FORMULATION OF THE RESISTIVE WALL BOUNDARY CONDITION IN NIMROD.

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This is a report outlining the formulation of the resistive wall boundary condition that is suitable for a finite element code such as NIMROD. Because NIMROD plans to use a vacuum potential code (VACUUM) to obtain information of the magnetic field beyond the resistive wall boundary, the emphasis is on the Greene's function solution of the vacuum potential and the special care needed for the n=0 case. In brief, the resistive wall boundary condition is derived from the jump conditions that connect the magnetic fields in the vacuum and in the plasma across the resistive wall, along with Ohm's law and the thin wall approximation. The result is an expression for the tangential electric field at the wall in terms of the plasma fields at the wall and two constant fields that must be supplied from other information. These axisymmetric constant fields are related to the total currents in the plasma/resistive wall system and are critical for nonlinear simulations. The tangential electric field can then be used as a boundary condition directly in the NIMROD formulation to advance the induction equation in time.

This note builds on the significant work of others. The mathematical underpinnings of the vacuum solution in general toroidal geometry have been given in the pioneering paper by Lüst and Martensen[1] (with an unpublished English translation by Dewar[2]). This solution has been implemented numerically in the VACUUM code, described in an elegant paper by Chance[3]. The VACUUM code calculates the "vacuum response matrix" that depends only on geometry and provides the coupling between the vacuum magnetic fields and the fields within the plasma domain. Since the VACUUM code was originally designed to interface with linear stability codes that consider only individual modes, it does not provide the secular (non-single valued) part of the vacuum response that is responsible for the two constant fields discussed above. Gianakon[4] has implemented a coupling between NIMROD and the VACUUM code. While this successfully addresses the problem of the complex coupling between the poloidal decomposition assumed by the VACUUM code and the finite-element representation used in NIMROD, the formulation of the resistive wall boundary condition follows that of Bondeson and Ward[5] and is not easily assimilated into the general NIMROD boundary condition algorithm, nor does it contain the secular part of the response required for a complete description of the problems of interest. Pletzer[6] outlined more satisfactory formulation of the resistive wall boundary condition, but that used a different formulation of the induction equation in NIMROD, was never implemented, and contains some errors regarding the secular terms.

Here a formulation is developed that is both general and consistent with the NIMROD boundary condition implementation. In the first section, the "natural boundary conditions" for the induction equation in a finite-element discretization are outlined. In the second section, the resistive wall boundary conditions using the thin-shell approximation are presented and related to the boundary conditions in the first section. Because these conditions require the solution of the vacuum, the Greene's function method is also reviewed in this section. The third section focuses exclusively on the importance of the secular terms, and the mathematical formalism needed for the axisymmetric toroidal case. After concluding, we discuss the effects of the non-solenoidal representation of the magnetic field in NIMROD in Appendix A, and give a concrete example of the resistive wall formalism in Appendix B.

1. BOUNDARY CONDITIONS AND THE FINITE ELEMENT METHOD

The NIMROD code uses the finite element method (FEM) for spatial discretization. We seek a formulation of the resistive wall boundary condition in axially symmetric geometry that is consistent with this implementation. The induction equation is

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E} \quad , \tag{1}$$

where $\mathbf{B}$ is the magnetic field and $\mathbf{E}$ is the electric field. In the finite-element method the dependent variables are expanded in a set of basis functions $a_{\mathbf{q}}(\mathbf{x})$. Here, the subscript $\mathbf{q}$ stands in place of the multiple indices that describe the mesh. The resulting equations are then multiplied by another member of the set, $a_{\mathbf{p}}(\mathbf{x})$, and integrated over all space to obtain the so-called weak form of the equations. Applying this *ansatz* to Equation (1), we have

$$\begin{aligned} \sum_{\mathbf{q}} \frac{d\mathbf{B}_{\mathbf{q}}}{dt} \int_V a_{\mathbf{p}}(\mathbf{x}) a_{\mathbf{q}}(\mathbf{x}) d^3\mathbf{x} &= -\int_V a_{\mathbf{p}}(\mathbf{x}) \nabla \times \mathbf{E} d^3\mathbf{x} \quad , \\ &= \int_V \nabla a_{\mathbf{p}}(\mathbf{x}) \times \mathbf{E} d^3\mathbf{x} - \oint_S a_{\mathbf{p}}(\mathbf{x}) \hat{\mathbf{n}} \times \mathbf{E} dS \quad , \end{aligned} \tag{2}$$

where $\mathbf{n}$ is the outward drawn normal to the bounding surface S, and we have integrated by parts. This can be written symbolically as

$$\sum_{\mathbf{q}} M_{\mathbf{p},\mathbf{q}} \frac{d\mathbf{B}_{\mathbf{q}}}{dt} = \mathsf{E}_{\mathbf{p}} - \mathsf{S}_{\mathbf{p}} \quad , \tag{3}$$

