Unified form for parallel ion stress in magnetized plasmas [1]

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Introduction

- **In this work, an integral (nonlocal) closure for the parallel ion stress is presented,**

  $$\pi_{\|} \equiv m \int d^3v (v_{\|}^2 - v_{\perp}^2/2) F,$$

  where $v_{\|}$ and $v_{\perp}$ are the parallel and perpendicular particle speeds.

Features of $\pi_{\|}$ closure

- ***Integral* or *nonlocal* closure implies analytic forms involving integrations along characteristics of the perturbed distribution function, F**
- **Allowing for arbitrary collisionality and requiring momentum conservation among ion species couples $\pi_{\|}$ to a nonlocal momentum restoring term.**

Importance of $\pi_{\|}$ closure

- **Unified $\pi_{\|}$ needed to capture anisotropic nature of momentum transport in ion (plasma) flow evolution equation which has large parallel ion stress force density, $\vec{\nabla} \cdot \Pi_{\|} = \vec{\nabla} \cdot (\hat{b}\hat{b} - I/3)\pi_{\|}$, in moderately collisional to nearly collisionless plasmas.**
- **Unified $\pi_{\|}$ may account for anomalous ion heating, $\vec{\nabla}\vec{V} : \Pi_{\|}$, in moderately collisional to nearly collisionless plasmas.**

Solve simplified Chapman-Enskog-like (CEL) drift kinetic equation.

- Use following Ansatz:

$$f = f_M + F = n\left(\frac{m}{2\pi T}\right)^{\frac{3}{2}} \exp\left(-\frac{m(\vec{v}-\vec{u})^2}{2T}\right) + F,$$

and average full CEL kinetic equation over gyroangle to write

$$\vec{v}_{\|} \cdot \vec{\nabla} F^1 - \left\langle C(F^1 + f_M^1)\right\rangle = -\frac{m}{T^0}(\hat{\mathrm{b}}\hat{\mathrm{b}} - \frac{\mathrm{I}}{3}) : \vec{\nabla}_{\|}\vec{u}_{\|}^1 \left(v_{\|}^2 - \frac{v_{\perp}^2}{2}\right) f_M^0$$
$$+ v_{\|}\left(\hat{\mathrm{b}} \cdot \vec{\nabla} \cdot \tilde{\Pi}_{\|}^1 - R_{\|}^1\right)\frac{f_M^0}{p^0}.$$

<u>Assumptions</u>

1. sheared slab magnetic geometry (ignores particle drifting and trapping),
2. steady-state limit ($\omega \rightarrow 0$),
3. omission of heat flow term and associated temperature gradient drive (focus on flow gradient drive).

Employ pitch-angle scattering operator with momentum restoring term.

- **The Lorentz scattering operator plus momentum restoring term is given by**

$$\begin{aligned}\left\langle C(F^1 + f_M^1)\right\rangle &\approx \mathcal{L}(F^1 + f_M^1) + v_\parallel \mathcal{N}_\parallel f_M^0 \\ &= \frac{1}{2}\frac{\partial}{\partial \xi}\left(1-\xi^2\right)\frac{\partial F^1}{\partial \xi} - 2\frac{v_\parallel u_\parallel^1}{v_{th}^2} f_M^0 + v_\parallel \mathcal{N}_\parallel f_M^0,\end{aligned}$$

where pitch-angle-type variable, $\xi \equiv \pm\sqrt{1 - v_\perp^2/v^2}$

- **Setting $\int d^3v v_\parallel \left\langle C(F^1 + f_M^1)\right\rangle = 0$ to ensure momentum conservation within ion species yields**

$$\mathcal{N}_\parallel = \int d^3v \nu v_\parallel \left(2(v_\parallel u_\parallel^1/v_{th}^2) f_M^0 - \mathcal{L}(F^1)\right) / \int d^3v \nu v_\parallel^2 f_M^0,$$

hence,

$$\left\langle C(F^1 + f_M^1)\right\rangle \approx \nu \mathcal{L}(F^1) - \frac{\nu v_\parallel}{t_\parallel} U_\parallel f_M^0,$$

where

$$t_\parallel \equiv \int d^3v \nu v_\parallel^2 f_M^0, \ \text{ and } \ \mathrm{U}_\parallel \equiv \int \mathrm{d}^3\mathrm{v} \nu \mathrm{v}_\parallel \mathcal{L}(\mathrm{F}^1).$$

