{\mathbf{b}}\cdot\boldsymbol{\nabla})\hat{\mathbf{b}} = -\,\hat{\mathbf{b}}\times(\boldsymbol{\nabla}\times\hat{\mathbf{b}})$ with $\hat{\mathbf{b}} \equiv \mathbf{B}/B$, it can be shown that

$$\begin{aligned}
\boxed{\mathbf{B}\cdot\mathsf{W}_V\cdot\mathbf{B}/2} &= \mathbf{B}\cdot\boldsymbol{\nabla}\mathbf{V}\cdot\mathbf{B} - (B^2/3)\,(\boldsymbol{\nabla}\cdot\mathbf{V}) \\
&= B\,(\mathbf{B}\cdot\boldsymbol{\nabla})(\mathbf{B}\cdot\mathbf{V}/B) + [\mathbf{B}\times(\mathbf{B}\times\mathbf{V})]\cdot\boldsymbol{\kappa} - (B^2/3)\,\boldsymbol{\nabla}\cdot\mathbf{V} \\
&= \boxed{B\,\mathbf{V}\cdot\boldsymbol{\nabla}B + \mathbf{B}\cdot\boldsymbol{\nabla}\times(\mathbf{V}\times\mathbf{B}) + (2B^2/3)\,\boldsymbol{\nabla}\cdot\mathbf{V} - (\mathbf{B}\cdot\mathbf{V})(\boldsymbol{\nabla}\cdot\mathbf{B}).} \qquad (10)
\end{aligned}$$

The form on the last line will be used below — because, as discussed below, its first term is the only term that survives on longer than collision times scales, after the faster MHD-type compressional Alfvén and sound wave relaxation processes come into quasi-equilibrium. The representation of $\mathbf{B}\cdot\mathsf{W}_q\cdot\mathbf{B}$ is similar with $\mathbf{V}\to\mathbf{q}$.

In order of appearance in the last line of (10), contributions to $\mathbf{B}\cdot\mathsf{W}_V\cdot\mathbf{B}$ have the following effects. The first, $\mathbf{V}\cdot\boldsymbol{\nabla}B$ term indicates parallel strain induced by flow in directions in which the magnitude of the magnetic field varies. Since the lowest order flows are within a flux surface, this term mainly produces poloidal flow damping, at a rate proportional to the collision frequency ν (for each species). The second term represents MHD-type advection of the parallel component of the magnetic field, as is evident from its linearized form $\mathbf{B}_0\cdot\boldsymbol{\nabla}\times(\tilde{\mathbf{V}}\times\mathbf{B}_0) \sim B_0\,\partial\tilde{B}_\parallel/\partial t$. Together with part of $\boldsymbol{\nabla}\cdot\mathbf{V}$, it provides viscous damping of "fast" compressional Alfvén waves, which to lowest order relax $\tilde{P} + B_0\tilde{B}_\parallel/\mu_0$. The third, $\boldsymbol{\nabla}\cdot\mathbf{V}$ term represents plasma compressibility. Its residual after the fast relaxation of compressional Alfvén waves provides viscous damping of sound waves on the ion collision time scale. Because there are no magnetic monopoles in the universe, the final, $\boldsymbol{\nabla}\cdot\mathbf{B}$ term vanishes; however, this term could be kept in extended MHD codes to assist in "divergence $\mathbf{B}$ cleaning" — i.e., for relaxing away via viscous damping any numerical errors that cause $\boldsymbol{\nabla}\cdot\mathbf{B} \neq 0$. For simplicity the analogous effects due to heat flows $\mathbf{V}_q \equiv -\,(2\mathbf{q}/5nT)$ will be neglected on the MHD compressional Alfvén and sound wave time scales; however, heat flow effects will be retained on the collision (poloidal flow damping) time scales.

The M3D and NIMROD extended MHD codes use semi-implicit numerical algorithms to take time steps longer than MHD wave time scales while capturing the constraints imposed

by these processes. Thus, for analytic purposes, on collision time scales the last form of (10) can be simplified by taking $\boldsymbol{\nabla}\cdot\mathbf{V}=0$ and $\boldsymbol{\nabla}\cdot\mathbf{B}=0$. In addition, since on the collision time scale perpendicular flows are $\mathbf{B}\times\boldsymbol{\nabla}\phi/B^2$ plus diamagnetic flows in approximately the form $\mathbf{V}_\perp=(1/B^2)\mathbf{B}\times\boldsymbol{\nabla}f$, in which $f=f(\psi_p)$ is a scalar flux function, one has $\mathbf{B}\cdot\boldsymbol{\nabla}\times(\mathbf{V}\times\mathbf{B})=B(\mathbf{B}\cdot\boldsymbol{\nabla}f)(\hat{\mathbf{b}}\cdot\boldsymbol{\nabla}\times\hat{\mathbf{b}})\sim(k_\parallel a)\,\beta$. As indicated, this term is small for typical plasma responses that are highly extended along field lines and for low β plasmas; thus, this contribution will be neglected henceforth. With all these simplifications, on the collision time scale the "residual" pressure anisotropy induced by the combination of flows and heat flows is (see Appendix A for how the viscosity coefficients are determined for impure plasmas)

$$\pi_\parallel\simeq-\frac{3}{B}\left(\eta_{00}\mathbf{V}\cdot\boldsymbol{\nabla}B-\eta_{01}\,\frac{2}{5\,nT}\,\mathbf{q}\cdot\boldsymbol{\nabla}B\right),\quad\text{on collision or longer time scales.}\tag{11}$$

The viscous force density caused by the parallel stress tensor $\boldsymbol{\pi}_\parallel$ defined in (6) is in general

$$\boxed{\mathbf{F}_\pi\equiv-\boldsymbol{\nabla}\cdot\boldsymbol{\pi}_\parallel=-\,\pi_\parallel\,[\,\boldsymbol{\kappa}-\mathbf{B}\,(\mathbf{B}\cdot\boldsymbol{\nabla}B)/B^3]-(1/B^2)\,\mathbf{B}\,(\mathbf{B}\cdot\boldsymbol{\nabla})\pi_\parallel+(1/3)\,\boldsymbol{\nabla}\pi_\parallel.}\tag{12}$$

And the parallel ($\mathbf{B}\cdot\ $) component of this viscous force is

$$\mathbf{B}\cdot\mathbf{F}_\pi\equiv-\,\mathbf{B}\cdot\boldsymbol{\nabla}\cdot\boldsymbol{\pi}_\parallel=\pi_\parallel\,(\hat{\mathbf{b}}\cdot\boldsymbol{\nabla}B)-(2/3)\,(\mathbf{B}\cdot\boldsymbol{\nabla})\,\pi_\parallel,\quad\text{parallel viscous force.}\tag{13}$$

The last term will be annihilated below by averaging this parallel force over a flux surface.

