A Study of Linear Ideal MHD Waves with the SEL Spectral Element Code

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Presented at the CEMM Meeting
Missoula, Montana, April 25, 2004

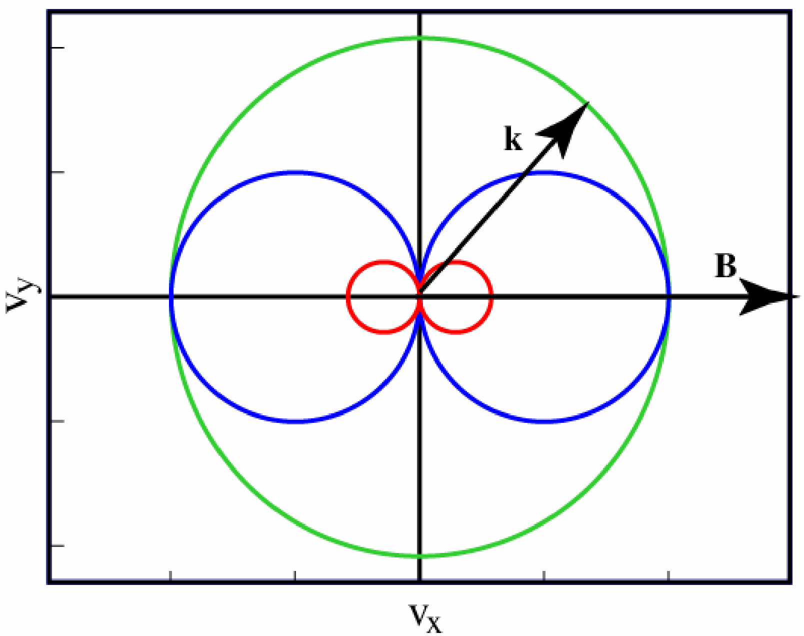
Ideal MHD Waves

$$\beta \equiv \frac{P}{2\mu_0 B^2}, \quad \gamma = \frac{5}{3}, \quad \cos\theta \equiv \frac{\mathbf{k}\cdot\mathbf{B}}{kB}$$

$$c_A^2 \equiv \frac{B^2}{\mu_0 \rho}, \quad c_S^2 \equiv \frac{\gamma P}{\rho}, \quad \frac{c_A^2}{c_s^2} = \frac{\gamma\beta}{2}, \quad v \equiv \frac{\omega}{k}$$

$$v_\pm^2 = \frac{1}{2}\left\{\left(c_A^2 + c_S^2\right) \pm \left[\left(c_A^2 + c_S^2\right)^2 - 4c_A^2 c_S^2 \cos^2\theta\right]^{1/2}\right\}, \quad v_A^2 = c_A^2 \cos^2\theta$$

Friedrichs Diagram, β = 10%



Consequences

For $\mathbf{k}\cdot\mathbf{B} \to 0$, $\omega_-, \omega_A \ll \omega_+$

These lowest-frequency modes are the most easily destabilized by small effects like pressure gradients, bootstrap currents, and resistivity. Accurate treatment of such subtle effects requires accurate representation of

$$k_{||} \ll a, R \ll k_\perp$$

Spatial Discretization

Flux-Source Form of Equations

$$\frac{\partial u^i}{\partial t} + \nabla \cdot \mathbf{F}^i = S^i$$

$$\mathbf{F}^i = \mathbf{F}^i(t, \mathbf{x}, u^j, \nabla u^j)$$

$$S^i = S^i(t, \mathbf{x}, u^j, \nabla u^j)$$

Galerkin Expansion

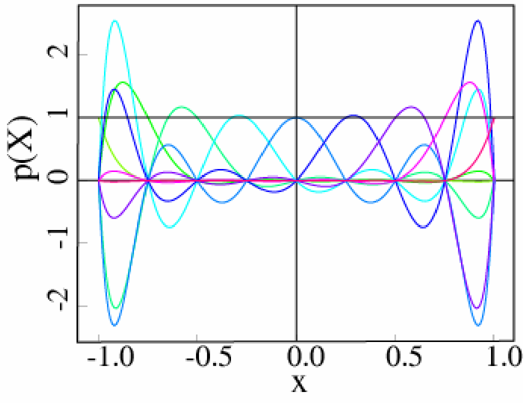
$$u^i(t, \mathbf{x}) \approx \sum_{j=0}^{n} u^i_j(t)\alpha_j(\mathbf{x})$$

