Implicit Low-Moment Closure

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Motivation

- Current Gyrokinetic-Maxwell Equations in use are not fully electromagnetic
  - The $A_\parallel - \phi$ field model does not have $\delta B_\parallel$
- GK equations, if derivable, might not be solvable in the edge or ITB with strong "equilibrium" variations in
  - $\mathbf{E} \times \mathbf{B}$ flow of thermal speed
  - density or temperature over $\sim 10\rho_i$
- For ETG simulations with non-adiabatic ion, N-point averaging with $N \gg 4$ is needed. It could be easier to follow the gyro-motion.

The Vlasov ion/Drift kinetic electron model

Vlasov ions:

$$\frac{d\mathbf{v}_i}{dt} = \frac{q}{m_i}(\mathbf{E} + \mathbf{v}_i \times \mathbf{B}), \qquad \frac{d\mathbf{x}_i}{dt} = \mathbf{v}_i$$

Drift kinetic electrons: $\varepsilon = \frac{1}{2}m_e v^2$

$$\frac{d\mathbf{x}}{dt} = \mathbf{v}_G \equiv v_\parallel \left(\mathbf{b} + \frac{\delta\mathbf{B}_\perp}{B_0}\right) + \mathbf{v}_D + \mathbf{v}_E$$

$$\frac{d\varepsilon}{dt} = -e\mathbf{v}_G \cdot \mathbf{E} + \mu\frac{\partial B}{\partial t}, \qquad \frac{d\mu}{dt} = 0$$

Ampere's equation

$$\nabla \times \mathbf{B} = \mu_0(\mathbf{J}_i - en_e(\mathbf{V}_{\mathrm{e}\perp} + u_{\parallel\mathrm{e}}\mathbf{b}))$$

$$\mathbf{V}_{\mathrm{e}\perp} = \frac{1}{B}\mathbf{E} \times \mathbf{b} - \frac{1}{enB}\mathbf{b} \times \nabla P_{\perp\mathrm{e}}$$

$$\mathbf{J}_i = \int f_i\mathbf{v}\, d\mathbf{v}, \qquad u_{\parallel\mathrm{e}} = \int f_e\, v_\parallel\, d\mathbf{v}, \qquad P_{\perp\mathrm{e}} = \int f_e\frac{1}{2}m_e v^2\, d\mathbf{v}$$

Faraday's equation,

$$\frac{\partial\mathbf{B}}{\partial t} = -\nabla \times \mathbf{E}$$

Convert Ampere's Equation into Ohm's Law

Starting with Ampere's equation

$$\nabla \times \mathbf{B} = \mu_0(\mathbf{J}_i - en_e(\mathbf{V}_{\mathrm{e}\perp} + u_{\parallel\mathrm{e}}\mathbf{b}))$$

Taking derivative w.r.t. time,

$$\nabla \times \frac{\partial \mathbf{B}}{\partial t} = \mu_0 \left(\frac{\partial \mathbf{J}_i}{\partial t} - en_e(\frac{\partial \mathbf{V}_{\mathrm{e}\perp}}{\partial t} + \frac{\partial u_{\parallel\mathrm{e}}}{\partial t}\mathbf{b}) \right)$$

Only use the parallel component of this equation! Use Faraday's equation for LHS,

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E}$$

and electron momentum equation,

$$m_e n_e \frac{\partial u_{\parallel\mathrm{e}}}{\partial t} + \nabla_\parallel \delta P_{\parallel e} + \delta \mathbf{B} \cdot \nabla P_{\parallel\mathrm{e}0} + en_e E_\parallel = 0$$

And neglect $\mathbf{b} \cdot \frac{\partial J_i}{\partial t}$ (smaller by mass ratio), we then obtain parallel Ohm's Law

$$enE_\parallel + \frac{m_e}{\mu_0 e}\mathbf{b} \cdot \nabla \times \nabla \times \mathbf{E} = -\nabla_\parallel \delta P_{\parallel\mathrm{e}} - \frac{\delta \mathbf{B}}{B} \cdot \nabla P_{\parallel\mathrm{e}0}$$