Up to now neither the magnetic field structure nor a coordinate system have been specified. However, they are needed to connect these results with axisymmetric neoclassical transport theory [7, 8]. The axisymmetric equilibrium magnetic field $\mathbf{B}_0\equiv\mathbf{B}_t+\mathbf{B}_p$ has toroidal and poloidal components. It is written in terms of the poloidal magnetic flux ψ_p:

$$\mathbf{B}_0(\psi_p,\theta)=I\boldsymbol{\nabla}\zeta+\boldsymbol{\nabla}\zeta\times\boldsymbol{\nabla}\psi_p=\boldsymbol{\nabla}\psi_p\times\boldsymbol{\nabla}[\,q(\psi_p)\,\theta-\zeta\,],\quad I(\psi_p)\equiv RB_t.\tag{14}$$

The radial, poloidal straight-field-line and toroidal axisymmetry coordinates will be taken to be ψ_p,θ,ζ for which the poloidal rotation of a field line per unit toroidal rotation is $d\theta/d\zeta=1/q(\psi_p)\equiv\mathbf{B}_0\cdot\boldsymbol{\nabla}\theta/\mathbf{B}_0\cdot\boldsymbol{\nabla}\zeta$. The Jacobian for transforming from the laboratory ($\mathbf{x}$) to these (non-orthogonal) curvilinear coordinates is $\sqrt{g}\equiv1/(\boldsymbol{\nabla}\psi_p\cdot\boldsymbol{\nabla}\theta\times\boldsymbol{\nabla}\zeta)=1/\mathbf{B}_0\cdot\boldsymbol{\nabla}\theta$. The flux surface average (FSA) of a scalar function $f(\mathbf{x})$ on a ψ_p flux surface is defined by

$$\langle f(\mathbf{x})\rangle\equiv\frac{\int_0^{2\pi}d\zeta\int_0^{2\pi}f(\mathbf{x})\;d\theta/\mathbf{B}_0\cdot\boldsymbol{\nabla}\theta}{2\pi\int_0^{2\pi}d\theta/\mathbf{B}_0\cdot\boldsymbol{\nabla}\theta},\quad\text{flux surface average of }f(\mathbf{x}).\tag{15}$$

The FSA is an annihilator for parallel derivatives of scalar functions: $\langle\mathbf{B}_0\cdot\boldsymbol{\nabla}f\rangle=0$.

On the collision and longer time scales the flow and heat flow are incompressible [7, 8]: $\boldsymbol{\nabla}\cdot\mathbf{V} = 0$, $\boldsymbol{\nabla}\cdot\mathbf{q} = 0$. Since to first order in the small gyroradius expansion ($\delta \ll 1$) the flows and heat flows lie within flux surfaces, using the general relation for the divergence of a vector in an axisymmetric system one can show for each plasma species that [8]

$$0 = \boldsymbol{\nabla}\cdot\mathbf{V} = (\mathbf{V}\cdot\boldsymbol{\nabla}\theta)\,\frac{\partial}{\partial\theta}\left(\frac{\mathbf{V}\cdot\boldsymbol{\nabla}\theta}{\mathbf{B}_0\cdot\boldsymbol{\nabla}\theta}\right) \implies \boxed{U_\theta(\psi_p) \equiv \frac{\mathbf{V}\cdot\boldsymbol{\nabla}\theta}{\mathbf{B}_0\cdot\boldsymbol{\nabla}\theta},} \text{ poloidal flow function.} \quad (16)$$

Similarly, $\boldsymbol{\nabla}\cdot\mathbf{q} = 0$ yields $Q_\theta(\psi_p) \equiv \mathbf{q}\cdot\boldsymbol{\nabla}\theta/\mathbf{B}_0\cdot\boldsymbol{\nabla}\theta$. Thus, to lowest order in δ the FSA of (13) yields the residual parallel viscous force for each species (here, $\hat{\mathbf{b}} \equiv \mathbf{B}_0/B_0$):

$$\langle \mathbf{B}_0\cdot\mathbf{F}_\pi\rangle \equiv -\,\langle \mathbf{B}_0\cdot\boldsymbol{\nabla}\cdot\boldsymbol{\pi}_\parallel\rangle \simeq -\,3\,\langle\,(\hat{\mathbf{b}}\cdot\boldsymbol{\nabla}B_0)^2\rangle\,[\,\eta_{00}\,U_\theta + \eta_{01}\,(-\,2/5nT)\,Q_\theta\,]. \quad (17)$$

IV. VISCOUS FORCES FOR EXTENDED MHD CODES

Extensions of the flux surface average (FSA) collisional parallel viscous force in (17) to the low collisionality regimes of axisymmetric tokamak plasmas have been developed [8]; these "neoclassical" results are discussed and summarized in Appendix B. For the present purposes it is convenient to specify the FSA of the residual parallel viscous force in terms of a damping force on the poloidal flow U_θ to an "intrinsic" or "offset" flow velocity U_θ^0 in the general form (for each plasma species, subscript s)

$$\boxed{\langle \mathbf{B}_0\cdot\mathbf{F}_{s\pi}\rangle \equiv -\,\langle \mathbf{B}_0\cdot\boldsymbol{\nabla}\cdot\boldsymbol{\pi}_{s\parallel}\rangle = -\,m_s n_s\,\mu_s\,\langle B_0^2\rangle(U_{s\theta} - U_{s\theta}^0),} \quad \text{FSA} \parallel \text{viscous force,} \quad (18)$$

$$\boxed{U_{s\theta}^0 \equiv k_s\,\frac{I(\psi_p)}{q_s\langle B_0^2\rangle}\,\frac{dT_{s0}}{d\psi_p},} \quad \text{offset poloidal flow.} \quad (19)$$

Formulas for the poloidal viscous damping frequency μ_s and offset poloidal flow coefficient k_s, which depend on the species and collisionality regime, are specified in Appendix B.

In general, the offset poloidal flow $U_{s\theta}^0$, which is proportional to the poloidal heat flow, does not depend solely on the temperature gradient of the species being considered. Rather, in tokamak plasmas with small admixtures of impurities in addition to the dominant hyrogenic species, it depends on impurity density and temperature gradients, and the impurity collisionality regime [9]. In transport codes the offset poloidal flow $U_{s\theta}^0$ is often evaluated using the NCLASS code [10]. However, as discussed in [9], if the impurities are in the intermediate (plateau) or high (Braginskii, Pfirsch-Schlüter) collisionality regime, $U_{i\theta}^0$ depends predominantly on the ion temperature gradient, as indicated in (18). Thus, as discussed

further in Appendix B, it will be assumed that the form of (19) provides a sufficient representation of the effects of small admixtures of impurities in extended MHD codes.

The general form for the viscous force induced by pressure anisotropy is given in (12). The key question is: what should be used for the pressure anisotropy $\pi_\parallel \equiv p_\parallel - p_\perp$? The answer for MHD-type responses is the combination of $\pi_\parallel$ given by (7) with the $\mathbf{B}\cdot\mathbf{W}_V\cdot\mathbf{B}/2$ given by (10) since on the fast compressional Alfvén and sound wave relaxation time scales the heat flows $\mathbf{V}_q$ can be neglected. After dissipation of MHD-type compressional Alfvén and sound waves with the Braginskii parallel viscous damping coefficient η_{00}, the residual contribution to (10) on collision and longer time scales is the first term, $B\,\mathbf{V}\cdot\boldsymbol{\nabla}B$, as indicated in (11). For that term the general multi-collisionality regime result for the flux surface average of the residual parallel viscous force can be written in the form given in (18), which is a generalization of the corresponding Braginskii form given in (17). To capture all these properties within a single viscous force it is proposed to leave the viscous force effects on compressional Alfvén and sound waves unchanged with the Braginskii η_{00} coefficient but to modify the $B\,\mathbf{V}\cdot\boldsymbol{\nabla}B$ term in (10) and (11) so the flux surface average it produces is given by (18). Thus, the proposed pressure anisotropy for each species is (here $\hat{\mathbf{b}} \equiv \mathbf{B}_0/B_0$)

$$\boxed{\begin{aligned}\pi_\parallel \;=\; & -\,mn\,\mu\,\langle B_0^2\rangle \frac{\hat{\mathbf{b}}\cdot\boldsymbol{\nabla}B_0}{\langle\,(\hat{\mathbf{b}}\cdot\boldsymbol{\nabla}B_0)^2\rangle}\left(U_\theta - U_\theta^0\right)\\ & -\,3\,\eta_{00}\left(\frac{\mathbf{B}\cdot\boldsymbol{\nabla}\times(\mathbf{V}\times\mathbf{B})}{B^2} + \frac{2}{3}\boldsymbol{\nabla}\cdot\mathbf{V} - \frac{(\mathbf{B}\cdot\mathbf{V})(\boldsymbol{\nabla}\cdot\mathbf{B})}{B^2}\right),\end{aligned}} \quad \text{proposed form.} \tag{20}$$

