CEMM Meeting

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EXTENDED FLUID MODELS AT LOW COLLISIONALITY

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EXACT FLUID MOMENT EQUATIONS

FOR EACH SPECIES α:

$$\frac{d_\alpha n_\alpha}{dt} + n_\alpha \bar{\nabla}\cdot\bar{u}_\alpha = 0 \qquad \left(\frac{d_\alpha}{dt} \equiv \frac{\partial}{\partial t} + \bar{u}_\alpha\cdot\bar{\nabla}\right)$$

$$m_\alpha n_\alpha \frac{d_\alpha \bar{u}_\alpha}{dt} + \bar{\nabla} p_\alpha + \bar{\nabla}\cdot \overline{\overline{\Pi}}_\alpha = e_\alpha n_\alpha (\bar{E} + \bar{u}_\alpha \times \bar{B}) + \bar{F}_\alpha^{coll}$$

$$\frac{3}{2}\frac{d_\alpha p_\alpha}{dt} + \frac{5}{2} p_\alpha \bar{\nabla}\cdot\bar{u}_\alpha + \bar{\nabla}\cdot\bar{q}_\alpha + \overline{\overline{\Pi}}_\alpha : (\bar{\nabla}\bar{u}_\alpha) = H_\alpha^{coll}$$

$$p_\alpha = \frac{1}{3}(p_{\alpha\parallel} + 2 p_{\alpha\perp})$$

$$\overline{\overline{\Pi}}_\alpha = \overline{\overline{\Pi}}_{\alpha\parallel} + \overline{\overline{\Pi}}_\alpha^{GV} + \overline{\overline{\Pi}}_{\alpha\perp}^{coll}$$

$$\overline{\overline{\Pi}}_{\alpha\parallel} = (p_{\alpha\parallel} - p_{\alpha\perp})\left(\bar{b}\bar{b} - \frac{1}{3}\mathbb{1}\right)$$

GENERAL GYROVISCOUS STRESS

(Dropping the species index)

$$\Pi_{ij}^{GV} = \frac{1}{4}\,\varepsilon_{\{ik\ell}\left(\delta_{mj\}} + 3 b_m b_{j\}}\right) b_k \hat{\mathbb{K}}_{\ell m}$$

WHERE:

$$\hat{\mathbb{K}}_{ij} = \frac{m}{eB}\Big\{ p_\perp \frac{\partial u_{\{i}}{\partial x_{j\}}} + \frac{\partial (q_{T\parallel} b_{\{i})}{\partial x_{j\}}} + \frac{\partial \hat{M}_{ijk}}{\partial x_k} +$$

$$+ (p_\parallel - p_\perp)\left[\frac{\partial}{\partial t}(b_i b_j) + u_k \frac{\partial}{\partial x_k}(b_i b_j) + b_{\{i}\, b_k \frac{\partial u_{j\}}}{\partial x_k}\right] +$$

$$+ (2 q_{B\parallel} - 3 q_{T\parallel})\, b_k \frac{\partial}{\partial x_k}(b_i b_j) +$$

$$+ \frac{\partial \Pi_{ij}^{GV}}{\partial t} + u_k \frac{\partial \Pi_{ij}^{GV}}{\partial x_k} + \Pi_{ij}^{GV} \frac{\partial u_k}{\partial x_k} + \Pi_{\{ik}^{GV} \frac{\partial u_{j\}}}{\partial x_k} \Big\}$$

COLLISIONLESS PERPENDICULAR STRESS-FLUX TENSOR

$$\hat{\mathbb{M}}_{ijk} = \frac{1}{2}\left(\delta_{\{ij} - b_i b_j\right) q_{T\perp k\}} + 2\, b_{\{i}\, b_j\, q_{B\perp k\}} +$$

$$+\, \varepsilon_{\{ilm}\, b_j\, b_l\, (\delta_{nk\}} - b_n\, b_{k\}})\, \Pi_{mn}$$

WHERE:

$$\bar{q}_{T\perp} = \frac{m}{eB}\, \bar{b} \times \left[\frac{2P_\perp}{m}\, \bar{\nabla}\left(\frac{P_\perp}{n}\right) + \cdots\right]$$

$$\bar{q}_{B\perp} = \frac{m}{eB}\, \bar{b} \times \left[\frac{P_\perp}{2m}\, \bar{\nabla}\left(\frac{P_\parallel}{n}\right) + \frac{P_\parallel (P_\parallel - P_\perp)}{mn}\, \bar{\kappa} + \cdots\right]$$

$$\Pi_{ij} = \frac{m}{eB}\left[\frac{P_\perp (P_\parallel - P_\perp)}{4mnB}\, \frac{\partial B_{\{i}}{\partial x_{j\}}} + \cdots\right]$$

"SAFE" SIMPLIFYING ASSUMPTIONS

* NEGLECT COLLISIONAL PERPENDICULAR VISCOUS TENSORS $\Pi_{\alpha\perp}^{coll}$.