Simplified kinetic equation captures dominant physics of parallel ion dynamics

- Kinetic equation of interest becomes:

$$v\xi\frac{\partial F}{\partial L} - \frac{\nu}{2}\frac{\partial}{\partial\xi}\left(1-\xi^2\right)\frac{\partial F}{\partial\xi} = vP_1(\xi)\left(\frac{2}{3p}\frac{\partial\pi_\parallel}{\partial L} - \frac{\nu}{t_\parallel}U_\parallel\right)f_M - v^2P_2(\xi)\frac{2m}{3T}\frac{\partial u_\parallel}{\partial L}f_M,$$

where $\vec{v}_\parallel \cdot \vec{\nabla}$ has been written $v\xi\partial/\partial L$ and $P_1(\xi) = \xi$ and $P_2(\xi) = (3\xi^2 - 1)/2$ are Legendre polynomials.

Approximations

1. pitch-angle scattering with momentum restoring term,
2. sheared slab magnetic geometry,
3. steady-state limit,
4. focus on flow gradient drive.

Solve kinetic equation by expanding in pitch-angle basis.

- **Expand F in set of N Legendre polynomials:**

$$\mathcal{L}(F) = \mathcal{L} \sum_{n=1}^{N} F_n(v, L) P_n(\xi) = \sum_{n=1}^{N} F_n(v, L) \lambda_n P_n(\xi),$$

with associated eigenvalues, λ_n.

- **Write $\vec{F} = [F_1, F_2, ..., F_N]$ and apply orthogonality:**

$$\mathrm{I}\vec{F} + \frac{v}{\bar{\nu}} \mathrm{A} \frac{\partial \vec{F}}{\partial L} = -\frac{\vec{G}}{\bar{\nu}},$$

where $\bar{\nu} = \nu/2$.

- **I is the identity matrix, the tridiagonal matrix A contains free streaming couplings between different eigenfunctions and $\vec{G}$ is projection of the drives onto eigenfunctions.**

- **Invert ODE operator, $\mathrm{I}\vec{F} + \frac{v}{\bar{\nu}} \mathrm{A} \frac{\partial \vec{F}}{\partial L}$ to write:**

$$F_i = \sum_{i=1}^{N} \sum_{j=1}^{N} \int^{L} dL' \left[a_{i,j} \left(\frac{2}{3p} \frac{\partial \pi_{\|}}{\partial L'} - \frac{\nu U_{\|}}{t_{\|}} \right) + b_{i,j} \frac{4v}{3v_{th}^2} \frac{\partial u_{\|}}{\partial L'} \right] f_M e^{-k_{\| j}(L-L')},$$

where $a_{i,j}$ and $b_{i,j}$ and effective inverse collision lengths $k_{\| j} \equiv \bar{\nu}/(\gamma_i v)$ are generated upon inverting the ODE operator.

Construct unified form for $\pi_{\|}$.

- **Employ the following moment definitions**

$$\pi_{\|} \equiv m \int d^3v (v_{\|}^2 - \frac{v_{\perp}^2}{2}) F = m \int d^3v v^2 P_2(\xi) F,$$
$$u_{\|F} \equiv \frac{1}{n} \int d^3v v_{\|} F = \frac{1}{n} \int d^3v v P_1(\xi) F = 0,$$

where second moment forces parallel flow moment of F to vanish.

- **Integrate over pitch-angle dependence to write:**

$$\pi_{\|} = \frac{4\pi m}{5} \int_0^{\infty} dv v^4 f_M \sum_{j=1}^{N} \int^{L} dL' \left[a_{2,j} \left(\frac{2}{3p} \frac{\partial \pi_{\|}}{\partial L'} - \frac{\nu U_{\|}}{t_{\|}} \right) + b_{2,j} \frac{4v}{3v_{th}^2} \frac{\partial u_{\|}}{\partial L'} \right] e^{-k_{\|j}(L-L')}, \quad (1)$$

$$u_{\|F} = \frac{4\pi}{3n} \int_0^{\infty} dv v^3 f_M \sum_{j=1}^{N} \int^{L} dL' \left[a_{1,j} \left(\frac{2}{3p} \frac{\partial \pi_{\|}}{\partial L'} - \frac{\nu U_{\|}}{t_{\|}} \right) + b_{1,j} \frac{4v}{3v_{th}^2} \frac{\partial u_{\|}}{\partial L'} \right] e^{-k_{\|j}(L-L')} = 0.$$