The remaining two components of the Ampere's equation are rewritten as

$$en\mathbf{E}_\perp = -\frac{1}{\mu_0}\mathbf{b} \times (\nabla \times \mathbf{B}) - \mathbf{J}_i \times \mathbf{b} - \nabla_\perp \delta P_{\perp\mathrm{e}}$$

Explicit time evolving is unstable at small $k_\perp$ due to the compressional Alfvén wave

Combine the momentum equation and the Maxwell equations to obtain Ohm's law:

$$
\begin{aligned}
en\mathbf{E}_\perp &= -\frac{1}{\mu_0}\mathbf{b} \times (\nabla \times \mathbf{B}) - \mathbf{J}_i \times \mathbf{b} - \nabla_\perp \delta P_{\perp \mathrm{e}} \\
enE_\parallel + \frac{m_e}{\mu_0 e}\mathbf{b} \cdot \nabla \times \nabla \times \mathbf{E} &= -\nabla_\parallel \delta P_{\parallel \mathrm{e}} \\
\frac{\partial \mathbf{B}}{\partial t} &= -\nabla \times \mathbf{E}
\end{aligned}
$$

δf method for ions and electrons

$$
\begin{aligned}
\frac{d}{dt}\delta f_i &= -q\mathbf{E} \cdot \mathbf{v}_i f_{0\mathrm{i}} \\
\frac{d}{dt}\delta f_e &= -\left(\mathbf{v}_E + v_\parallel \frac{\delta \mathbf{B}_\perp}{B_0}\right) \cdot \nabla f_{0\mathrm{e}} + \left(-eE_\parallel v_\parallel + \mu \frac{\partial B}{\partial t}\right) f_{0\mathrm{e}}
\end{aligned}
$$

For ρ_i scale instabilities $k_\perp \rho_i \sim 1$, $\beta \sim 1\%$, the compressional wave frequency $\omega/\Omega_i \geq 10$, $\Omega_i \triangle t << 0.01$ is needed! We would like to be able to use $\Omega_i \triangle t \sim 0.1$, i.e., just small enough to get the gyro-motion.

- Quasi-neutral
  - No displacement current in the Faraday's equation
- No transverse electron inertia (no electron polarization current). Electron FLR and polarization current can be added for reconnection problems.
- The magnetic field perturbation is 3-D, whereas in the $A_\parallel - \phi$ model $\delta\mathbf{B} = \nabla \times (A_\parallel \mathbf{b})$ is 2-D
- Unable to combine $A_\parallel - \phi$ field model with Vlasov ions. With GK ions ϕ is obtained from GK Poisson equation. With Vlasov ions the equation

$$n_i = n_e$$

  does not determine ϕ! However, taking time derivative of this equation to the second order results in

$$\frac{n_0 q}{m_i}\nabla_\perp^2\phi - \frac{q}{m_i}\nabla\phi\cdot\nabla n_0 + \frac{e}{m_e}\nabla_\parallel E_\parallel$$
$$= \nabla\cdot\left(\frac{1}{m_e}\nabla\cdot\mathbf{P}_i - \frac{qn_0}{m_i}\mathbf{V}_i\times\mathbf{B}\right) - \frac{1}{m_e}\nabla_\parallel\left(\frac{\delta\mathbf{B}}{B}\cdot\nabla(n_0T_0) + \nabla_\parallel\delta P_{\parallel\mathrm{e}}\right) + \frac{\dot{\mathbf{E}}\times\mathbf{b}}{B}\cdot\nabla n_0$$

  We have not been able to produce the Alfven waves solving this equation.