* NEGLECT ELECTRON MASS.

* ASSUME A SINGLE ION SPECIES OF UNIT CHARGE.

* ASSUME QUASINEUTRALITY:

$$n_e = n_i \equiv n$$

$$\bar{u}_e = \bar{u}_i - \frac{1}{ne}\bar{J} \equiv \bar{u} - \frac{1}{ne}\bar{J}$$

$$\bar{J} = \bar{\nabla} \times \bar{B}$$

TWO-FLUID EXTENDED-MHD

$$\frac{dn}{dt} + n\,\bar{\nabla}\cdot\bar{u} = 0$$

$$\bar{E} = -\bar{u}\times\bar{B} + \frac{1}{en}\left(\bar{j}\times\bar{B} - \bar{\nabla}p_e - \bar{\nabla}\cdot\Pi_{e\parallel} + \bar{F}_e^{coll}\right)$$

$$m_i n\,\frac{d\bar{u}}{dt} + \bar{\nabla}(p_i + p_e) + \bar{\nabla}\cdot\left(\Pi_{i\parallel} + \Pi_{e\parallel} + \Pi_i^{G.V}\right) = \bar{j}\times\bar{B}$$

$$\frac{3}{2}\,\frac{dp_e}{dt} - \frac{3}{2ne}\,\bar{j}\cdot\bar{\nabla}p_e + \frac{5}{2}\,p_e\,\bar{\nabla}\cdot\left(\bar{u} - \frac{1}{ne}\bar{j}\right) + \bar{\nabla}\cdot\bar{q}_e +$$

$$+\,\Pi_{e\parallel} : \left[\bar{\nabla}\left(\bar{u} - \frac{1}{ne}\bar{j}\right)\right] = H_e^{coll}$$

$$\frac{3}{2}\,\frac{dp_i}{dt} + \frac{5}{2}\,p_i\,\bar{\nabla}\cdot\bar{u} + \bar{\nabla}\cdot\bar{q}_i + \left(\Pi_{i\parallel} + \Pi_i^{G.V}\right) : \bar{\nabla}\bar{u} = H_i^{coll}$$

CLOSURE VARIABLES:

$(p_{e\parallel} - p_{e\perp})$, $(p_{i\parallel} - p_{i\perp})$,

$\bar{q}_e$, $\bar{q}_i$,

$$\bar{F}_e^{coll} = -\bar{F}_i^{coll}$$

$$H_e^{coll} = -H_i^{coll} + \frac{1}{ne}\,\bar{j}\cdot\bar{F}_e^{coll}$$

REGARDLESS OF THE CLOSURE MODEL, THE ABOVE SYSTEM POSSESSES AN EXACT ENERGY CONSERVATION LAW:

$$\frac{\partial}{\partial t}\left(\frac{1}{2} m_i n u^2 + \frac{3}{2} p_i + \frac{3}{2} p_e\right) + \bar{\nabla}\cdot\bar{\varphi} =$$
$$= \bar{j}\cdot\bar{E} = -\frac{\partial}{\partial t}\left(\frac{B^2}{2}\right) - \bar{\nabla}\cdot\left(\bar{E}\times\bar{B}\right),$$

WHERE $\bar{\varphi}$ IS THE TOTAL PLASMA ENERGY FLUX:

$$\bar{\varphi} = \frac{1}{2} m_i n u^2 \bar{u} + \left(2p_{i\perp} + \frac{1}{2} p_{i\parallel}\right)\bar{u} + \left(p_{i\parallel} - p_{i\perp}\right) u_\parallel \bar{b} +$$
$$+ \left(2p_{e\perp} + \frac{1}{2} p_{e\parallel}\right)\left(\bar{u} - \frac{1}{en}\bar{j}\right) + \left(p_{e\parallel} - p_{e\parallel}\right)\left(u_\parallel - \frac{1}{en} j_\parallel\right)\bar{b} +$$
$$+ \Pi_i^{GV}\cdot\bar{u} + \bar{q}_i + \bar{q}_e$$

FURTHER REDUCED MODELS

A: FAST DYNAMICS ORDERING

$\delta \sim \rho_i / L \ll 1$, $\partial/\partial t \sim \delta\, \omega_{ci}$, $u \sim v_{thi}$,