where $M_{\mathbf{p},\mathbf{q}}$ are the elements of the mass matrix, $\mathsf{E}_{\mathbf{p}}$ is the source of inductance, and $\mathsf{S}_{\mathbf{p}}$ is a boundary term representing the last integral on the right hand side of Equation (2). The boundary conditions enter through this term. Neglecting the surface term results in a solution that is consistent with zero tangential electric field (constant loop voltage); thus, perfectly conducting boundary conditions are the *natural* conditions for this problem. Other boundary conditions, such as those of a resistive wall, should be cast in this form to be consistent with the internal structure of the NIMROD algorithm. That is, other boundary should specify an appropriate tangential electric field, by which a surface integral can be taken. Note that due to the non-solenoidal nature of the magnetic field in the finite-element representation, NIMROD actually solves an equation with additional terms that change the equation from first-order to second-order. The full implications of this are discussed in Appendix A.

2. RESISTIVE WALL BOUNDARY CONDITION AND VACUUM RESPONSE

The boundary conditions at a resistive wall are determined by matching the internal solution (called the *plasma solution*) to the external solution (called the *vacuum solution*) using the well-known *jump conditions* on the magnetic field. (See Figure 1.) By using a thin wall approximation for the resistive wall and the method of Green's functions to obtain the vacuum response, the tangential electric field at the wall can be expressed completely in terms the internal (or plasma) magnetic field at the wall and the material properties (resistivity and thickness) of the wall. Equation (3) then allows the internal magnetic field at the wall to be advanced in time. In general, this expression will couple all points on the boundary so that an explicit advance of this equation is preferred. This should be satisfactory as long as the time step is significantly smaller than the wall time constant.

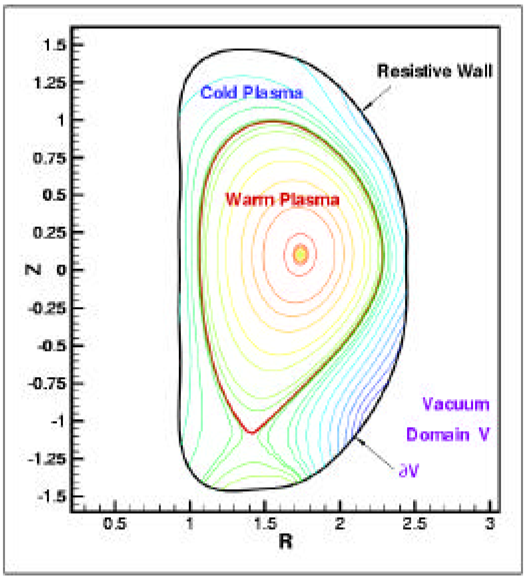


Figure 1. The NIMROD computational domain with a resistive wall boundary condition requires a vacuum solution for the region beyond the resistive wall. This solution is provide either analytically (for slab and cylindrical geometry) or computationally (VACUUM code for toroidal geometry).

The conditions satisfied by magnetic field across any surface are

$$\mathbf{n}\times[\mathbf{B}] = m_0\mathbf{K} \quad , \tag{4a}$$

$$\mathbf{n}\cdot[\mathbf{B}] = 0 \quad , \tag{4b}$$

where $\mathbf{K}$ is the surface current flowing in the interface, and the brackets [...] represent the difference between quantities on either side of the interface. For our case, $\mathbf{n}$ is the outward drawn normal to the computational domain, and

$$[f] = f_v - f_p \quad , \tag{5}$$

is the difference between vacuum (external) and plasma (internal) quantities.

In the *thin shell approximation*, the surface current $\mathbf{K}_w$ in the wall is related to the current density by $\mathbf{K}_w = d_w\mathbf{J}_w$, where d_w is the wall thickness. Ohm's law relates $\mathbf{J}_w$ to the wall resistivity h_{w} and the tangential electric field. Equation (4a) then yields an expression for the tangential electric field at the wall:

$$\mathbf{E}_{t_w} = \frac{h_w}{m_0 d_w} \mathbf{n} \times \left(\mathbf{B}_{v_w} - \mathbf{B}_{p_w} \right) \quad . \tag{6}$$

The formulation will be complete once we have independently expressed $\mathbf{B}_v$ in terms of $\mathbf{B}_p$. This will be done by writing the vacuum field as $\mathbf{B}_v = \nabla c + \mathbf{B}_{\mathrm{sec}}$. where c is the magnetic scalar potential (since $\nabla \times \mathbf{B}_v = 0$), and $\mathbf{B}_{\mathrm{sec}}$ is the *secular magnetic field.* The necessity of the secular magnetic field comes from considering the calculation of the toroidal current contained within the resistive wall and plasma:

$$\begin{aligned} I_z = \oint_{C_q} \mathbf{B}_v \bullet \mathbf{dl}_q &= \oint_{C_q} \frac{\partial c}{\partial q} dq + \oint_{C_q} \mathbf{B}_{\mathrm{sec}} \bullet \mathbf{dl}_q \\ &= \oint_{C_q} \mathbf{B}_{\mathrm{sec}} \bullet \mathbf{dl}_q \end{aligned} \tag{7}$$

If the secular, or non-periodic, terms are need if the resistive wall contains plasma current. In a nonlinear simulation, total currents will be set to zero by the boundary condition if it is not taken into account.