Important features of $\pi_{\|}$ closure

1. **Nonlocal $\pi_{\|}$ couples to nonlocal momentum restoring term, $U_{\|} \equiv \int d^3v \nu v_{\|} \mathcal{L}(F)$.**
2. **Nonlocality of both terms results from deriving closures for arbitrary collisionality.**

Integrate by parts to derive symmetric form.

- Interchange order of integration over v and L' and integrate by parts to write:

$$\pi_\parallel = \frac{4\pi m}{5}\int^{L} dL' \int_0^\infty dv v^4 f_M \sum_{j=1}^{N}\left[a_{2,j}\left(\frac{2}{3p}\frac{\partial \pi_\parallel}{\partial L'} - \frac{\nu U_\parallel}{t_\parallel}\right) + b_{2,j}\frac{4v}{3v_{th}^2}\frac{\partial u_\parallel}{\partial L'}\right] e^{-k_{\parallel j}(L-L')},$$

$$U_\parallel = A\int^{L} dL' \int_0^\infty dv v^3 f_M \sum_{j=1}^{N} a_{1,j}\left[\left(\frac{2}{3p}\frac{\partial \pi_\parallel}{\partial L'} + \frac{\nu}{k_{\parallel j} t_\parallel}\frac{\partial U_\parallel}{\partial L'}\right) + b_{1,j}\frac{4v}{3v_{th}^2}\frac{\partial u_\parallel}{\partial L'}\right] e^{-k_{\parallel j}(L-L')},$$

where A is

$$A = \left(\int_0^\infty dv v^3 f_M \sum_{j=1}^{N} a_{1,j}\frac{\nu}{|k_{\parallel j}| t_\parallel}\right)^{-1}.$$

- Unified form for $\pi_\parallel$ which introduces concept of nonlocal momentum conservation that results when collision lengths are long compared to parallel flow gradient scale lengths.

Can also write $\pi_{\|}$ closure as coupled Volterra equations.

- **Again interchange the order of integration and integrate the differentiated terms in Eq. (1) by parts to write**

$$\mathrm{K}_{11}(U_{\|}) + \mathrm{K}_{12}(\pi_{\|}) = \int_0^{\infty} d\bar{L}\left(u_{\|}(L+\bar{L}) + u_{\|}(L-\bar{L})\right)\frac{\partial K_1(\bar{L})}{\partial \bar{L}} + B_1 u_{\|}(L),$$

$$\mathrm{K}_{21}(U_{\|}) + (1+B_2)\pi_{\|} + \mathrm{K}_{22}(\pi_{\|}) = \int_0^{\infty} d\bar{L}\left(u_{\|}(L+\bar{L}) - u_{\|}(L-\bar{L})\right)\frac{\partial K_2(\bar{L})}{\partial \bar{L}},$$

where the boundary terms are

$$B_1 = \sum_{k_{\|j}>0} 2b^{+}_{1,j}\int d^3v P_1^2 \frac{4v^2}{3v_{th}^2} f_M, \quad \text{and}$$

$$B_2 = \sum_{k_{\|j}>0} 2|a^{+}_{2,j}|\int d^3v P_2^2 \frac{2v^2}{3p} f_M,$$

and

$$\frac{\partial K_i}{\partial \bar{L}} = \sum_{k_{\|j}<0}\int d^3v v^i P_i^2 b^{-}_{i,j}|k_{\|j}|\frac{4v}{3v_{th}^2} f_M e^{-|k_{\|j}|\bar{L}},$$

$$\mathrm{K}_{i1}(U_{\|}) = \int_0^{\infty} d\bar{L}\left(U_{\|}(L-\bar{L}) + (-1)^{i+1}U_{\|}(L+\bar{L})\right)\sum_{k_{\|j}>0} a^{+}_{i,j}\int d^3v v^i P_i^2 \frac{\nu}{t_{\|}} f_M e^{-k_{\|j}\bar{L}},$$

$$\mathrm{K}_{i2}(\pi_{\|}) = \int_0^{\infty} d\bar{L}\left(\pi_{\|}(L-\bar{L}) + (-1)^{i}\pi_{\|}(L+\bar{L})\right)\sum_{k_{\|j}>0} a^{+}_{i,j}\int d^3v v^i P_i^2 k_{\|j}\frac{2}{3p} f_M e^{-k_{\|j}\bar{L}}.$$

Nonlocal closure unifies ion stress.