Weak Form of Equations

$$(\alpha_i, \alpha_j)\dot{u}^k_j = \int_\Omega d\mathbf{x}\left(S^k\alpha_i + \mathbf{F}^k \cdot \nabla\alpha_i\right) - \int_{\partial\Omega} d\mathbf{x}\alpha_i\mathbf{F}^k \cdot \hat{\mathbf{n}}$$

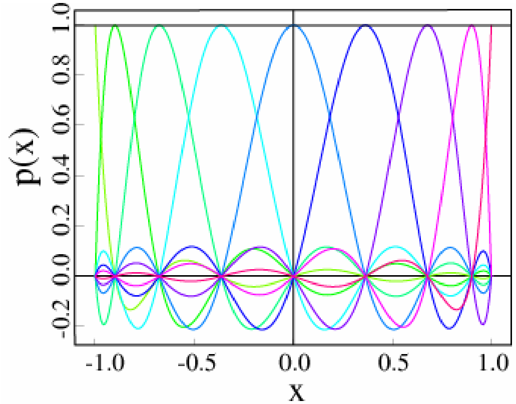
Alternative Polynomial Bases

Uniform Nodal Basis



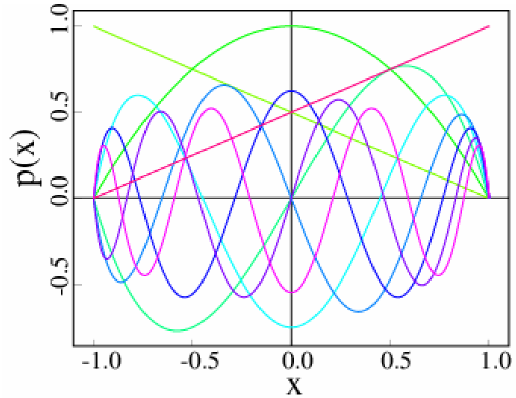
- Lagrange interpolatory polynomials
- Uniformly-spaced nodes
- Diagonally subdominant

Jacobi Nodal Basis



- Lagrange interpolatory polynomials
- Nodes at roots of *$(1-x^2)\ P_n^{(0,0)}(x)$*
- Diagonally dominant

Spectral (Modal) Basis



- Jacobi polynomials *$(1+x)/2$, $(1-x)/2$, $(1-x^2)\ P_n^{(1,1)}(x)$*
- Nearly orthogonal
- Manifest exponential convergence

Fully Implicit Newton-Krylov Time Step

$$\mathbf{M}\dot{\mathbf{u}} = \mathbf{r}$$

$$\mathbf{M}\left(\frac{\mathbf{u}^{+} - \mathbf{u}^{-}}{h}\right) = \theta\mathbf{r}^{+} + (1-\theta)\mathbf{r}^{-}$$

$$\mathbf{R}\left(\mathbf{u}^{+}\right) \equiv \mathbf{M}\left(\mathbf{u}^{+} - \mathbf{u}^{-}\right) - h\left[\theta\mathbf{r}^{+} + (1-\theta)\mathbf{r}^{-}\right] = 0$$

$$\mathbf{J} \equiv \mathbf{M} - h\theta\left\{\frac{\partial r_i^{+}}{\partial u_j^{+}}\right\}$$

$$\mathbf{R} + \mathbf{J}\delta\mathbf{u}^{+} = 0, \quad \delta\mathbf{u}^{+} = -\mathbf{J}^{-1}\mathbf{R}\left(\mathbf{u}^{+}\right), \quad \mathbf{u}^{+} \rightarrow \mathbf{u}^{+} + \delta\mathbf{u}^{+}$$

- Nonlinear Newton-Krylov iteration.
- Elliptic equations: **M** = 0.
- Static condensation, fully parallel.
- PETSc: GMRES with Schwarz ILU, overlap of 3, fill-in of 5.

Static Condensation

$$\mathbf{L}\mathbf{u} = \mathbf{r} \tag{1}$$

Partition: (1) element edges: (2) element interiors

$$\mathbf{u} = \begin{pmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \end{pmatrix}, \quad \mathbf{r} = \begin{pmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \end{pmatrix}, \quad \mathbf{L} = \begin{pmatrix} \mathbf{L}_{11} & \mathbf{L}_{12} \\ \mathbf{L}_{21} & \mathbf{L}_{22} \end{pmatrix} \tag{2}$$

$$\begin{aligned} \mathbf{L}_{11}\mathbf{u}_1 + \mathbf{L}_{12}\mathbf{u}_2 &= \mathbf{r}_1 \\ \mathbf{L}_{21}\mathbf{u}_1 + \mathbf{L}_{22}\mathbf{u}_2 &= \mathbf{r}_2 \end{aligned} \tag{3}$$