Thus, the suggested prescription for introducing viscous force effects due to parallel stresses into the M3D and NIMROD extended MHD codes is to implement the viscous force given by (12) with the pressure anisotropy $\pi_\parallel$ given in (20) using μ_s from (B8), (B12), (B14) and k_s from (B11), (B13), (B15) in Appendix B. Formulas for the needed Braginskii-type collisional viscosity coefficients η_{s00} for electrons, ions in impure plasmas are given in (A11), (A17).

Various heuristic viscous forces have been implemented and tested in reduced MHD codes [12–14]. Heuristic forms previously tested in the NIMROD code are discussed in [15]. The most successful heuristic form used in NIMROD is [15] $\mathbf{F}_\pi = -mn\mu\langle B^2\rangle(\mathbf{V}\cdot\mathbf{e}_\theta)\,\mathbf{e}_\theta/(\mathbf{B}\cdot\mathbf{e}_\theta)^2$, in which $\mathbf{e}_\theta = \sqrt{g}\,\boldsymbol{\nabla}\zeta\times\boldsymbol{\nabla}\psi_p = \mathbf{B}_p/(\mathbf{B}_0\cdot\boldsymbol{\nabla}\theta)$ is the covariant base vector. This form is obviously simpler than the form proposed in (18)–(20). However, the more general viscous force proposed here has the advantage that it captures the usual Braginskii viscous force as well as the residual multi-collisionality effects and offset flows $U_{s\theta}^0$ on long time scales.

V. VERIFICATION TESTS FOR VISCOUS FORCE EFFECTS

In implementing this suggested procedure for including the viscous force effects due to multi-collisionality parallel stresses in extended MHD codes, three categories of verification tests are suggested: fast MHD-type processes, relaxation of flows, and transport-time-scale effects — as discussed in the succeeding paragraphs.

Fast MHD-type processes: Generally speaking, viscous force effects should be negligible and have no significant effects on ideal-MHD-type plasma responses. Thus, the first verification test is to make sure adding the viscous force effects "do no harm" to these responses.

Relaxation of flows: The electron viscous force effects add to the magnetic field evolution equation a fourth order derivative, diffusive- (parabolic-) type term (via $\mathbf{V}_e = \mathbf{V}_i - \mathbf{J}/n_e e$ with $\mu_0 \mathbf{J} \equiv \boldsymbol{\nabla}\times\mathbf{B}$), which is sometimes called a "hyper-resistivity" effect. While this is useful in dissipating magnetic field structures that are highly localized radially, it should not affect other physical processes much. Similarly, the ion viscous force adds a diffusive effect to the momentum equation, primarily to its parallel component. These effects should become significant on the electron and ion collision time scales, respectively; they should dissipate compressional Alfvén and sound waves on the ion collision time scale. On time scales longer than the respective collisional time scales the electron and ion flows should relax to being: incompressible, of first order in the gyroradius, and flowing within an equilibrium or average flux surface. Specifically, they should be described by Eqs. (18) to (47) in [11]. Suggested verification tests would be to check that, on time scales longer than the respective collision times: 1) electron and ion flows are incompressible to $\mathcal{O}\{\delta^2\}$ as indicated in (16) above and Eq. (22) in [11]; 2) the poloidal flow function $U_\theta \equiv \mathbf{V}\cdot\boldsymbol{\nabla}\theta/\mathbf{B}_0\cdot\boldsymbol{\nabla}\theta$ is approximately constant on flux surfaces (to order $\mathcal{O}\{\delta\}$); 3) the current density $\mathbf{J}$ is as given in Eqs. (26) and (29) in [11]; 4) the FSA neoclassical Ohm's law given in Eq. (39) in [11] is obtained with the parallel resistivity given by (40) and bootstrap current by (41); 5) the poloidal ion flow is as indicated in Eqs. (44) and (45) in [11]; 6) the radial flows $\mathbf{V}\cdot\boldsymbol{\nabla}\psi_p$ for both electrons and ions are order δ smaller than the corresponding poloidal and toroidal flows within flux surfaces; and 7) for an axisymmetric equilibrium $\mathbf{B}_0$ field and a pressure anisotropy $\pi_\parallel$ that is independent of the toroidal angle, the viscous force does not cause a toroidal torque on either plasma species (i.e., $\langle R^2\boldsymbol{\nabla}\zeta\cdot\boldsymbol{\nabla}\cdot\boldsymbol{\pi}_\parallel\rangle = 0$). Ultimate verification and validation of the proposed form for $\pi_\parallel$ given by (20) and the viscous force it causes will be through detailed

comparisons for a wide range of applications with results from kinetic-based approaches, such as those being developed by Held et al. [16].

Transport-time-scale effects: The main residual viscous force effects induced by collision-induced parallel viscous stresses on the long transport time scale are (analytically) to enforce ambipolarity through first order in the gyroradius (via Eqs. (42)–(45) of [11]) and to cause the banana-plateau (ambipolar) neoclassical radial particle flux, as indicated in Eq. (89) in [11]. In addition, their inclusion facilitates the analytic derivation of a toroidal flow (rotation) equation on the transport time scale, Eq. (119) of [11]. Suggested verification tests would be to: 1) obtain the banana-plateau radial particle flux indicated in Eq. (89) of [11] and check that it is ambipolar; and 2) show that the toroidal flow obeys the toroidal rotation evolution equation given by Eq. (119) in [11], after irrelevant terms there are eliminated.

VI. DISCUSSION AND SUMMARY

As indicated by the boxed equations above and the paragraph after (20), a new procedure has been proposed for including viscous force effects caused by collision-induced parallel viscous stresses in high (Braginskii [4], Pfirsch-Schlüter), intermediate (plateau) and low (banana) collisionality regimes. The proposed procedure uses standard Braginskii collisional viscous forces with coefficients η_0 for relaxing MHD-type responses, but a new multi-collisionality regime residual viscous force on collision time scales. Specifically, the new procedure uses the normal CGL pressure tensor form in (6) and viscous force definition in (12), but suggests a new form for the pressure anisotropy given by (20). The multi-collisionality regime poloidal flow viscous damping frequency μ_s is given in (B8), (B12), (B14) and the constant k_s for the intrinsic poloidal flow $U_{s\theta}^0$ defined in (19) is given by (B11), (B13), (B15). The relevant collisional (Braginskii) regime viscosity coefficients η_{s00} for electrons, ions in impure plasmas, as is typical in tokamaks, are defined in (A11), (A17).