To complete the boundary condition, we need the solution of the magnetic vacuum field in the region beyond the resistive wall. We first consider the magnetic scalar potential. Since $\nabla \cdot \mathbf{B}_v = 0$, the vacuum potential c satisfies Laplace's equation in the external region

$$\nabla^2 c = 0 \tag{8}$$

subject to the boundary condition given by Equation (4b), ie,

$$\mathbf{n} \cdot \nabla c = \mathbf{n} \cdot \mathbf{B}_p - \mathbf{n} \cdot \mathbf{B}_{\mathrm{sec}} \tag{7}$$

on the surface S. For simplicity the case where the outer boundary is at infinity is chosen.

We will use the method of Green's functions, which relies on Green's second identity

$$\int_V \left(u \nabla^2 v - v \nabla^2 u \right) d^3\mathbf{x} = \oint_S \left(u \mathbf{n} \cdot \nabla v - v \mathbf{n} \cdot \nabla v \right) dS \quad . \tag{13}$$

We introduce the *free space Green's function*

$$G(\mathbf{x} \,|\, \mathbf{x}') = \frac{1}{|\mathbf{x} - \mathbf{x}'|} \quad , \tag{14}$$

which satisfies the equation

$$\nabla^2 G(\mathbf{x} \,|\, \mathbf{x}') = -4 p d\left(\mathbf{x} - \mathbf{x}' \right) \quad . \tag{15}$$

Here, $\mathbf{x}$ is called the observation point, and $\mathbf{x}'$ is called the source point. Identifying u in Equation (13) with c and v with G, using Equations (7), (8) and (15), and integrating over the entire external region yields

$$-4p\,c(\mathbf{x}) = \int_{S_w} \left[c(\mathbf{x}'_w)\mathbf{n}'\cdot\nabla' G(\mathbf{x}\,|\,\mathbf{x}'_w) - G(\mathbf{x}\,|\,\mathbf{x}'_w)\mathbf{n}'\cdot\mathbf{B}_p(\mathbf{x}'_w) \right] dS' \quad , \tag{16}$$

where $\mathbf{x}'_w$ ranges over the resistive wall surface S_w. Evaluating Equation (16) on the wall results in the expression

$$4pc(\mathbf{x}_w) + \int_{S_w} \mathbf{n}'\cdot\nabla' G(\mathbf{x}_w\,|\,\mathbf{x}'_w)c(\mathbf{x}'_w)dS' = \int_{S_w} G(\mathbf{x}_w\,|\,\mathbf{x}'_w)\mathbf{n}'\cdot\mathbf{B}_p(\mathbf{x}'_w)dS' \quad , \tag{17}$$

which expresses the potential on the wall in terms of the normal component of the magnetic field on the plasma and the (known) Green's function. This can be written symbolically as

$$\sum_{w'} Q(w,w')c(w') = \sum_{w'} G(w,w')B_n(w') \quad , \tag{18}$$

or

$$c(w) = \sum_{w'} V(w,w')B_n(w') \quad , \tag{19}$$

where

$$V(w,w') = \sum_{p} Q^{-1}(w,p)G(p,w') \tag{20}$$

is the *vacuum response function.* NIMROD obtains this response matrix analytically for slab and cylindrical cases, and by the VACUUM code[3] for toroidal cases.

The desired resistive wall boundary condition follows immediately. We have

$$\begin{aligned}
\mathbf{E}_{t_w} &= \frac{h_w}{m_0 d_w}\mathbf{n}\times\left[\nabla c^*(\mathbf{x}_w) + \mathbf{B}_{\text{sec}}(\mathbf{x}_w) - \mathbf{B}_p(\mathbf{x}_w)\right] \\
&= \frac{h_w}{m_0 d_w}\mathbf{n}\times\left[\nabla\sum_{w} V(w,w')B_n(w') + \mathbf{B}_{\text{sec}}(\mathbf{x}_w) - \mathbf{B}_p(\mathbf{x}_w)\right]
\end{aligned} \tag{21}$$

All quantities on the right hand are known from the geometry, from the internal solution, or from externally imposed conditions. Only derivatives tangential to the wall need be computed. Equation (21) can be used with Equation (3) to advance the internal field at the wall provided that we specify the secular magnetic field, $\mathbf{B}_{sec}$. The secular magnetic field is discussed in the next section.