$q_{\parallel} \sim p\, v_{thi} \sim p u \quad (q_{\perp} \sim \delta\, p\, v_{thi})$

A1: FINITE ION LARMOR RADIUS

A2: ZERO ION LARMOR RADIUS

A2a: SINGLE FLUID (RESISTIVE-MHD-LIKE)

A2b: HALL-MHD-LIKE

B: SLOW DYNAMICS ORDERING

$\delta \sim \rho_i / L$, $\partial/\partial t \sim \delta^2 \omega_{ci}$, $u \sim \delta\, v_{thi}$,

$q_{\parallel} \sim \delta\, p\, v_{thi} \sim p u \sim q_{\perp}$

FINITE ION LARMOR RADIUS

A1: FAST DYNAMICS WITH FINITE ION LARMOR RADIUS

THE SYSTEM HAS THE SAME FORM AS THE GENERAL ONE, ONLY USING THE FIRST ORDER, FAST DYNAMICS ION GYROVISCOSITY:

$$\hat{K}_{ij}^{(1)} = \frac{m}{eB}\left\{ p_\perp \frac{\partial u_{\{i}}{\partial x_{j\}}} + \frac{\partial (q_{T\parallel} b_{i\}})}{\partial x_{j\}}} + (2q_{B\parallel} - 3q_{T\parallel}) b_k \frac{\partial (b_i b_j)}{\partial x_k} + (p_\parallel - p_\perp)\left[\frac{\partial (b_i b_j)}{\partial t} + u_k \frac{\partial (b_i b_j)}{\partial x_k} + b_{\{i} b_k \frac{\partial u_{j\}}}{\partial x_k} \right] \right\}$$

$$\Rightarrow \Pi_{ij}^{GV} = \frac{1}{4}\, \varepsilon_{\{ikl}\, (\delta_{mj\}} + 3 b_m b_{j\}})\, b_k\, \hat{K}_{lm}^{(1)} =$$

$$= b_{\{i} h_{\perp j\}}^{(1)} + \varepsilon_{\{ikl}\, (\delta_{mj\}} - b_m b_{j\}})\, b_k\, S_{lm}^{(1)}$$

WHERE

$$\bar{h}_\perp^{(1)} = \frac{m}{eB}\, \bar{b} \times \left[\bar{\nabla} q_{T\parallel} + 2(q_{B\parallel} - q_{T\parallel})\bar{\kappa} + 2p_\parallel (\bar{b}\cdot\bar{\nabla})\bar{u} + p_\perp \bar{b} \times \bar{\omega} \right],$$

$$S_{ij}^{(1)} = \frac{m}{4eB}\left[p_\perp \frac{\partial u_{\{i}}{\partial x_{j\}}} + \frac{q_{T\parallel}}{B} \frac{\partial B_{\{i}}{\partial x_{j\}}} \right]$$

A2a: SINGLE FLUID (RESISTIVE-MHD-LIKE)

* FAST DYNAMICS ORDERING.

* NEGLECT ALL FINITE ION GYRORADIUS, ION SOUND GYRORADIUS AND ION INERTIAL SKIN DEPTH EFFECTS:

$$\frac{\rho_i}{L} \to 0, \quad \frac{\rho_s}{L} \to 0, \quad \frac{d_i}{L} \to 0.$$

THEN, $\frac{1}{en} j \ll u \implies \bar{u}_e \simeq \bar{u}_i$,

$\frac{1}{en} |\bar{\nabla} \cdot \Pi_{e\parallel}| \lesssim \frac{1}{en} |\bar{\nabla} p_e| \ll |\bar{u} \times \bar{B}|$

AND $|\Pi_i^{GV}| \ll p_i$, SO THE SYSTEM REDUCES TO THE SINGLE FLUID FORM:

$$\bar{E} = -\bar{u} \times \bar{B} + \frac{1}{en} \bar{F}_e^{coll}$$

$$m_i n \frac{d\bar{u}}{dt} + \bar{\nabla}(p_i + p_e) + \bar{\nabla} \cdot (\Pi_{i\parallel} + \Pi_{e\parallel}) = \bar{j} \times \bar{B}$$

$$\frac{3}{2} \frac{d}{dt}(p_i + p_e) + \frac{5}{2}(p_i + p_e) \bar{\nabla} \cdot \bar{u} + \bar{\nabla} \cdot (\bar{q}_i + \bar{q}_e) + $$

$$+ (\Pi_{i\parallel} + \Pi_{e\parallel}) : (\bar{\nabla} \bar{u}) = \frac{1}{en} \bar{j} \cdot \bar{F}_e^{coll}$$

A2b: HALL-MHD-LIKE

* FAST DYNAMICS ORDERING.