Some issues and limitations regarding use of this procedure to represent viscous force effects due to collisional parallel viscous stresses in various collisionality regimes are: 1) Is the form of the viscous force given by the combination of (12) and (20) really the best or most appropriate form? 2) Are long scale ($\gg \pi R_0 q$) parallel variations in flow components appropriately and adequately relaxed with the Braginskii η_{00} viscosity coefficients even in low collisionality regimes where "collisionless" closures [16, 17] become relevant? 3) While

the exponential temporal decay of poloidal flows resulting from (18) is not precisely correct in low collisionality regimes [18], is this residual viscous force sufficient, except perhaps for applications where the poloidal flow dynamics is critical, since it produces the correct equilibrium flows? 4) Are the approximations that lead to the "offset" poloidal flows $U_{s\theta}^0$ sufficiently accurate for extended MHD modeling in the M3D and NIMROD codes? and 5) While the poloidal variation of the viscous force in the banana collisionality regime [19] is not correct, is this residual viscous force sufficient, except perhaps for applications that depend critically on the poloidal variation of viscous force effects, since its flux-surface average is correct?

The fate of this proposal in extended MHD codes will depend on the practicality of its implementation and its usefulness in capturing the most important viscous force effects in low collisionality toroidal plasmas in the M3D and NIMROD codes.

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Appendix A: Collisonal (Braginskii-type) Viscosities

The Braginskii [4] closures for the parallel viscous stress $\boldsymbol{\pi}_{\|}$ were developed for collisional plasmas (i.e., $|\lambda\boldsymbol{\nabla}\mathbf{V}| \ll 1$) and MHD-type applications where the flow velocity $\mathbf{V}$ is of order the $\mathbf{E}\times\mathbf{B}$ flow velocity and large compared to the diamagnetic flows. In particular, $\mathbf{V}$ is assumed to be large compared to the heat flow velocity $\mathbf{V}_q \equiv (-2\mathbf{q}/5nT)$ and higher order flow-type moments (energy-weighted heat flow etc.) — but still gyroradius small compared to thermal speeds: $|\mathbf{V}|/v_T \sim \delta \ll 1$. However, as noted at the end of Section II, since two-fluid treatments include diamagnetic flows, the diamagnetic-type heat flow $\mathbf{V}_q$ is comparable to the diamagnetic flow $\mathbf{V}_*$ and cannot be neglected. Then, the rate of strain

tensor is modified: $\mathsf{W}_V \rightarrow \mathsf{W}_V + \mathsf{W}_q$ in which the rate of strain tensor for heat flows is given in (9). Similarly, the stress tensor gets modified: $\boldsymbol{\pi} \rightarrow \boldsymbol{\pi}_0 + \boldsymbol{\pi}_1 + \cdots \equiv \boldsymbol{\pi}_V + \boldsymbol{\pi}_q + \cdots$. Here, the subscript indicates the order j of the energy weighting Laguerre polynomial in the relevant moments of the distribution function ($\mathbf{v}' \equiv \mathbf{v} - \mathbf{V}$):

$$\boldsymbol{\pi}_j \equiv \int d^3v'\, m\, [\mathbf{v}'\mathbf{v}' - (v'^2/3)\, \mathsf{I}\,]\, L_j^{5/2}(mv'^2/2T)\, f(\mathbf{x}, \mathbf{v}, t), \tag{A1}$$

in which $L_j^{5/2}(x)$ are Laguerre polynomials: $L_0^{5/2} = 1$, $L_1^{5/2} = 7/2 - x, \cdots$.

Taking the $\int d^3v'\, m\, \hat{\mathbf{b}} \cdot [\mathbf{v}'\mathbf{v}' - (v'^2/3)\, \mathsf{I}\,] \cdot \hat{\mathbf{b}}\, L_j^{5/2}(mv'^2/2T)$ moments with $j = 0, 1$ of a Chapman-Enskog form of the collisional equilibrium ($\partial/\partial t < \nu_s$) plasma kinetic equation and neglecting higher order (in a collisional regime) $\hat{\mathbf{b}} \cdot (\boldsymbol{\pi} \cdot \boldsymbol{\nabla}\mathbf{V}) \cdot \hat{\mathbf{b}}$ viscous-dissipation-type terms yields a matrix equation for each plasma species:

$$n_s\, T_s \begin{pmatrix} \hat{\mathbf{b}} \cdot \mathsf{W}_V^s \cdot \hat{\mathbf{b}} \\ \hat{\mathbf{b}} \cdot \mathsf{W}_q^s \cdot \hat{\mathbf{b}} \end{pmatrix} = -\, \frac{6}{5\, \tau_{ss}}\, \mathsf{G}_s \cdot \begin{pmatrix} (2/3)\, \pi_{s0\|} \\ (2/3)\, \pi_{s1\|} \end{pmatrix}. \tag{A2}$$

Here, τ_{ss} is a reference self-collision frequency for a plasma species s that is defined by

$$\boxed{\frac{1}{\tau_{ss}} \equiv \frac{4}{3\sqrt{\pi}}\, \frac{4\pi\, n_s q_s^4 \ln \Lambda}{\{4\pi\epsilon_0\}^2\, m_s\, v_{Ts}^3},} \quad \text{reference collision frequency.} \tag{A3}$$

The matrix G_s is a 2×2 matrix of Coulomb collisional "drag" coefficients on the stresses that result from the parallel stress moments of the collision operator. The parallel viscous stresses $\pi_{s0\|}$ and $\pi_{s1\|}$ will be obtained by multiplying this equation by the inverse G_s^{-1} of the matrix G_s for each species s. [If higher order energy moments are included (i.e., $j \geq 2$), they yield $(j{+}1) \times (j{+}1)$ matrices and $j{+}1$ equations; however, after inverting the larger G_s matrices the results obtained below change less than the $1/\ln\Lambda \sim 6$ % intrinsic accuracy of the Fokker-Planck Coulomb collision operator and hence are not warranted.)

The collisional matrix and its inverse can be written in general as [8, 20]

$$\mathsf{G}_s = Z \begin{pmatrix} 1 & \frac{3}{2} \\ \frac{3}{2} & \frac{17}{4} \end{pmatrix} + \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & \frac{3}{4} \\ \frac{3}{4} & \frac{205}{48} \end{pmatrix}, \quad \mathsf{G}_s^{-1} = \frac{\begin{pmatrix} \frac{17\,Z}{4} + \frac{205}{48\sqrt{2}} & -(\frac{3\,Z}{2} + \frac{3}{4\sqrt{2}}) \\ -(\frac{3\,Z}{2} + \frac{3}{4\sqrt{2}}) & Z + \frac{1}{\sqrt{2}} \end{pmatrix}}{2\, Z^2 + 301\, Z/48\sqrt{2} + 89/48}. \tag{A4}$$

Here, the first matrix in G_s represents collisions of a species s with a species s' of charge Z that has a much larger mass ($m_{s'} >> m_s$) and the second matrix represents self-collisions

within the s species. Dotting the inverse G_s^{-1} with (A2) yields the collisional pressure anisotropy induced by the flow and heat flow rates of strain within the s species of

$$\pi_{s\|} = -\,(3/2)\,[\,\eta_{00}^s\,\hat{\mathbf{b}}\cdot\mathbf{W}_V^s\cdot\hat{\mathbf{b}} + \eta_{01}^s\,\hat{\mathbf{b}}\cdot\mathbf{W}_q^s\cdot\hat{\mathbf{b}}\,. \tag{A5}$$