$$\mathbf{L}_{22}\mathbf{u}_2 = \mathbf{r}_2 - \mathbf{L}_{21}\mathbf{u}_1 \tag{4}$$

$$\begin{aligned} \bar{\mathbf{L}}_{11} &\equiv \mathbf{L}_{11} - \mathbf{L}_{12}\mathbf{L}_{22}^{-1}\mathbf{L}_{21} \\ \bar{\mathbf{r}}_1 &\equiv \mathbf{r}_1 - \mathbf{L}_{12}\mathbf{L}_{22}^{-1}\mathbf{r}_2 \end{aligned} \tag{5}$$

$$\bar{\mathbf{L}}_{11}\mathbf{u}_1 = \bar{\mathbf{r}}_1 \tag{6}$$

- Equation (4) solved by local LU factorization and back substitution.
- Equation (6), substantially reduced, solved by global Newton-Krylov.

Linear Ideal MHD

$$\mathbf{B} = (\sin\theta\cos\phi,\ \sin\theta\sin\phi,\ \cos\theta)$$

$$\rho\frac{\partial \mathbf{v}}{\partial t} = \mathbf{j}\times\mathbf{B} - \nabla p, \quad \mathbf{j} = \nabla\times\mathbf{b}$$

$$\frac{\partial \mathbf{b}}{\partial t} = \nabla\times(\mathbf{v}\times\mathbf{B}), \quad \nabla\cdot\mathbf{b} = 0$$

$$\frac{\partial p}{\partial t} + \gamma P \nabla\cdot\mathbf{v} = 0, \quad \mathbf{b} = b_z\hat{\mathbf{z}} + \hat{\mathbf{z}}\times\nabla\psi$$

Flux-Source Form

$$\mathbf{u} \equiv (\rho v_x,\ \rho v_y,\ \rho v_z,\ \psi,\ b_z,\ p)$$

$$\frac{\partial}{\partial t}(\rho\mathbf{v}) + \nabla\cdot\mathsf{T} = 0, \quad \mathsf{T} \equiv (p + \mathbf{b}\cdot\mathbf{B})\,\mathsf{I} - (\mathbf{bB} + \mathbf{Bb})$$

$$\frac{\partial \psi}{\partial t} = \hat{\mathbf{z}}\cdot\mathbf{v}\times\mathbf{B}, \quad \frac{\partial b_z}{\partial t} + \nabla\cdot(\mathbf{v}B_z - \mathbf{B}v_z) = 0$$

$$\frac{\partial p}{\partial t} + \nabla\cdot(\gamma P\mathbf{v}) = 0$$

Frequencies and Polarizations

$$\frac{\partial^2 \mathbf{v}}{\partial t^2} = c_A^2 \left\{\nabla \times \left[\nabla \times (\mathbf{v} \times \mathbf{n})\right]\right\} \times \mathbf{n} + c_S^2 \nabla\nabla \cdot \mathbf{v}$$

$$\mathbf{v}(\mathbf{x}, t) = \mathbf{v}_0 \exp\left[i\left(\mathbf{k} \cdot \mathbf{x} - \omega t\right)\right], \quad \mathsf{L} \cdot \mathbf{v}_0 = 0$$

$$\mathsf{L} = c_A^2 \left[k_\parallel^2 \mathsf{I} - k_\parallel (\mathbf{kn} + \mathbf{nk}) + \mathbf{kk}\right] + c_S^2 \mathbf{kk} - \omega^2 \mathsf{I}$$

$$\det \mathsf{L} = \left(k_\parallel^2 c_A^2 - \omega^2\right)\left[k^2 k_\parallel^2 c_A^2 c_S^2 - \omega^2 k^2 \left(c_A^2 + c_S^2\right) + \omega^4\right] = 0$$

$$\hat{\mathbf{e}}_1 \equiv \mathbf{n} = \frac{\mathbf{B}}{B}, \quad \hat{\mathbf{e}}_2 \equiv \frac{\mathbf{k} \times \mathbf{B}}{|\mathbf{k} \times \mathbf{B}|}, \quad \hat{\mathbf{e}}_3 \equiv \hat{\mathbf{e}}_1 \times \hat{\mathbf{e}}_2, \quad \mathbf{B} = B\hat{\mathbf{e}}_1, \quad \mathbf{k} = k_\parallel \hat{\mathbf{e}}_1 + k_\perp \hat{\mathbf{e}}_3$$

$$\mathsf{L} = \left(c_A^2 k_\parallel^2 - \omega^2\right) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \left(c_A^2 + c_S^2\right) \begin{pmatrix} k_\parallel^2 & 0 & k_\parallel k_\perp \\ 0 & 0 & 0 \\ k_\parallel k_\perp & 0 & k_\perp^2 \end{pmatrix} - c_A^2 k_\parallel \begin{pmatrix} 2k_\parallel & 0 & k_\perp \\ 0 & 0 & 0 \\ k_\perp & 0 & 0 \end{pmatrix}$$