Implicit Scheme

$$\frac{\delta \mathbf{B}^{n+1} - \delta \mathbf{B}^{n}}{\triangle t} = -\nabla \times \mathbf{E}^{n+1}$$

$$\mathbf{E}_{\perp}^{n+1} - \frac{\triangle t}{\beta}\mathbf{b} \times (\nabla \times \nabla \times \mathbf{E}^{n+1}) = -\nabla_{\perp}\delta P_{\perp \mathrm{e}}^{n+1} - \frac{1}{\beta}\mathbf{b} \times (\nabla \times \delta \mathbf{B}^{n}) - \mathbf{J}_{\mathrm{i}\perp}^{*} \times \mathbf{b} - \delta \mathbf{J}_{\perp \mathrm{i}} \times \mathbf{b}$$

$$E_{\parallel}^{n+1} + \frac{m_e}{m_i}\frac{1}{\beta}\mathbf{b} \cdot \nabla \times \nabla \times \mathbf{E}^{n+1} = -\nabla_{\parallel}\delta P_{\parallel \mathrm{e}}^{n+1}$$

- Particle coordinates and electron weights (hence pressure) explicitly advanced.
- Ions weights first advanced without $\mathbf{E}_{\perp}$ then used to gather $\mathbf{J}_{\mathrm{i}\perp}^{*}$

$$w_i^{*} = w_i^{n} + qE_{\parallel}^{n} v_{\parallel} \triangle t$$

- After fields solved, update ion weights

$$w_i^{\mathrm{n+1}} = w_i^{*} + qE_{\perp}^{\mathrm{n+1}} \cdot \mathbf{v}_{\perp} \triangle t$$

Implicit Scheme (cont'd)

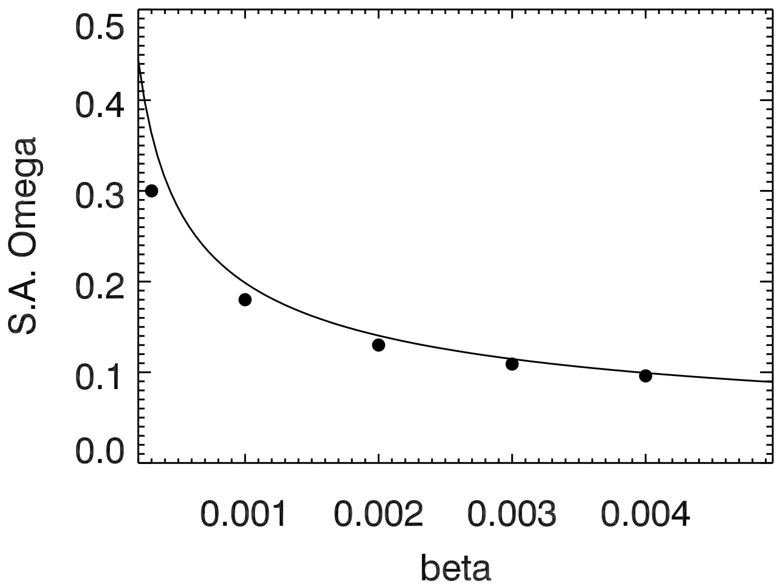
- It turns out necessary to treat the increment to $\mathbf{J}_{\perp \mathrm{i}}$ due to $\mathbf{E}_{\perp}^{\mathrm{n+1}}$ fully implicitly