- **When collisions localized integrals along magnetic field:**

$$\pi_{\|} = -nm\mu_{\|}\frac{\partial u_{\|}}{\partial L},$$

where the viscosity, $\mu_{\|}$, is

$$\begin{aligned}\mu_{\|} &= \left(\frac{32}{15\sqrt{\pi}}\int_0^{\infty} ds s^5 e^{-s^2}\sum_{k_{\|j}>0}\frac{|b_{2,j}|\nu_{ii}}{k_{\|j}v_{th}}\right)\frac{v_{th}^2}{\nu_{ii}} \\ &= 2.75\frac{v_{th}^2}{\nu_{ii}}.\end{aligned}$$

Coefficient 2.75 lies between the collisional viscosity coefficients of Braginskii [2], 1.81, and Chang/Callen [3], 3.13.

- **In nearly collisionless limit, $\mu_{\|}$ becomes**

$$\mu_{\|} = 1.04\frac{v_{th}^2}{k_{\|}v_{th}}.$$

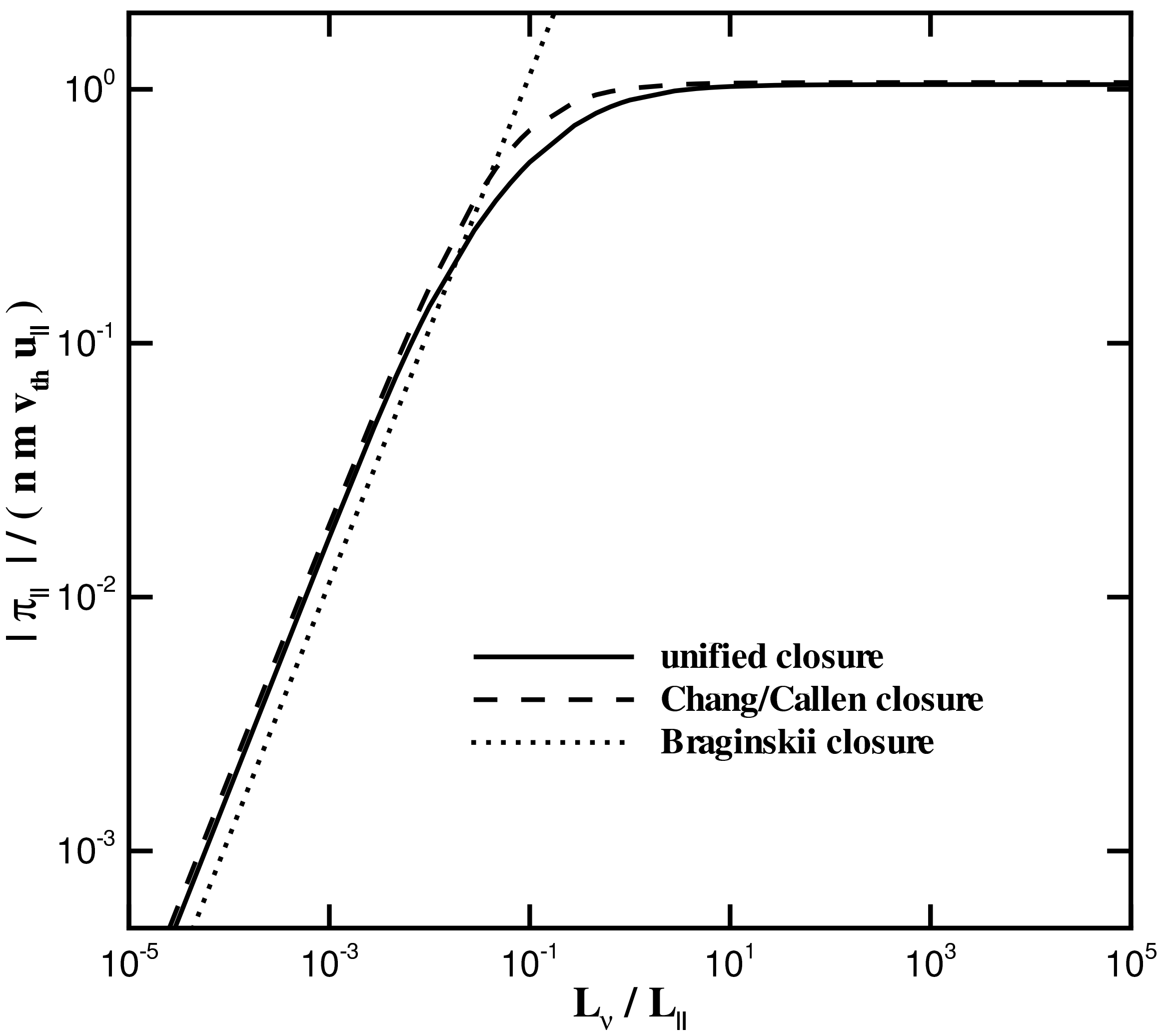
In Chang/Callen the coefficient in front of this expression is $(3/5)\sqrt{\pi} = 1.06$. Here the stress is due solely to wave-particle Landau interactions.