3. THE SECULAR MAGNETIC FIELD

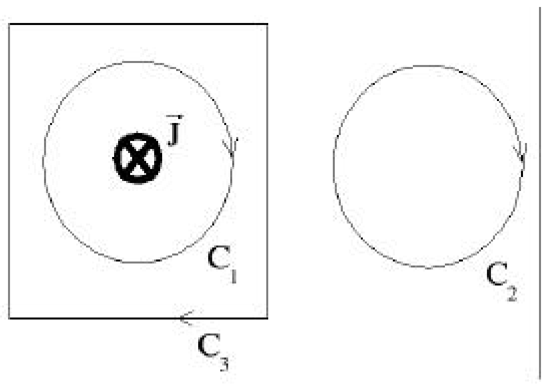
Before beginning the mathematical discussion of the formulation in the general case, the simpler two-dimension case is discussed to elucidate a subtle point necessary for understanding the mathematical formulation to be written later. Consider the (vacuum) magnetic field of a delta-function current source in two dimensions as shown if Figure 2 for the three contours shown.

As discussed above, the vacuum magnetic field can be composed into two parts: $\mathbf{B}_v = \nabla\chi + \mathbf{B}_{\sec}$. Recall from Eq. (7) that Ampere's Law was used to show $I_z = \oint_{C_\theta} \mathbf{B}_v \bullet \mathbf{dl}_\theta$ where I_z is the amount of current enclosed by the contour C_θ. We first consider the contour C_1 shown in Figure 2, which is a circle whose center is the current source. From an elementary application of Ampere's Law, the magnetic field is given by

$$\mathbf{B}_v = \frac{I}{2\pi r}\hat{\mathbf{e}}_\theta = \frac{I}{2\pi r} r\nabla\theta = \nabla\left(\frac{I}{2\pi}\theta\right) \tag{22}$$

Comparing this expression with $\mathbf{B}_v = \nabla\chi + \mathbf{B}_{\sec}$[1], it is apparent that for this contour, the magnetic field is completely described by the secular magnetic field, that is $\chi=0$ and $\mathbf{B}_v = \mathbf{B}_{\sec}$. Now consider contour, C_2. Because no current is enclosed by this contour, $\mathbf{B}_{\sec} = 0$ and $\mathbf{B}_v = \nabla\chi$ which is the opposite case of contour C_1. Note that χ will be a non-trivial solution to Poisson's equation.

The third contour is the most relevant to toroidal cases. Because the contour encloses the current source, it has a secular magnetic field component. However, it is not clear that converting Eq. (22) into a form describing the magnetic field along the contour, such that the magnetic field is completely described by the secular magnetic field, is the most advantageous. In general, the magnetic field will have both secular and single-valued components, and the decomposition into each is somewhat arbitrary. The decomposition chosen will determine the boundary conditions for the Poisson Equation for χ. .



Figur 2. Three different contours leads to three different descriptions of the magnetic field.

The decomposition chosen in this paper follows the work of Lüst and Martensen that was developed for the mathematical description of the vacuum response for the δW formulation. Their notation and formalism is used but reworked to make it more apparent for our problem. We consider a doubly-periodic system with the periodic coordinates θ and ζ. The vacuum magnetic field can be written as

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[1] This suggests the common expression $\mathbf{B}_v = \nabla\chi$ and $\chi = \chi^* + \alpha\theta$, where χ^* representing the single-valued potential and the second term representing the non-periodic (secular) part of the potential. This is a very special case, so we avoid this notation. In this paper, χ always represents the single-valued potential.

$$\mathbf{B}_v = \nabla c + g_q \mathbf{Y}_q + g_z \mathbf{Y}_z \tag{23}$$

The constants γ_i's are called the periods of the potential χ. Based on the vacuum equations for the magnetic field, $\nabla \cdot \mathbf{B}_v = 0$ and $\nabla \times \mathbf{B}_v = 0$, the vectors $\mathbf{Y}_i$ must satisfy

$$\nabla \cdot \mathbf{Y}_v = 0 \tag{24a}$$

$$\nabla \times \mathbf{Y}_v = 0 \tag{24b}$$

with the appropriate boundary conditions to be discussed below. For convenience, we will choose the normalization:

$$\oint_{C_i} \mathbf{Y}_j \bullet \mathbf{dl}_i = d^i_j \tag{25}$$

where δ^i_j is the Kronecker delta. With this normalization, the periods are easily found using Ampere's Law:[2]

$$g_q = I_z; \quad g_z = I_q \tag{26}$$

where the currents, I_j are the amount of current enclosed by contours C_i.

Up to this point, the coordinate system used has not been specified. Assuming axisymmetry, we will choose $\zeta = -\phi$, where ϕ is the symmetric toroidal angle in cylindrical coordinates. To choose, θ, we consider the resistive wall shown in Figure 1. Let the resistive wall be parameterized as $R(\theta)$, $Z(\theta)$. This defines θ. The wall defines a r=*constant* surface, where ρ is a generalized radial coordinate. The unit normal vector for this surface is given conveniently as

$$\hat{n} = \frac{\nabla r}{|\nabla r|} . \tag{27}$$

This ρ,θ,ζ coordinate system is not a flux coordinate system, nor is it an orthogonal coordinate system in general.