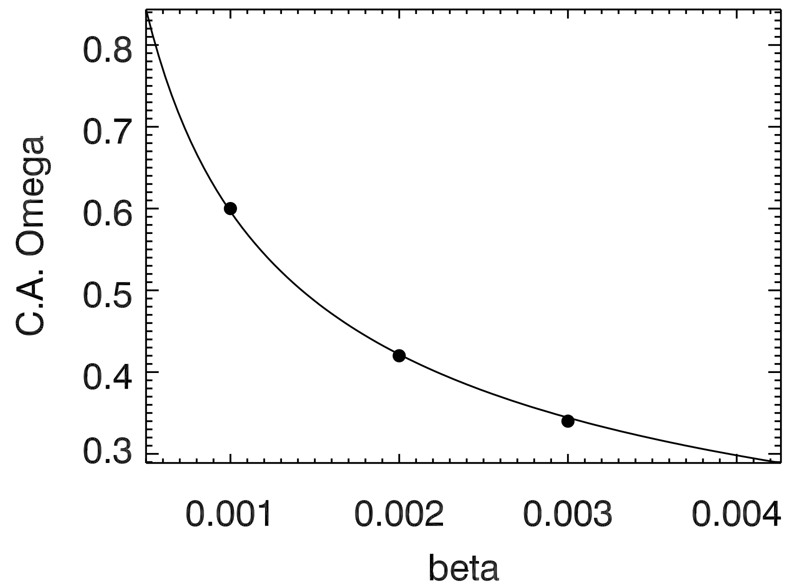
$$\delta \mathbf{J}_{\perp \mathrm{i}}(\mathbf{x}) = \triangle t \sum_j q \mathbf{E}_{\perp}^{\mathrm{n+1}}(\mathbf{x}_j^{\mathrm{n+1}}) \cdot \mathbf{v}_j^{\mathrm{n+1}} S(\mathbf{x} - \mathbf{x}_j^{\mathrm{n+1}})$$

$$\delta \mathbf{J}_{\perp \mathrm{i}}(\mathbf{x}) \approx q\, n_i \triangle t\, \mathbf{E}_{\perp}^{\mathrm{n+1}}(\mathbf{x}) \equiv \delta \mathbf{J}'_{\perp \mathrm{i}}$$

- Iterate on the difference between $\delta \mathbf{J}_{\perp \mathrm{i}}$ and $\delta \mathbf{J}'_{\perp \mathrm{i}}$
- $3 \sim 4$ iterations are accurate enough

3-D Shearless Slab Alfven Wave Simulation





$32 \times 32 \times 32$ grids, $1,048,576$ particles per species
For shear Alfvén wave, $k_\perp = 0$, $k_\parallel \rho_i = 0.00626$, initialize with $\delta \mathbf{B}_\perp$.
For compressional Alfven wave, $k_\parallel = 0$, $k_\perp \rho_i = 0.019$, initiallize with $\delta \mathbf{B}_\parallel$.

∇T_e Driven Whistler Instability

Neglect ions, $\nabla \delta P_{\perp \mathrm{e}}$ in perpendicular Ohm's law,

$$\frac{\partial f_1}{\partial t} + v_\parallel \nabla_\parallel f_1 = \kappa (E_y + v_\parallel B_x) f_0 + (-E_\parallel v_\parallel + \frac{\partial B_\parallel}{\partial t}) f_0$$

$$E_\parallel + \mathbf{b} \cdot \nabla \times \nabla \times \mathbf{E} - \kappa_T B_x = -\nabla_\parallel \delta P_{\parallel \mathrm{e}}$$

$$\mathbf{E}_\perp + \frac{1}{\beta} \mathbf{b} \times (\nabla \times \mathbf{B}_1) = 0$$

$$\frac{\partial \mathbf{B}_1}{\partial t} = -\nabla \times \mathbf{E}$$

$\kappa = \kappa_T (mv^2/2 - 3/2)$, $\kappa_T = -\dfrac{1}{T} \dfrac{\partial f_0}{\partial x}$

Dispersion relation

$$\begin{vmatrix} 1-\frac{i}{\omega\beta}k_xk_y & \frac{i}{\omega\beta}(k_x^2+k_\parallel^2) & -\frac{i}{\omega\beta}k_\parallel k_y \\ -\frac{i}{\omega\beta}(k_y^2+k_\parallel^2) & 1+\frac{i}{\omega\beta}k_xk_y & \frac{i}{\omega\beta}k_\parallel k_x \\ ik_\parallel S_1-\frac{m_e}{m_i}\frac{1}{\beta}k_\parallel k_x & ik_\parallel S_2+\kappa_T\frac{k_\parallel}{\omega}-\frac{m_e}{m_i}\frac{1}{\beta}k_\parallel k_y & 1+ik_\parallel S_3-\kappa_T\frac{k_y}{\omega}+\frac{m_e}{m_i}\frac{1}{\beta}k_\perp^2 \end{vmatrix}=0$$