Here, the viscosity coefficient for each species s are

$$\eta_{00}^s \;=\; \frac{5}{6}\,\mathsf{G}_{s00}^{-1}\,n_sT_s\,\tau_{ss} = \frac{5}{6}\,\frac{17\,Z/4 + 205/48\sqrt{2}}{2\,Z^2 + 301\,Z/48\sqrt{2} + 89/48}\,n_sT_s\,\tau_{ss}, \tag{A6}$$

$$\eta_{01}^s \;=\; \frac{5}{6}\,\mathsf{G}_{s01}^{-1}\,n_sT_s\,\tau_{ss} = -\,\frac{5}{6}\,\frac{3\,Z/2 + 3/4\sqrt{2}}{2\,Z^2 + 301\,Z/48\sqrt{2} + 89/48}\,n_sT_s\,\tau_{ss}. \tag{A7}$$

For an electron-ion plasma with a hydrogenic ion species (i.e., $Z=1$), $\nu_e = 1/\tau_{ee}$ and the numerical coefficient in η_{00}^e is 0.73, in agreement with the Braginskii coefficient η_0^e given in (5). For only one species of ions $Z\to 0$ in (A6), $\tau_{ii}\equiv\tau_i = 1/(\sqrt{2}\,\nu_i)$ and the numerical coefficient in η_{00}^i from (A6) is $(5/6)(205/89\sqrt{2})(1/\sqrt{2}) = 0.96$, in agreement with η_0^i in (5).

Typical tokamak plasmas have small admixtures of impurity (non-hydrogenic) ions. For collisions of electrons with hydrogenic ions (subscript i, $Z_i=1$) and various types of impurity ions (subscript I, charge Z_I), the effective ion charge is

$$\boxed{Z_{\rm eff} \equiv \frac{n_i + \sum_I n_I Z_I^2}{n_e},} \quad \text{ion charge for electron collisions in an impure plasma.} \tag{A8}$$

The electron collision frequency and length are defined for an impure plasma by

$$\boxed{\nu_e \equiv \frac{4\sqrt{2\pi}\,(\sum_i n_i Z_i^2)\,e^4\,\ln\Lambda}{\{4\pi\epsilon_0\}^2\,3\,m_e^{1/2}T_e^{3/2}} \;\simeq\; \frac{5\times 10^{-11}\,n_e(\mathrm{m}^{-3})\,Z_{\rm eff}}{[T_e(\mathrm{eV})]^{3/2}}\left(\frac{\ln\Lambda}{17}\right),} \tag{A9}$$

$$\lambda_e = \frac{v_{Te}}{\nu_e} \;\simeq\; 1.2\times 10^{16}\,\frac{[T_e(\mathrm{eV}]^2}{Z_{\rm eff}\,n_e(\mathrm{m}^{-3})}. \tag{A10}$$

For an impure plasma the electron viscosity coefficients obtained from (A6) and (A7) are

$$\eta_{00}^e \;=\; \frac{5}{12}\,\frac{17\,Z_{\rm eff}^2/4 + 205\,Z_{\rm eff}/48\sqrt{2}}{2\,Z_{\rm eff}^2 + 301\,Z_{\rm eff}/48\sqrt{2} + 89/48}\,m_en_e\,\nu_e\lambda_e^2, \tag{A11}$$

$$\eta_{01}^s \;=\; -\,\frac{5}{12}\,\frac{3\,Z_{\rm eff}^2/2 + 3\,Z_{\rm eff}/4\sqrt{2}}{2\,Z_{\rm eff}^2 + 301\,Z_{\rm eff}/48\sqrt{2} + 89/48}\,m_en_e\,\nu_e\lambda_e^2. \tag{A12}$$

With only hydrogenic ions ($Z_{\rm eff}\to 1$), η_{00}^e reduces to the Braginskii η_0^e in (5).

For collisions of hydrogenic ions (subscript i, $Z_i=1$) with various heavier impurity ions (subscript I, charge Z_I) the effective ion charge is

$$\boxed{Z_* \equiv \frac{\sum_I n_I Z_I^2}{n_i},} \quad \text{ion charge for hydrogenic ion collisions in an impure plasma.} \tag{A13}$$

Note that the effective ion charge for electron collisions can be written in terms of this effective charge for hydrogenic ions via $Z_{\rm eff} = (n_i/n_e)(1+Z_*)$ or $Z_* = [(n_e/n_i)Z_{\rm eff} - 1]$. The ion collision frequency and length can be defined for an impure plasma in terms of the conventional deuterium (mass m_D) ion collision frequency $[1/\tau_{ii} = \nu_i\sqrt{2}/(1+\sqrt{2}Z_*)]$:

$$\boxed{\nu_i \equiv \frac{4\sqrt{\pi}(n_i + \sqrt{2}\sum_I n_I Z_I^2)e^4 \ln\Lambda}{\{4\pi\epsilon_0\}^2\, 3\, m_i^{1/2} T_i^{3/2}} \simeq \frac{5.8\times 10^{-13} n_i({\rm m}^{-3})(1+\sqrt{2}Z_*)}{(m_i/m_D)^{1/2}\,[T_i({\rm eV})]^{3/2}}\left(\frac{\ln\Lambda}{17}\right),} \tag{A14}$$

$$\lambda_i = \frac{v_{Ti}}{\nu_i} \simeq 1.7\times 10^{16}\frac{[T_i({\rm eV}]^2}{(1+\sqrt{2}Z_*)\, n_i({\rm m}^{-3})}\ {\rm m}. \tag{A15}$$

Neglecting the small variations in the $\ln\Lambda$ coefficients, the ion collision frequency in an impure plasma can be written in terms of the electron collision frequency in (A9):

$$\nu_i \;=\; \frac{n_i}{n_e}\left(\frac{m_e}{m_i}\right)^{1/2}\left(\frac{T_e}{T_i}\right)^{3/2}\frac{1+\sqrt{2}\,Z_*}{\sqrt{2}\,Z_{\rm eff}}\,\nu_e \;\sim\; 10^{-2}\,\nu_e. \tag{A16}$$

For an impure plasma the ion viscosity coefficients obtained from (A6) and (A7) are

$$\eta_{00}^i \;=\; \frac{5(1+\sqrt{2}Z_*)}{12\sqrt{2}}\;\frac{17\,Z_*/4 + 205/48\sqrt{2}}{2\,Z_*^2 + 301\,Z_*/48\sqrt{2} + 89/48}\,m_i n_i\,\nu_i\lambda_i^2, \tag{A17}$$

$$\eta_{01}^i \;=\; -\,\frac{5(1+\sqrt{2}Z_*)}{12\sqrt{2}}\;\frac{3\,Z_*/2 + 3/4\sqrt{2}}{2\,Z_*^2 + 301\,Z_*/48\sqrt{2} + 89/48}\,m_i n_i\,\nu_i\lambda_i^2. \tag{A18}$$

With only hydrogenic ions ($Z_* \to 0$), η_{00}^i reduces to the Braginskii η_0^i in (5).

Appendix B: Viscosity Coefficients For Multi-Collisionality Regimes

There is one fundamental approximation used in obtaining the Braginskii viscous stresses that is not appropriate for extended MHD descriptions of tokamak plasmas. Namely, the collision length is assumed to be shorter than parallel inhomogeneity scale lengths of the flow velocity $\mathbf{V}$ (i.e., $|\lambda\,\boldsymbol{\nabla}_\parallel \mathbf{V}| \ll |\mathbf{V}|$). It is proposed here that this shortcoming be rectified by incorporating neoclassical-based closures for collisional effects on the residual parallel viscous force [8]. The lowest (banana) collisionality regime will be discussed first; next, scalings of low, intermediate and high collisionality regimes will be discussed. Finally, multi-collisionality forms for the flux surface average (FSA) of the residual parallel viscous force will be developed.