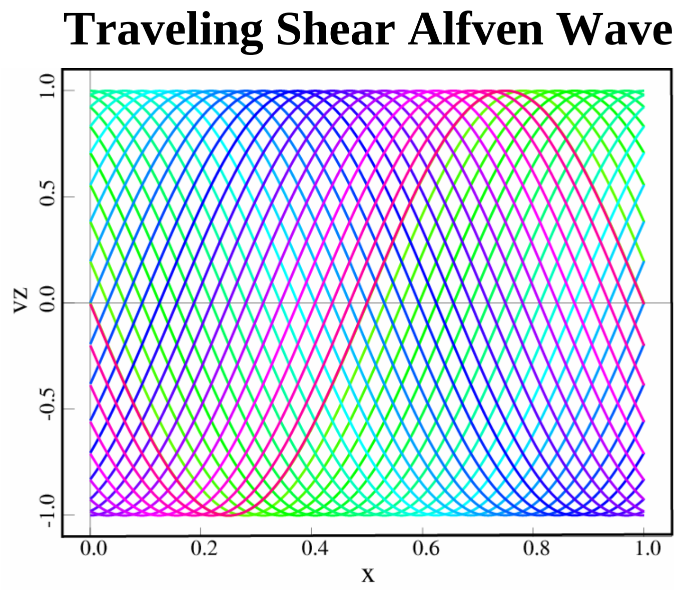
$$\text{For } \omega^2 = c_A^2 k_\parallel^2, \quad \mathbf{v}_0 = (0, 1, 0)$$

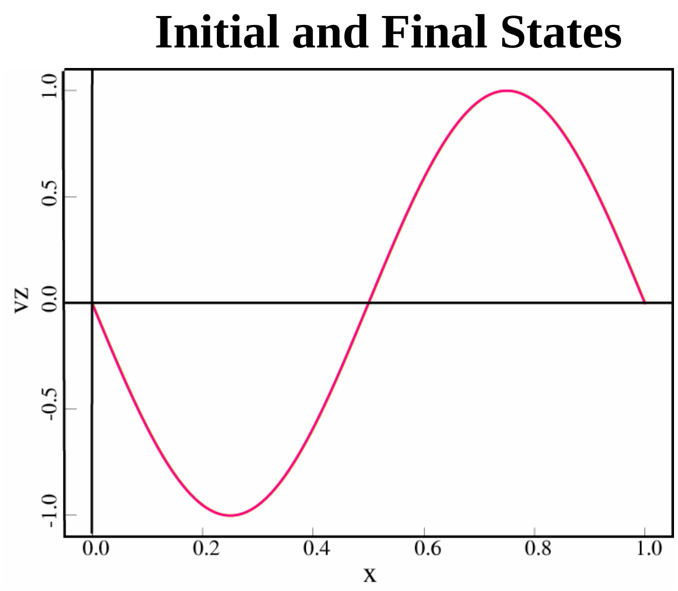
$$\text{For } \omega^2 = \frac{1}{2}\left\{\left(c_A^2 + c_S^2\right) k^2 \pm \left[\left(c_A^2 + c_S^2\right)^2 k^4 - 4 c_A^2 c_S^2 k^2 k_\parallel^2\right]^{1/2}\right\}, \quad \mathbf{v}_0 = (v_\parallel, 0, 1), \quad v_\parallel \equiv \frac{c_S^2 k_\parallel k_\perp}{\omega^2 - c_S^2 k_\parallel^2}$$