* NEGLECT FINITE ION GYRORADIUS, BUT RETAIN FINITE ION SOUND GYRORADIUS AND FINITE ION INERTIAL SKIN DEPTH:

$$\frac{\rho_i}{L} \to 0 \,, \quad \frac{\rho_s}{L} \text{ AND } \frac{d_i}{L} \text{ FINITE.}$$

RECALL

$$\rho_s^2 = \frac{m_i (T_i + T_e)}{e^2 B^2} = \left(\frac{T_i + T_e}{T_i} \right) \rho_i^2 \,,$$

$$d_i^2 = \frac{m_i}{e^2 n} = \frac{2}{\beta_i} \rho_i^2 \,,$$

SO THIS ORDERING IS CONSISTENT IN THE LIMIT OF COLD IONS.

THE SYSTEM IS THE SAME AS THE GENERAL ONE WITHOUT THE ION GYROVISCOSITY TERMS, BECAUSE $|\Pi_i^{GV}| \sim \frac{\rho_i}{L} p_i \to 0$ BUT ALL OTHER TWO-FLUID EFFECTS ARE KEPT.

B: SLOW DYNAMICS ORDERING

$\delta \sim \rho_i/L \ll 1$, $\partial/\partial t \sim \delta^2 \omega_{ci}$, $u \sim \delta v_{thi}$, $q_{\parallel} \sim q_{\perp} \sim \delta p v_{thi}$

NOW WE HAVE:

$\Pi_i^{GV} \sim \delta^2 p_i$ AND $\bar{\nabla} \cdot \Pi_i^{GV} \sim m_i n \frac{d\bar{u}}{dt} \Rightarrow$

$\Rightarrow$ F-L-R ION GYROVISCOUS EFFECTS MUST BE RETAINED.

$\Rightarrow$ FINITE-d_i AND FINITE-ρ_s HALL EFFECTS (HENCE WHISTLER AND/OR KAW'S) MUST BE KEPT SINCE

$$\rho_s^2 = \left(\frac{T_i + T_e}{T_i}\right) \rho_i^2 \text{ AND } d_i^2 = \frac{2}{\beta_i} \rho_i^2 .$$

* NO CONSISTENT ARGUMENT KNOWN TO THIS SPEAKER WHY $(p_{\parallel} - p_{\perp})/p$ SHOULD BE ORDERED SMALL AT LOW COLLISIONALITY (EXCEPT FOR SPECIAL SITUATIONS SUCH AS AXISYMMETRIC EQUILIBRIA).

SECOND ORDER GYROVISCOUS STRESS IN SLOW DYNAMICS

(Dropping the species index)

$$\Pi_{ij}^{GV} = \frac{1}{4}\,\varepsilon_{\{ik\ell}\left(\delta_{mj\}} + 3\,b_m b_{j\}}\right) b_k\, \hat{K}_{\ell m}^{(2)}$$

WHERE:

$$\hat{K}_{ij}^{(2)} = \frac{m}{eB}\Big\{ p_\perp \frac{\partial u_{\{i}}{\partial x_{j\}}} + \frac{\partial (q_{T\parallel} b_{\{i})}{\partial x_{j\}}} + \frac{\partial \hat{M}_{ijk}^{(1)}}{\partial x_k} +$$

$$+ (p_\parallel - p_\perp)\left[\frac{\partial}{\partial t}(b_i b_j) + u_k \frac{\partial}{\partial x_k}(b_i b_j) + b_{\{i} b_k \frac{\partial u_{\{j}}{\partial x_{k\}}}\right] +$$

$$+ (2q_{B\parallel} - 3q_{T\parallel})\, b_k \frac{\partial}{\partial x_k}(b_i b_j)\Big\}$$

$$\hat{M}_{ijk}^{(1)} = \frac{1}{2}\left(\delta_{\{ij} - b_i b_j\right) q_{T\perp k\}}^{(1)} + 2\, b_{\{i} b_j\, q_{B\perp k\}}^{(1)} +$$

$$+ \varepsilon_{\{i\ell m}\, b_j b_\ell \left(\delta_{nk\}} - b_n b_{k\}}\right) \Pi_{mn}^{(1)}$$

$$\bar{q}_{T\perp}^{(1)} = \frac{m}{eB}\,\bar{b}\times\left[\frac{2p_\perp}{m}\,\bar{\nabla}\left(\frac{p_\perp}{n}\right) + \cdots\right]$$

$$\bar{q}_{B\perp}^{(1)} = \frac{m}{eB}\,\bar{b}\times\left[\frac{p_\perp}{2m}\,\bar{\nabla}\left(\frac{p_\parallel}{n}\right) + \frac{p_\parallel(p_\parallel - p_\perp)}{mn}\,\bar{\kappa} + \cdots\right]$$

$$\Pi_{ij}^{(1)} = \frac{m}{eB}\left[\frac{p_\perp(p_\parallel - p_\perp)}{4mnB}\,\frac{\partial B_{\{i}}{\partial x_{j\}}} + \cdots\right]$$