The low collisionality "banana" regime [7, 8] is defined to be that where trapped particles circumnavigate their banana drifts orbits without suffering enough collisions to scatter them

out of the trapped particle region of velocity space. In a fluid description the consequences of this physics are captured [8] through the residual parallel viscous force induced by collisions of untrapped (circulating) particles that carry the flow with "immobile" trapped particles. In the banana collisionality regime the residual parallel viscous force can be written [8] for each species in the form of the corresponding FSA of the Braginskii closure relation in (17):

$$\langle \mathbf{B}_0 \cdot \mathbf{F}_\pi \rangle \equiv - \langle \mathbf{B}_0 \cdot \boldsymbol{\nabla} \cdot \boldsymbol{\pi}_\parallel \rangle = - mn \, \langle B_0^2 \rangle \, [\, \mu_{00} \, U_\theta - \mu_{01} \, (2/5nT) \, Q_\theta \,]. \tag{B1}$$

The poloidal flow damping frequencies on the flows and heat flows can be written for electrons colliding with ions of charge Z as [see Eqs. (4.18), (4.20), (4.21) and (4.61)–(4.63) in [8]]:

$$\mu_{00}^e = \left[Z + \sqrt{2} - \ln(1+\sqrt{2}) \right] \frac{f_t}{\tau_{ee} f_c} \simeq \frac{Z + 0.533}{Z} \, (1.46\sqrt{\epsilon}\,) \, \nu_e, \tag{B2}$$

$$\mu_{01}^e = - \left[\left(Z + \frac{1}{\sqrt{2}} \right) - \frac{5}{2} [Z + \sqrt{2} - \ln(1+\sqrt{2})] \right] \frac{f_t}{\tau_{ee} f_c} \simeq \frac{1.5Z + 0.625}{Z} \, (1.46\sqrt{\epsilon}\,) \, \nu_e. \tag{B3}$$

As for the collisional viscosity coefficients in (A6) and (A7), the ion banana regime poloidal flow damping coefficients are obtained by setting $Z \to 0$, $\tau_{ee} \to \tau_{ii}$ in the first forms. In (B2) and (B3) the flow-weighted fraction of circulating particles f_c is defined by [8, 9]

$$\boxed{f_c \equiv \frac{3}{4} \langle B_0^2 \rangle \int_0^{1/B_{\max}} \frac{\lambda \, d\lambda}{\langle \sqrt{1 - \lambda \, B_0(\theta)} \, \rangle} \simeq 1 - 1.46 \, \sqrt{\epsilon} + 0.46 \, \epsilon \sqrt{\epsilon},} \quad \text{circulating particle fraction.} \tag{B4}$$

The complementary fraction of trapped particles is $f_t \equiv 1 - f_c \simeq 1.46\sqrt{\epsilon} - 0.46 \, \epsilon\sqrt{\epsilon}$. In the approximate forms at the end of all these formulas the variation of the magnetic field strength on a magnetic flux surface has been approximated by $B_0 \simeq B_0 R_0 / R \simeq B_{00}(1 - \epsilon \cos\theta)$ in which $\boxed{\epsilon = (B_{\max} - B_{\min})/(B_{\max} + B_{\min}) \simeq r/R_0 \ll 1.}$

If the residual parallel viscous force in (B1) is dominant in the FSA parallel momentum equation of a plasma species, it forces the poloidal flow to an "offset" value determined by the poloidal heat flow: $U_\theta^0 = (\mu_{01}/\mu_{00}) \, (2/5nT) \, Q_\theta$. In the banana collisionality regime, to lowest order in $\sqrt{\epsilon}$, collisions force the parallel ion heat flow to zero and poloidal heat flow is induced by the diamagnetic ion heat flow; then, it is given by [7, 8, 11] $(2/5n_iT_i) \, Q_{i\theta} \simeq [I/(q_i \langle B_0^2 \rangle)] \, dT_{i0}/d\psi_p$. Thus, the poloidal ion offset flow velocity is (for $\sqrt{\epsilon} \ll 1$)

$$U_{i\theta}^0 \simeq \frac{\mu_{i01}}{\mu_{i00}} \frac{2}{5n_iT_i} Q_{i\theta} \simeq \frac{0.625}{0.533} \frac{I}{q_i \langle B_0^2 \rangle} \frac{dT_{i0}}{d\psi_p} \simeq 1.17 \frac{1}{B_p} \frac{1}{Z_i e B_0} \frac{dT_{i0}}{dr}, \quad \text{banana regime.} \tag{B5}$$

This result indicates a diamagnetic-type offset poloidal ion flow speed of $V_{ip} \simeq B_p U_{i\theta}^0 \simeq [1.17/(Z_i e B_0)](dT_{i0}/dr)$ driven by the ion radial temperature gradient, a familiar asymptotic limit in neoclassical transport theory [7, 8].

Collisionality regimes in tokamak plasmas are defined by the ratio of the effective collision frequency of trapped particles $\nu_{\rm eff} \sim \nu/f_t^2 \sim \nu/\epsilon$ to their bounce frequency $\omega_b \sim \sqrt{\epsilon}\, v_T/R_0 q$:

$$\boxed{\nu_* \equiv \frac{\nu_{\rm eff}}{\omega_b} = \frac{\nu}{\epsilon^{3/2}(v_T/R_0 q)} = \frac{R_0 q}{\epsilon^{3/2}\lambda}, \quad \text{collisionality regime parameter.}} \tag{B6}$$

The three relevant collisionality regimes (for each species) are

$$\begin{aligned} &\nu_* \ll 1, && \text{low (banana) collisionality regime,} \\ &1 \ll \nu_* \ll \epsilon^{-3/2}, && \text{intermediate (plateau) collisionality regime,} \\ &\epsilon^{-3/2} \ll \nu_*, && \text{high (Braginskii, Pfirsch-Schlüter) collisionality regime.} \end{aligned} \tag{B7}$$

The FSA residual parallel viscous force in the banana regime was defined in (B1). The Pfirsch-Schlüter regime is [7, 8] the high collisionality Braginskii regime whose FSA parallel viscous force, which is of the same form as (B1), was given in (17). The plateau regime is an intermediate collisionality regime where typical untrapped particles are collisionless, but trapped and low parallel velocity particles drift radially off flux surfaces, which causes radial plasma transport and parallel viscous forces. The FSA of the residual parallel viscous force in the plateau collisionality regime can also be written in the same form as (B1) with viscous damping coefficients that can be obtained from (4.65) in [8]. Their poloidal flow damping frequencies scale as $\mu \sim \epsilon^2 v_T/R_0 q$, i.e., independent of collision frequency.

For extended MHD simulations it is convenient to specify the FSA of the residual parallel viscous force in terms of the damping of the poloidal flow to an "offset" flow velocity U_θ^0 in the form (for each plasma species) indicated in (18) and (19). Multi-collisionality forms of the parallel viscosity coefficients μ_e, μ_i for electrons, ions have been developed [7–9]. They can be written in the form [7] $\mu \sim \sqrt{\epsilon}\,\nu/[(1+\nu_*^{1/2}+\nu_*)(1+\epsilon^{3/2}\nu_*)]$, with various order unity numerical factors in front of each of the factors. Here, the $\nu_*^{1/2}$ factor in the denominator arises from [7] collisional boundary layer effects in the vicinity of the velocity-space boundary between trapped and untrapped (circulating) particles.