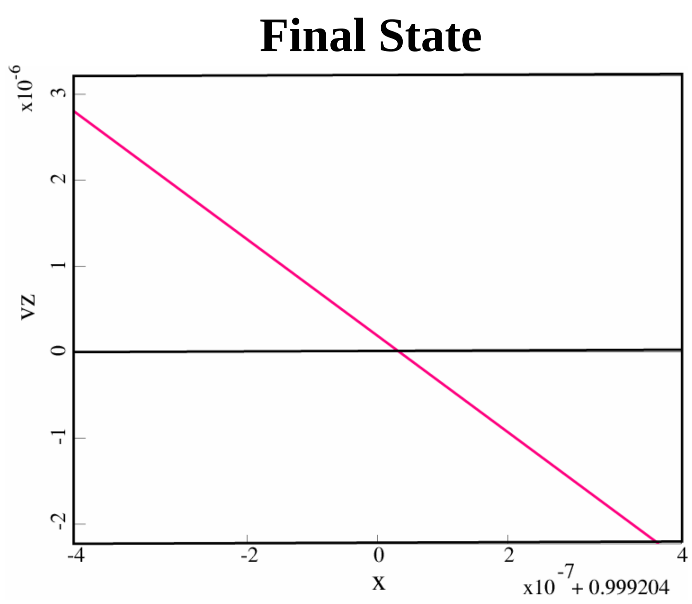
Selection of Parameters

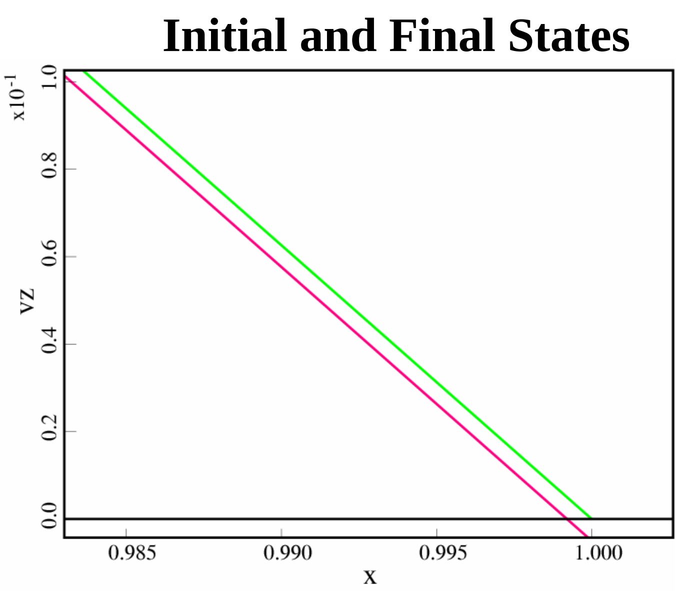
- **Domain:** x, y = (0,1); **wave numbers:** kx = ky = 2π.
- **Grid parameters**: (nx, ny, np, procs) = (8, 8, 2, 8), (4, 4, 4, 8), (2, 2, 8, 4).
- **Magnetic field:** spherical coordinates about z axis:
  - Bx = $\sin\theta \cos\varphi$, By = $\sin\theta \cos\varphi$, Bz = $\cos\theta$
  - $\theta = 90°$ (in x-y plane), $\varphi \to 135°$ (approaching transverse propagation).
- **Beta:** 10%.
- **Time:** tmax = shear Alfven period, dt = tmax/64, nt = 64, polarization = Shear Alfven.
- **Solution procedure:** GMRES, ILU-5 preconditioning + static condensation for np > 2.
- **Reported Results**
  - Ksp: number of GMRES iterations in 64 time steps
  - Cpu: run time on Linux cluster, 3.1 GHz Xeons, Gigabit network
  - Errt: relative error in wave period.
  - Errx: relative spatial truncation error, from convergence of polynomials
  - CFL: Courant number relative to fast wave frequency, ω_f dt.

Measurement of Wave Period









Summary of Numerical Results

- **(nx, ny, np, procs) = (8, 8, 2, 8)**
  - No static condensation
  - For φ = 130, 131, 132, 133:
    - Ksp = 770, 1023, 1472, 2792
    - Cpu = 8.6, 9.9, 13.4, 22.7 s
  - Errx = 10%, Errt ~ $8x10^{-4}$, CFL = 2.9

- **(nx, ny, np, procs) = (4, 4, 4, 8)**
  - Static condensation
  - For φ = 130 – 134.9999
    - Ksp = 64 - 128
    - Cpu ~ 4 s
  - Errx ~ $3x10^{-4}$, Errt ~ $8x10^{-4}$, CFL = $5x10^{4}$

- **(nx, ny, np, procs) = (2, 2, 8, 4)**
  - Static condensation
  - For φ = 130 – 134.9999:
    - Ksp = 64 - 137
    - Cpu ~ 6 - 8 s
  - Errx ~ 10^{-8}, Errt ~ $8x10^{-4}$, CFL = $5x10^{4}$

Conclusions

- Accurate computation of shear Alfven and slow waves as $\mathbf{k} \cdot \mathbf{B} \rightarrow 0$ is essential for study of tokamak stability.

- Stiffness makes this numerically challenging.
  - Explicit methods require very small time steps.
  - Implicit methods have very large condition numbers.

- Spectral elements provide rapid convergence + parallelization.

- Static condensation is very effective in treating large condition numbers.

- Precise study of linear wave motion provides an important test bed for numerical properties.