The toroidal secular term is easiest to derive due to our use of axisymmetry; it is essentially the same as the 2 dimensional case discussed above. Let $\mathbf{Y}_z = \nabla z$. Then the vacuum requirements of Eq. 24 are trivially satisfied. Let $g_z = 2pF$ be the enclosed poloidal current in the toroidal field coils and plasma. This is the same as the toroidal flux function evaluated at the separatrix, $F(y_{sep})$. The secular term for the for the z component is therefore $\mathbf{B}^z_{\sec} = F(y_{sep})\nabla z$.

The poloidal secular term is more difficult to calculate. Let the vector $\mathbf{Y}_\theta$ be given by

$$\mathbf{Y}_q = Y^q \mathbf{e}_q = Y^q \, \mathrm{J} \nabla z \times \nabla r \tag{28}$$

where J is the Jacobian of the coordinate system. The direction vector is chosen such that is perpendicular to the ζ=constant plane, $\mathbf{Y}_q \cdot \nabla z = 0$. More importantly, and what

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[2] This is only true for axisymmetric systems. For stellarators, the periods and the currents will be related by a matrix.

essentially distinguishes this decomposition, is that that this secular term is perpendicular to the surface of interest, $\mathbf{Y}_q \cdot \hat{n} = 0$, which implies that the flux through the wall will be calculated strictly from the single-valued part of the secular magnetic field. That is, the boundary condition for B_n, (Eq. 7) will come from the scalar potential:

$$\mathbf{n} \cdot \mathbf{B}_v = \mathbf{n} \cdot \nabla c = \mathbf{n} \cdot \mathbf{B}_p . \tag{29}$$

It now remains to find the contravariant component Y^q given the conditions on $\mathbf{Y}_i$ (Eq. 24). Taking the divergence:

$$\begin{aligned} \nabla \cdot \mathbf{Y}_q &= \nabla \cdot \left(Y^q \, \mathrm{J} \nabla z \times \nabla r \right) \\ &= \frac{\partial \left(Y^q \, \mathrm{J} \right)}{\partial q} \nabla q \cdot \nabla z \times \nabla r \\ &= \mathrm{J}^{-1} \frac{\partial \left(Y^q \, \mathrm{J} \right)}{\partial q} \quad = 0 \end{aligned} \tag{30}$$

which implies

$$Y^q = \mathrm{J}^{-1} \, a(r) \tag{31}$$

such that $\mathbf{Y}_q = a(r) \nabla z \times \nabla r$. To find the coefficient α, the other vacuum condition is used:

$$\begin{aligned} \nabla \times \mathbf{Y}_q &= \nabla \times \left(Y^q \mathbf{e}_q \right) = \nabla \times \left(Y^q \, g_{rq} \nabla r + Y^q \, g_{qq} \nabla q \right) \\ &= \mathrm{J}^{-1} \, g_{qq} \frac{\partial a}{\partial r} + a \left[\frac{\partial}{\partial r} \left(\mathrm{J}^{-1} \, g_{qq} \right) + \frac{\partial}{\partial q} \left(\mathrm{J}^{-1} \, g_{rq} \right) \right] = 0 \end{aligned} \tag{32}$$

The solution to this equation is $a(r) = C \exp[-\int_{r_w}^{r} E(r', q') dr']$ where $E(r', q') = (\partial / \partial r \left(\mathrm{J}^{-1} g_{qq} \right) + \partial / \partial q \left(\mathrm{J}^{-1} g_{rq} \right)) / \mathrm{J}^{-1} g_{qq}$. This equation is to be parametrically solved for each θ. Before discussing this further, we wish to verify that we get the same result as in the cylindrical case. For a cylinder, the Jacobian is given by $J=r$ and the metric elements are given by $g_{rq}=0$ and $g_{rr}=r^2$ which give $E=1/r$. Evaluating the integral, we obtain $a = 1/(2pr)$. This gives the secular term,

$$I_z \mathbf{Y}_q = I_z \, \mathrm{J}^{-1} \, a g_{qq} \nabla q = \frac{I_z}{2p} \nabla q \tag{34}$$

which agrees with Eq. 22.

For the toroidal case, it is generally not possible to solve for α as a function of ρ because it would require constructing a coordinate system out to infinity and evaluating numerically. On the surface of interest however, α is a constant and can be absorbed into the normalization which relates the period to the current. That is, using Ampere's Law,

$$I_z = \oint_{C_q} \mathbf{Y}_q \cdot \mathbf{dl}_? = g_q a(r_w) \oint_{C_q} \mathrm{J}^{-1} \mathbf{e}_q \cdot \mathbf{e}_q dq = g_q a(r_w) \oint_{C_q} \mathrm{J}^{-1} \, g_{qq} dq \tag{35}$$

such that $a(r_w) = 1 / \oint_{C_q} \mathrm{J}^{-1} \, g_{qq} dq$. This completely specifies the poloidal secular term.