[2] S. I. Braginskii, *Transport Processes in a Plasma*, Consultants Bureau, New York, edited by M. A. Leontovich, **1, 1965**

[3] Z. Chang and J. D. Callen, *Phys. Plasmas*, **4, 1167 (1992)**

Unified $\pi_{||}$ approximate for all collisionalities.

- **Parallel stress for sinusoidal flow perturbations of scale length, $L_{||}$, $\tilde{u}_{||}(L) = u_{||} \sin\left(\frac{2\pi L}{L_{||}}\right)$, shows behavior as collision length L_ν is varied:**



Nonlocal $\boldsymbol{\pi}_{||}$ contains physics of pressure anisotropy.

- **Chew-Golberger-Low pressure tensor is [4]:**

  $$\mathbf{P} = \mathbf{p}_{||}\hat{\mathbf{b}}\hat{\mathbf{b}} + \mathbf{p}_{\perp}(\mathbf{I} - \hat{\mathbf{b}}\hat{\mathbf{b}}),$$

- **Using** $\boldsymbol{p} = (\boldsymbol{p}_{||} + 2\boldsymbol{p}_{\perp})/3$ **and** $\boldsymbol{\Pi} = \mathbf{P} - \mathbf{pI}$ **yields**

  $$\boldsymbol{\Pi} = (\mathbf{p}_{||} - \mathbf{p}_{\perp})(\hat{\mathbf{b}}\hat{\mathbf{b}} - \mathbf{I}/3),$$

  where $\boldsymbol{p}_{||} \equiv \boldsymbol{m} \int \boldsymbol{d^3 v v_{||}^2 f}$ **and** $\boldsymbol{p}_{\perp} \equiv \boldsymbol{m} \int \boldsymbol{d^3 v (v_{\perp}^2/2) f}$.

- **In this work**

  $$\boldsymbol{\Pi}_{||} \equiv \boldsymbol{m} \int \boldsymbol{d^3 v}(\boldsymbol{v}_{||}^2 - \frac{\boldsymbol{v}_{\perp}^2}{2})\boldsymbol{F}(\hat{\mathbf{b}}\hat{\mathbf{b}} - \mathbf{I}/3).$$

- **Note, however, that unlike purely collisionless form for CGL stress,** ***F*** **contains collisional information and is more general than bi-Maxwellian distribution associated with CGL form.**

[4] G. F. Chew, M. L. Goldberger and F. E. Low, *Proc. Roy. Soc. (London)*, **A 236, 112 (1956)**