In general the offset poloidal ion flow $U_{i\theta}^0$ in (19) and (B5) depends on impurity density and temperature gradients, and the collisionality regimes of impurities as well as that of the dominant ion species [9]. In transport codes the constant k_i is often evaluated using the NCLASS code [10]. However, since impurity collision frequencies are usually much larger than ion collision frequencies, impurities have higher ν_* values ($\nu_{*I}/\nu_{*i} \sim Z_I^2 \gg 1$); hence, they are often in the plateau or even Pfirsch-Schlüter collisionality regimes. There, it can

be shown [9] that to lowest order the offset poloidal ion flow $U_{i\theta}^0$ is proportional to the ion temperature gradient as indicated in (19) with the coefficient k_i determined mainly by the collisionality regime of the hydrogenic ions. Very deep in the ion banana collisionality regime (i.e., $\nu_{*i} \ll 1$) the impurity ion density and temperature gradient effects should be taken into account by using NCLASS [10] to obtain the offset flows $U_{i\theta}^0$. However, this very low hydrogenic ion collisionality regime is only barely reached in most present tokamak plasmas and when impurity flows become important one should really be solving three-fluid equations that include impurity density, momentum and energy equations. Thus, for the purpose of extended MHD codes it will be assumed that the offset poloidal flows $U_{s\theta}^0$ can be represented in terms of the respective temperature gradients as indicated in (19).

In this work descriptions for the residual parallel viscous forces are desired that both encompass all three collisionality regimes and asymptotically approach the low (banana) collisionality regime results for $\nu_* \ll 1$ and the high (Braginskii) collisionality regime when $\nu_* \gg \epsilon^{-3/2}$. Also, small admixtures of impurities should be allowed for since tokamak plasmas often have $Z_{\rm eff} \sim 2$–3. Such descriptions have been developed by Kim et al. [9] from the general formulas presented in [8]. Using those results (but correcting the coefficient of the Pfirsch-Schlüter term in the denominator from 1/6 to 2/3 to obtain the correct high collisionality limit) and adding collisional boundary layer effects [7], the poloidal damping frequency of each species s in (18) is proposed to be, in the spirit of a Padé approximation,

$$\boxed{\mu_s \;\equiv\; \mu_{00}^s \;=\; \frac{(f_t/f_c)\,\hat{K}_{00}^b/\tau_{ss}}{\left[1+\nu_{*s}^{1/2}+2.92\,\nu_{*s}\hat{K}_{00}^b/\hat{K}_{00}^p\right]\left[1+2\hat{K}_{00}^p/(3\,\omega_{ts}\tau_{ss}\hat{K}_{00}^{PS})\right]}.} \tag{B8}$$

Here, $\hat{K}_{00}^b$, $\hat{K}_{00}^p$ and $\hat{K}_{00}^{PS}$ quantities are dimensionless viscosity coefficients in the banana, plateau and Pfirsch-Schlüter collisionality regimes, respectively; they are given in Table I.

The collisionality parameter ν_{*s} is solely a function of the poloidal flux ψ_p and is specified for a general axisymmetric magnetic field geometry by [9] (for each species)

$$\boxed{\nu_{*s} \;\equiv\; \frac{f_t/f_c}{1.46}\,\frac{\omega_{ts}}{v_{Ts}^2\tau_{ss}}\,\frac{\langle B_0^2\rangle}{2\langle(\hat{\mathbf{b}}\cdot\boldsymbol{\nabla}B_0)^2\rangle} \;\sim\; \frac{\nu_s}{\epsilon^{3/2}v_{Ts}/R_0q},} \quad \text{general collisionality parameter.} \tag{B9}$$

Also, the characteristic transit frequency for each species of untrapped particles is

$$\omega_{ts} \equiv v_{Ts}/R_0q, \quad \text{transit frequency.} \tag{B10}$$

As indicated in (B9), in a large aspect ratio tokamak where $\sqrt{\epsilon} \ll 1$ one obtains $f_t/f_c \sim 1.46\sqrt{\epsilon}$, $\langle B_0^2\rangle \simeq B_0^2$ and $\langle(\hat{\mathbf{b}}\cdot\boldsymbol{\nabla}B_0)^2\rangle \simeq \epsilon^2 B_0^2/(2R_0^2q^2)$ and (B9) reduces to the usual (B6).

TABLE I: ASYMPTOTIC DIMENSIONLESS VISCOSITY COMPONENTS.

In impure plasmas $Z \to Z_{\rm eff}$ for electrons, $Z \to Z_*$ for ions and $Z \to 1/Z_*$ for impurities.

In the rightmost column $D \equiv (6/5)(2Z^2 + 301/48\sqrt{2} + 89/48) \simeq 2.40Z^2 + 5.32Z + 2.225$.

collisionality regime:	banana (b)	plateau (p)	Pfirsch-Schlüter (PS)
$\hat{K}_{00}$	$Z + \sqrt{2} - \ln(1+\sqrt{2})$	$\sqrt{\pi}$	$(17Z/4 + 205/48\sqrt{2})/D$
	$\simeq Z + 0.533$	$\simeq 1.77$	$\simeq (4.25Z + 3.02)/D$
$\hat{K}_{01}$	$Z + 1/\sqrt{2}$	$3\sqrt{\pi}$	$(7/2)(23Z/4 + 241/48\sqrt{2})/D$
	$\simeq Z + 0.707$	$\simeq 5.32$	$\simeq (20.13Z + 12.43)/D$
$\hat{K}_{11}$	$2Z + 9/4\sqrt{2}$	$12\sqrt{\pi}$	$(49/4)(33Z/4 + 325/48\sqrt{2})/D$
	$\simeq 2Z + 1.591$	$\simeq 21.27$	$\simeq (101.06Z + 58.65)/D$

In the asymptotic banana regime ($\nu_{*s} \to 0$) the μ_s in (B8) reduces to $(f_t/f_c)\hat{K}^b_{00}/\tau_{ss}$ which is the same result as was given in (B2). Similarly, in the asymptotic Pfirsch-Schlüter regime ($\nu_{*s} \gg \epsilon^{-3/2}$ or $\omega_{ts}\tau_{ss} \gg 1$) the μ_s in (B8) reduces to $(3T_s\tau_{ss}/m_s)\hat{K}^{PS}_{00}\langle(\hat{\mathbf{b}}\cdot\boldsymbol{\nabla}B_0)^2\rangle/\langle B_0^2\rangle$, which in turn yields $3\,\eta^s_{00}\langle(\hat{\mathbf{b}}\cdot\boldsymbol{\nabla}B_0)^2\rangle/(m_s n_s\langle B_0^2\rangle)$ in which the η^s_{00} coefficients are the generalized Braginskii coefficients given in (A6).