The complete description of the vacuum magnetic field including the secular terms for a doubly-periodic, axisymmetric system is therefore

$$\mathbf{B}_v = \nabla c + \frac{I_z}{\oint_{c_q} J^{-1} g_{qq} dq} \nabla z \times \nabla r + F(y_{sep}) \nabla z \,. \tag{36}$$

4. Conclusions

The key equation is given by Eq. 36, which can be used to determine the boundary conditions as expressed in Equations 21 and 2. For tokamak cases, the single-valued potential is determined by the VACUUM code and for slab or cylindrical cases, it can be specified analytically. For the secular terms, the constants for the contained currents, I_z and F, are extremely important in nonlinear simulations because they are required for the $n=0$ solution. These constants must be specified independently from some external (or initial) conditions, and can depend on time. For example, if one assumes that the currents contained within the resistive wall do not vary with time, then all of the necessary information is contained within the EQDSK (or similar) file. Assuming constant current, if a displacement of the plasma (resulting, for example, from a vertical displacement event (VDE)) causes the plasma current to decrease, the deficit of current will automatically flow in the resistive wall. This wall current, and its Ohmic power, can be computed directly from Equation (6) and Ohms law. The currents could also be specified as functions of time, as determined from some current and flux programming applied to an external circuit, or from experimental traces. This would be the case for feedback studies for example.

Acknowledgements

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APPENDIX A. BOUNDARY CONDITIONS REQUIRED BY THE DIVERGENCE-DIFFUSION TERMS IN NIMROD

The magnetic field representation in NIMROD is not guaranteed to be solenoidal. Terms are added to the induction equation in order to control numerically the growth of $\nabla \cdot \mathbf{B}$. These terms alter the mathematical form of the induction equation and may also affect the boundary conditions that are required to advance the magnetic field in time. Understanding the effect of these terms and their role in determining the allowable boundary conditions is important for a fully consistent implementation of the resistive wall boundary condition.

We investigate the boundary conditions imposed on the components of the magnetic field imposed by the "divergence-diffusion" terms that are added to the induction equation to control error in $\nabla \cdot \mathbf{B}$. We first deal directly with the differential form of the equations. Later we will introduce spatial discretization using the finite element method. The formulation of the induction equation in NIMROD is represented by the differential equation

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E} + k \nabla \nabla \cdot \mathbf{B} \quad . \tag{A1}$$

Note that $\mathbf{B}$ is no longer a pseudovector (derivable from the curl of a vector), so the additional terms may alter the required boundary conditions. The divergence of Equation (1) leads to a diffusion equation for the divergence of **B**, i.e.,

$$\frac{\partial}{\partial t} \nabla \cdot \mathbf{B} = \nabla \cdot k \nabla \nabla \cdot \mathbf{B} \quad . \tag{A2}$$

The idea is that errors in $\nabla \cdot \mathbf{B}$ (introduced by the discretization of $\nabla \times \mathbf{E}$, for example) will be "diffused" from short wavelength to long wavelength. As with all diffusive processes, their ultimate fate will be determined by the boundary conditions.

The volume integral of Equation (2) yields

$$\frac{d}{dt} \int \nabla \cdot \mathbf{B} dV = \frac{d}{dt} \oint \mathbf{B} \cdot \mathbf{n} dS = \oint k \mathbf{n} \cdot \nabla (\nabla \cdot \mathbf{B}) dS \quad . \tag{A3}$$

The first equality in Equation (3) states that the volume integral of $\nabla \cdot \mathbf{B}$ will be preserved if the boundary conditions enforce the constancy of the surface integral of the normal component of **B**. Since the surface integral of all periodic components of $\mathbf{B} \cdot \mathbf{n}$ automatically vanishes, this condition applies explicitly only to the ($m = 0$, $n = 0$) component of the field. Combined with the second equality in Equation (3), this implies that the surface integral of the normal derivative of $\nabla \cdot \mathbf{B}$ (i.e., the flux of $\nabla \cdot \mathbf{B}$ through the surface) must also vanish. [Again, this constraint applies only to the non-periodic (m=0, n=0) component, as the other components will automatically vanish upon integration.] This second equality implies that solutions of Equation (1) with $\mathbf{B} \cdot \mathbf{n} = constant$ (Dirichlet conditions) on the boundaries will *automatically* satisfy the surface condition (i.e., the vanishing of the divergence flux). Conversely, if this second constraint is *not* satisfied by the initial conditions, then it is *inconsistent* to impose the boundary condition $\mathbf{B} \cdot \mathbf{n} = constant$ on the fields. Since this concern arises only from the (m = 0, n = 0) component, it may not be a problem in practice, but it appears that some care should be taken in specifying the initial conditions to assure that all the constraints implied by Equation (3) are satisfied.