NEGLECTING $(p_{\parallel}-p_{\perp})$ AND KEEPING ONLY THE DIAMAGNETIC PART OF THE PERPENDICULAR HEAT FLUX:

$$\bar{q}_{\perp}^{(1)} = \frac{5p}{2eB}\,\bar{\nabla}\left(\frac{p}{n}\right)$$

AND

$$\hat{K}_{ij}^{(2)} = \frac{m}{eB}\left(p\,\frac{\partial u_{\{i}}{\partial x_{j\}}} + \frac{2}{5}\,\frac{\partial q_{\{i}}{\partial x_{j\}}}\right).$$

UNDER THE SLOW DYNAMICS ORDERING, $p\,\bar{\nabla}\bar{u}$ AND $\bar{\nabla}\bar{q}$ ARE COMPARABLE $\Longrightarrow$

$\Longrightarrow$ A) THE BRAGINSKII FORM $\hat{K}_{ij}^{(2)} = \frac{mp}{eB}\frac{\partial u_{\{i}}{\partial x_{j\}}}$ SHOULD NOT BE USED IN SLOW DYNAMICS (IT WAS INDEED DERIVED FOR FAST DYNAMICS, $u \sim V_{thi}$).

$\Longrightarrow$ B) THE "GYROVISCOUS CANCELLATION" IS INCONSEQUENTIAL.

IN THE SLOW-ORDERED MOMENTUM EQUATION:

$$m_i n \frac{d\bar{u}}{dt} + \bar{\nabla}(p_i + p_e) + \bar{\nabla}\cdot \Pi_i^{GV} - \bar{j}\times\bar{B} = \cdots,$$

$$m_i n \frac{d\bar{u}}{dt} \sim \bar{\nabla}\cdot \Pi_i^{GV} \sim \delta^2 \bar{\nabla} p_i \quad \Longrightarrow$$

$\Longrightarrow$ THE SCALAR PRESSURES MUST BE ACCURATE WITHIN $O(\delta^2)$ RELATIVE TO THEIR STATIC PARTS:

$$p_\alpha = p_\alpha^{(0)}[1 + O(\delta^2)] \quad \text{WITH} \quad \bar{\nabla} p_\alpha^{(0)} \sim \bar{j}\times\bar{B}$$

NOW, IN THE ENERGY EQUATIONS:

$$\frac{3}{2}\frac{dp_\alpha}{dt} + \frac{5}{2} p_\alpha \bar{\nabla}\cdot\bar{u} + \bar{\nabla}\cdot \bar{q}_\alpha = \cdots,$$

$$\frac{dp_\alpha}{dt} \sim p_\alpha \bar{\nabla}\cdot\bar{u} \sim p_\alpha^{(0)} \frac{v_{thi}}{L}\left[O(\delta) + O(\delta^3)\right]$$

$\Longrightarrow$ THE HEAT FLUXES MUST BE ACCURATE WITHIN $O(\delta^3)$ RELATIVE TO $p_\alpha^{(0)} v_{thi}$:

$$q_\alpha = p_\alpha^{(0)} v_{thi}\left[O(\delta) + O(\delta^3)\right]$$

SUMMARY OF CONDITIONS FOR A CONSISTENT, SLOW-ORDERED MODEL

* HALL TERM $\frac{1}{en}(\bar{J}\times\bar{B}-\bar{\nabla}p_e)$, ASSOCIATED WITH FINITE d_i AND ρ_s, KEPT IN OHM'S LAW.

* ION GYROVISCOSITY INCLUDING AT LEAST THE GRADIENTS OF THE $O(\delta p v_{thi})$ HEAT FLUXES.

  MANY MORE TERMS NEEDED IF $(p_\parallel - p_\perp)/p$ CANNOT BE ORDERED SMALL.

  NO USE MAKING THE "GYROVISCOUS CANCELLATION" EXPLICIT.

* HEAT FLUXES ACCURATE TO $O(\delta^3 p v_{thi})$ IN ENERGY EQUATIONS.