A multi-collisionality formula for the coefficient k in the intrinsic poloidal flow U^0_θ can be constructed similarly for each species from Eqs. (C.19)–(C.21) and Table 1 of [9]:

$$\boxed{k_s \equiv \frac{\mu^s_{01}}{\mu^s_{00}} = \frac{5}{2} - \frac{\hat{K}_{01}}{\hat{K}_{00}} = \frac{5}{2} - \frac{\hat{K}^b_{01}}{\hat{K}^b_{00}}\,\frac{1+\nu_{*s}^{1/2}+2.92\,\nu_{*s}\hat{K}^b_{00}/\hat{K}^p_{00}}{1+\nu_{*s}^{1/2}+2.92\,\nu_{*s}\hat{K}^b_{01}/\hat{K}^p_{01}}\,\frac{1+2\hat{K}^p_{00}/(3\,\omega_{ts}\tau_{ss}\hat{K}^{PS}_{00})}{1+2\hat{K}^p_{01}/(3\,\omega_{ts}\tau_{ss}\hat{K}^{PS}_{01})}.} \tag{B11}$$

To lowest order in $\sqrt{\epsilon}$, for hydrogenic ions and no impurities ($Z_{\rm eff} \to 1$), k_i is 1.17, -0.5 and -1.6 in the banana, plateau and Pfirsch-Schlüter collisionality regimes. Similarly, for electrons, hydrogenic ions and no impurities, k_e is 1.39, -0.5 and -2.0 in these regimes.

These coefficients for the poloidal flow damping frequency μ and coefficient k in the offset poloidal flow yield numerically correct FSA parallel viscous forces (B1) and (17) in the asymptotic limits of the banana and Pfirsch-Schlüter collisionality regimes. However, they overestimate them slightly in intermediate collisionality regimes — the μ value can be a factor of order 1.4 too large in the plateau collisionality regime (see Fig. 1 in [8]).

In the large aspect ratio limit $\sqrt{\epsilon} \ll 1$ the viscous damping frequency μ and offset flow coefficient k can be simplified. Namely, using the definition in (B8) and the specifications of the coefficients in Table I one obtains for electrons

$$\boxed{\mu_e} \simeq \frac{1.46\sqrt{\epsilon}(1+\frac{0.533}{Z_{\rm eff}})\,\nu_e}{\left[1+\nu_{*e}^{1/2}+1.65(1+\frac{0.533}{Z_{\rm eff}})\nu_{*e}\right]\left[1+1.18\frac{2.4Z_{\rm eff}^2+5.32Z_{\rm eff}+2.225}{Z_{\rm eff}(4.25Z_{\rm eff}+3.02)}\,\epsilon^{3/2}\nu_{*e}\right]}$$

$$\overset{Z_{\rm eff}=2.5}{\simeq} \boxed{\frac{1.77\sqrt{\epsilon}\,\nu_e}{(1+\nu_{*e}^{1/2}+2\,\nu_{*e})\left(1+1.06\,\epsilon^{3/2}\nu_{*e}\right)}.} \tag{B12}$$

The corresponding electron coefficient for the offset poloidal flow is

$$\boxed{k_e} \simeq \frac{5}{2}-\frac{Z_{\rm eff}+0.707}{Z_{\rm eff}+0.533}\,\frac{1+\nu_{*e}^{1/2}+1.65(1+\frac{0.533}{Z_{\rm eff}})\nu_{*e}}{1+\nu_{*e}^{1/2}+0.55(1+\frac{0.707}{Z_{\rm eff}})\nu_{*e}}\,\frac{1+1.18\frac{2.4Z_{\rm eff}^2+5.32Z_{\rm eff}+2.225}{Z_{\rm eff}(4.25Z_*+3.02)}\epsilon^{3/2}\nu_{*e}}{1+3.54\frac{2.4Z_{\rm eff}^2+5.32Z_{\rm eff}+2.225}{Z_{\rm eff}(20.13Z_{\rm eff}+12.43)}\epsilon^{3/2}\nu_{*e}}$$

$$\overset{Z_{\rm eff}=2.5}{\simeq} \boxed{\frac{5}{2}-1.06\,\frac{1+\nu_{*e}^{1/2}+2\,\nu_{*e}}{1+\nu_{*e}^{1/2}+0.70\,\nu_{*e}}\,\frac{1+1.06\,\epsilon^{3/2}\nu_{*i}}{1+0.69\,\epsilon^{3/2}\nu_{*e}}.} \tag{B13}$$

The last approximate forms are appropriate for most tokamak plasmas where $Z_{\rm eff}$ is often in the range of 2–3. The electron collision frequency ν_e is given in (A9) and the electron collisionality parameter ν_{*e} is defined in (B6) with $\nu \to \nu_e \equiv Z_{\rm eff}/\tau_{ee}$, $v_T \to v_{Te} \equiv \sqrt{2T_e/m_e}$ and $\lambda \to \lambda_e \equiv v_{Te}/\nu_e$, which is given in (A10). The boundary layer term does not have a $Z_{\rm eff}$ factor in it because the full ν_e influences pitch-angle scattering in velocity space.

The corresponding ion poloidal viscous damping frequency is given by (for $\sqrt{\epsilon} \ll 1$)

$$\boxed{\mu_i} \simeq \frac{1.46\sqrt{\epsilon}\,\frac{Z_*+0.533}{Z_*+0.707}\,\nu_i}{\left[1+\nu_{*i}^{1/2}+1.65\frac{Z_*+0.533}{Z_*+0.707}\nu_{*i}\right]\left[1+1.18\frac{2.4Z_*^2+5.32Z_*+2.225}{(Z_*+0.707)(4.25Z_*+3.02)}\,\epsilon^{3/2}\nu_{*i}\right]}$$

$$\overset{Z_*=3}{\simeq} \boxed{\frac{1.32\sqrt{\epsilon}\,\nu_i}{(1+\nu_{*i}^{1/2}+1.49\,\nu_{*i})\left(1+0.80\,\epsilon^{3/2}\nu_{*i}\right)}.} \tag{B14}$$

The corresponding ion coefficient for the offset poloidal flow is

$$\boxed{k_i} \simeq \frac{5}{2}-\frac{Z_*+0.707}{Z_*+0.533}\,\frac{1+\nu_{*i}^{1/2}+1.65\frac{Z_*+0.533}{Z_*+0.707}\nu_{*i}}{1+\nu_{*i}^{1/2}+0.55\nu_{*i}}\,\frac{1+1.18\frac{2.4Z_*^2+5.32Z_*+2.225}{(Z_*+0.707)(4.25Z_*+3.02)}\epsilon^{3/2}\nu_{*i}}{1+3.54\frac{2.4Z_*^2+5.32Z_*+2.225}{(Z_*+0.707)(20.13Z_*+12.43)}\epsilon^{3/2}\nu_{*i}}$$

$$\overset{Z_*=3}{\simeq} \boxed{\frac{5}{2}-1.05\,\frac{1+\nu_{*i}^{1/2}+1.57\,\nu_{*i}}{1+\nu_{*i}^{1/2}+0.55\,\nu_{*i}}\,\frac{1+0.80\,\epsilon^{3/2}\nu_{*i}}{1+0.52\,\epsilon^{3/2}\nu_{*i}}.} \tag{B15}$$

The last approximate forms are appropriate for most tokamak plasmas where Z_* is often in the range of 2–4. The ion collision frequency ν_i is given in (A14) and the ion collisionality parameter ν_{*i} is defined in (B6) with $\nu \to \nu_i \equiv (Z_* + 1/\sqrt{2})/\tau_{ii}$, $v_T \to v_{Ti} \equiv \sqrt{2T_i/m_i}$ and $\lambda \to \lambda_i \equiv v_{Ti}/\nu_i$, which is given in (A15).

[1] CEMM website: `http://w3.pppl.gov/CEMM`.

[2] M3D website: `http://w3.pppl.gov/m3d/index.php`.

[3] NIMROD website: `https://nimrodteam.org`.

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