A better measure of the divergence error in the magnetic field is the volume integral of $\left|\nabla \cdot \mathbf{B}\right|^2$. Writing $u \equiv \nabla \cdot \mathbf{B}$, we have, from Equation (2),

$$\frac{d}{dt}\int \left|u\right|^2 dV = -2\int k\left|\nabla u\right|^2 dV + 2\oint ku\mathbf{n}\cdot\nabla u dS \quad . \tag{A4}$$

Since $k > 0$, we are assured of diminishing error (in the absence of sources) if the surface integral on the right hand side vanishes. Note that this term is *nonlinear* in $\nabla \cdot \mathbf{B}$, and so may contain contributions from all of the Fourier modes. The vanishing of the last term in Equation (3) [imposed for the (m = 0, n = 0) component only] may *not* be sufficient to assure diminishing error. The consequences of this for practical computations are unclear.

We now introduce the finite element discretization of Equation (1). The first term on the right hand side requires that only the tangential components of the electric field be specified on the boundaries. These conditions were dealt with in a previous note. Here we deal only with the additional terms. After using the finite element *ansatz*, we find (symbolically)

$$\sum_{\mathbf{q}} \frac{d\mathbf{B}_{\mathbf{q}}}{dt}\int_V a_{\mathbf{p}}(\mathbf{x})a_{\mathbf{q}}(\mathbf{x})dV = \sum_{\mathbf{q}}\int_V k a_{\mathbf{p}}(\mathbf{x})\nabla\nabla\cdot\left(a_{\mathbf{q}}(\mathbf{x})\mathbf{B}_{\mathbf{q}}\right)dV \quad , \tag{A5a}$$

$$\begin{aligned} &= -k\sum_{\mathbf{q}}\int_V \nabla a_{\mathbf{p}}(\mathbf{x})\mathbf{B}_{\mathbf{q}}\cdot\nabla a_{\mathbf{q}}(\mathbf{x})dV \\ &\quad + k\sum_{\mathbf{q}}\oint_S \mathbf{n}a_{\mathbf{p}}(\mathbf{x})\mathbf{B}_{\mathbf{q}}\cdot\nabla a_{\mathbf{q}}(\mathbf{x})dS \quad , \end{aligned} \tag{A5b}$$

where we have assumed k is a constant. The last integral on the right hand side of Equation (5b) contains the boundary conditions to be applied to the "divergence-diffusion" terms. Clearly they only apply to the normal component of **B**. Ignoring this term yields solutions that satisfy the condition

$$\sum_{\mathbf{q}}\oint_S \mathbf{n}a_{\mathbf{p}}(\mathbf{x})\mathbf{B}_{\mathbf{q}}\cdot\nabla a_{\mathbf{q}}(\mathbf{x})dS = 0 \tag{A6}$$

on the boundary, which is the natural boundary condition for this problem. This is related to $\nabla \cdot \mathbf{B}$ on the boundary, and does not appear to be consistent with specifying Dirichlet conditions for $\mathbf{B}\cdot\mathbf{n}$. The relationship between Equation (6) and the constraints imposed by Equations (3) and (4) is also not clear. If Dirichlet conditions are specified, then it seems that the surface integral in Equation (5b), which involves *all* the spatial components of **B**, must be included in the solution. No such term appears in the NIMROD formulation.

APPENDIX B. EXAMPLE: A TWO-DIMENSIONAL CYLINDER

We consider the two-dimensional example of finding the resistive wall boundary for a z-independent cylinder of radius a in polar coordinates (r,q). To do this we must find the external potential at the boundary of the cylinder. This is a textbook problem, but it is nonetheless instructive to demonstrate the use of the formalism presented in the main body of the paper.

The problem reduces to finding the free space Greens' function, Equation (14), which in this geometry satisfies the equation

$$\nabla^2 G(r,q \,|\, r',q') = -\frac{2p}{r} d(r-r')d(q-q') \quad . \tag{B1}$$

Using the expansions

$$d(q-q') = \frac{1}{2p} \sum_{m=-\infty}^{\infty} e^{im(q-q')} \quad , \tag{B2}$$

and

$$G(r,q \,|\, r',q') = \sum_{m=-\infty}^{\infty} g_m(r \,|\, r') e^{im(q-q')} \quad , \tag{B3}$$

we find that, for $m \neq 0$, $g_m(r \,|\, r')$ satisfies the differential equation

$$\frac{1}{r}\frac{d}{dr}\left(r \frac{dg_m}{dr} \right) - \frac{m^2}{r^2} g_m = -\frac{1}{r} d(r-r') \quad . \tag{B4}$$

We will deal with the case $m = 0$ later in this section.