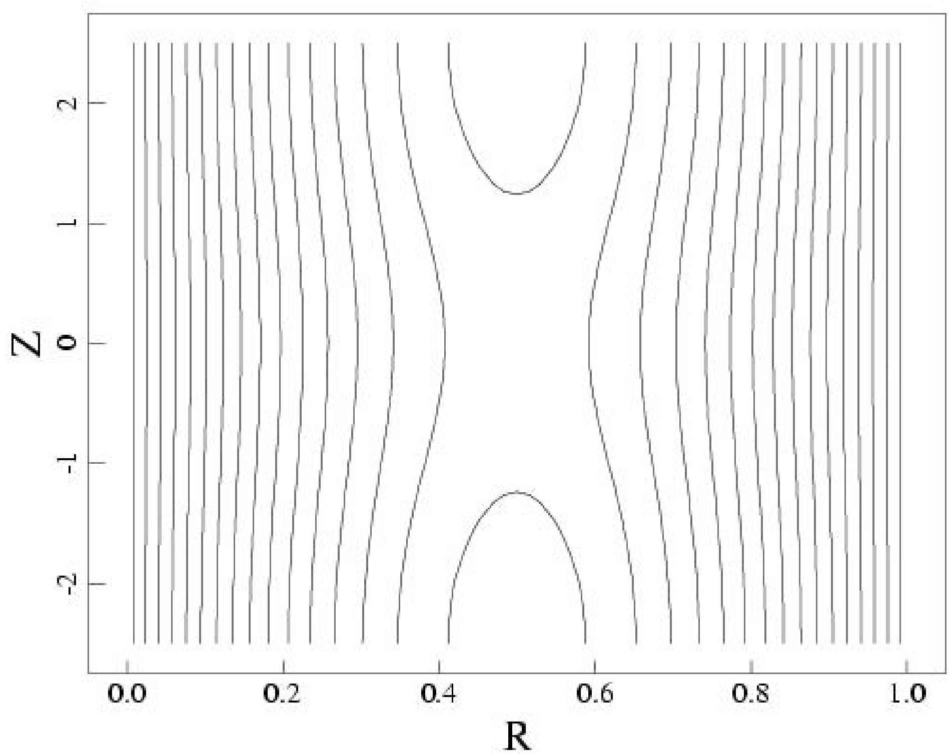
$\boldsymbol{Test}\ \boldsymbol{\pi}_{\|}$ closure in slab island geometry.

- **Evolve zero-β plasma momentum equation in slab island geometry to determine effect of stress anisotropy on steady-state flow profile:**

$$m_i n \left(\frac{\partial}{\partial t} + \vec{u} \cdot \vec{\nabla}\right) \vec{u} = \vec{J} \times \vec{B} - \vec{\nabla} \cdot \Pi_{\|_i} - \vec{\nabla} \cdot \Pi_{\perp}.$$

- **Evolve finite-β plasma momentum and temperature evolution equations to determine steady-state viscous heating:**

$$\frac{3}{2} n \left(\frac{\partial}{\partial t} + \vec{u} \cdot \vec{\nabla}\right) T = -p \vec{\nabla} \cdot \vec{u} - (\Pi_{\|_i} + \Pi_{\perp}) : \vec{\nabla} \vec{u} - \vec{\nabla} \cdot \vec{q} + \eta J^2$$



Conclusions

- Unified form for parallel ion stress, $\pi_\parallel$, has been constructed

- Viscosity coefficient, $\mu_\parallel$, maps continuously from collisional to nearly collisionless regimes with

  $$1.81_{\text{Braginskii}} < \frac{\mu_\parallel}{(v_{th}^2/\nu_{ii})} = 2.75 < 3.13_{\text{Chang/Callen}},$$

  in collisional regime and

  $$\frac{\mu_\parallel}{(v_{th}^2/k_\parallel v_{th})} = 1.04 \approx 1.06_{\text{Chang/Callen}}$$

  in nearly collisionless regime.

- Nonlocal $\pi_\parallel$ couples to nonlocal momentum restoring term introducing novel concept of nonlocal, momentum conserving collision operator.

- Addition of $\vec{\nabla}\cdot\Pi_\parallel = \vec{\nabla}\cdot(\hat{\mathrm{b}}\hat{\mathrm{b}} - \mathrm{I}/3)\pi_\parallel$ to plasma flow evolution equation captures anisotropic nature of momentum transport in moderately collisional to nearly collisionless plasmas

- Unified $\pi_\parallel$ may account for anomalous ion heating, $\Pi_\parallel : \vec{\nabla}\vec{V}$, in moderately collisional to nearly collisionless plasmas.