Equation (25) is valid for $0 \le r \le \infty$. This is divided into two regions, $0 \le r < r'$ (in which the solution will be denoted as $g^<$), and $r' < r \le \infty$ (in which the solution will be deonited as $g^>$). The solution in each region is

$$g_m = Ar^m + Br^{-m} \quad . \tag{B5}$$

(Here we will deal only with the case $m > 0$; the procedure for negative m is obvious.) The solution must remain finite at both 0 and ∞, so that

$$g^< = A^< r^m \quad , \tag{B6a}$$

and

$$g^> = B^> r^{-m} \quad . \tag{B7b}$$

The constants $A^<$ and $B^>$ are determined by the continuity of g, and the discontinuity of its derivative, at $r = r'$. The latter is found by integrating Equation (25) across the delta-function:

$$\left. \frac{dg^>}{dr} \right|_{r=r'} - \left. \frac{dg^<}{dr} \right|_{r=r'} = -\frac{1}{r'} \quad . \tag{B8}$$

The solution is then

$$g_m^{<} = \frac{1}{2m}\left(\frac{r}{r'}\right)^m \quad , \tag{B9a}$$

and

$$g_m^{>} = \frac{1}{2m}\left(\frac{r'}{r}\right)^m \quad . \tag{B9b}$$

We can now use Equations (29a,b) in Equation (17) to find the potential at the boundary, for which we assume the expansion

$$c(a,q) = \sum_{m=-\infty}^{\infty} c_m(a) e^{imq} \quad , \tag{B10}$$

We get (for $m > 0$)

$$c_m(a) = \frac{a}{m} B_{r_m}(a) \quad , \tag{B11}$$

where $B_{r_m}(a)$ is the m^{th} Fourier coefficient of the internal radial magnetic field at the boundary. The reality of c requires that $c_{-m} = c_m^*$ [here (..)* refers to the complex conjugate]. Equation (31) is verified by directly solving Laplace's equation in the external region subject to the boundary condition given by Equation (7) and evaluating the result at the boundary. It is also identical to the formulas used in the DEBS code. Solutions with z-dependence (which involve modified Bessel functions) have also been obtained with the formalism presented here and agree exactly with direct solutions of the boundary value problem.

Since the axisymmetric ($m = 0$) part of the internal radial magnetic field must vanish, the axisymmetric part of the periodic potential is a solution of Laplace's equation subject to the boundary conditions that $dc_0 / dr = 0$ at both the wall and at infinity. This part of the potential is thus a constant that we take to be zero without loss of generality.

We now turn to the resistive wall boundary condition, which is given by Equation (6) with

$$\mathbf{B}_v = \nabla c + \frac{a B_{q_{vac_0}}}{r} \mathbf{e}_q + B_{z_{vac_0}} \mathbf{e}_z \quad . \tag{B12}$$

Here $B_{q_{vac_0}}$ and $B_{z_{vac_0}}$ are the mean (axisymmetric) poloidal and axial magnetic field components just outside the resistive wall. They arise from the non-single valued terms as indicated in Equation (9). The poloidal electric field at the wall has only an axisymmetrix ($m = 0$) part that is given by

$$E_q(a) = \frac{h_w}{m_0 d_w}\left(-B_{z_{vac_0}}(a) + B_{z_{p0}}(a)\right) \quad . \tag{B13}$$

The mean axial field just outside the wall must be independently specified. Since this field vanishes outside an infinitely long solenoid we set this term to zero. (This implies that the coils that produce the internal axial field lie within the plasma/resistive wall system.) Then the $m = 0$ component of the poloidal electric field is

$$E_{\theta_0}(a) = \frac{\eta_w}{\mu_0 \delta_w} B_{z_{p0}}(a) \quad , \tag{B14}$$

with all other Fourier components vanishing.

The axial electric field at the wall is

$$E_z(a) = \frac{\eta_w}{\mu_0 \delta_w} \left[\frac{1}{a} \frac{\partial \chi}{\partial \theta}\bigg|_a + B_{\theta_{vac0}}(a) - B_{\theta_p}(a) \right] \quad . \tag{B15}$$

Since $B_{\theta_{vac0}}$ has only an axisymmetric ($m = 0$) part, the Fourier components of $E_z(a)$ for $m \neq 0$ are given by

$$E_{z_m}(a) = \frac{\eta_w}{\mu_0 \delta_w} \left[\frac{1}{a} im \chi_m(a) - B_{\theta_{pm}}(a) \right] \quad , \tag{B16a}$$

$$= \frac{\eta_w}{\mu_0 \delta_w} \left[iB_{r_{pm}}(a) - B_{\theta_{pm}}(a) \right] \quad , \tag{B16b}$$

where we have used Equation (31). For $m = 0$ we have

$$E_{z_0}(a) = \frac{\eta_w}{\mu_0 \delta_w} \left[B_{\theta_{vac0}}(a) - B_{\theta_{p0}}(a) \right] \quad . \tag{B17}$$

Equations (34), (36b), and (37) give the tangential field at the wall completely in terms of the internal field at the wall and the constant $B_{\theta_{vac0}}(a)$, and they constitute the resistive wall boundary condition for this case. They can be used in Equation (2) (or, equivalently, (3)) to advance the internal magnetic field to the next time level.

We remark that the analysis in this case was greatly simplified by the decoupling of the poloidal harmonics in cylindrical geometry. As indicated in Equations (18-21), in general expressions for the electric field at the wall will involve a matrix that couples all points on the boundary. The elements of this matrix are calculated by the VACUUM code.