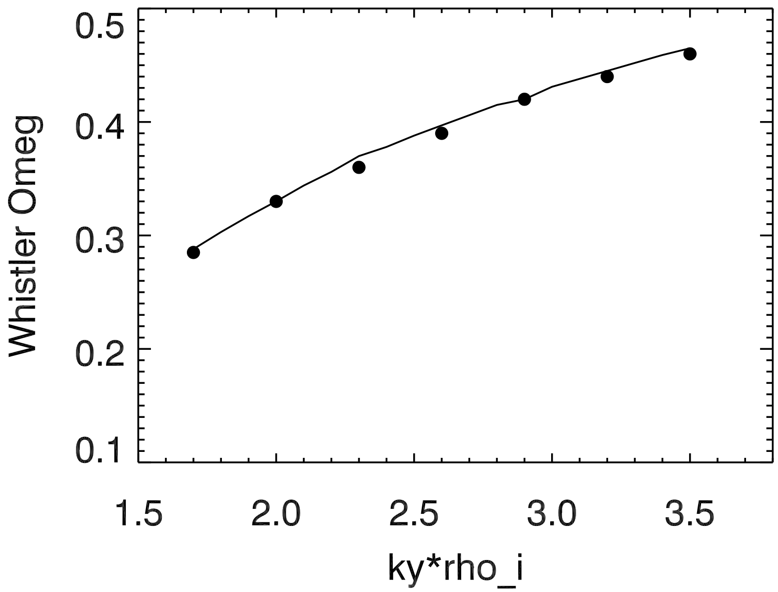
$$S_1=-m_ek_yI_2$$

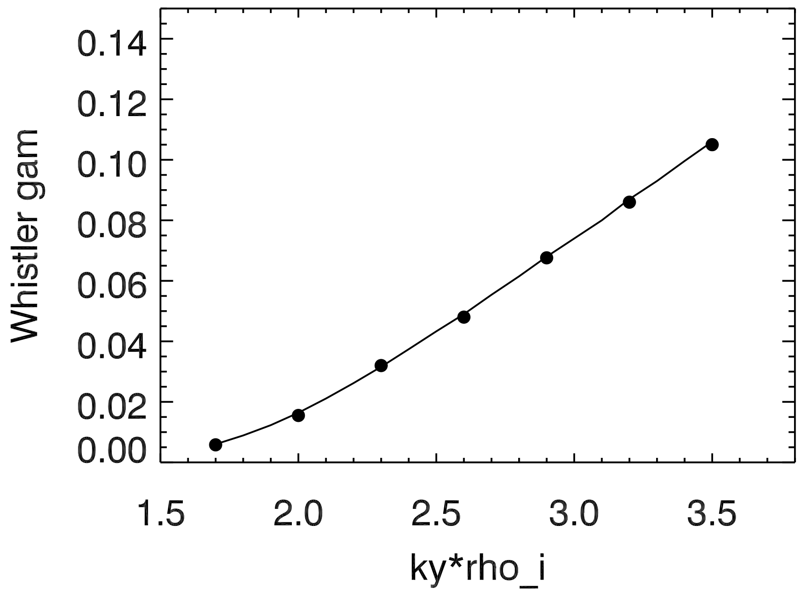
$$S_2=im_e\{-\frac{1}{2}\kappa_TI_2+\frac{1}{2}m_e\kappa_TI_4-\frac{k_\parallel}{\omega}(-\frac{1}{2}\kappa_TI_3+\frac{1}{2}m_e\kappa_TI_5)-ik_xI_2\}$$

$$S_3=im_e\{\frac{k_y}{\omega}(\kappa_TI_3/2+\frac{1}{2}m_e\kappa_TI_5)-I_3\}$$

$$I_n=\int\frac{v_\parallel^n}{\omega-k_\parallel v_\parallel}\frac{1}{\sqrt{2\pi}v_T}e^{-v_\parallel^2/2v_T^2}\,dv_\parallel$$

Whistler Instability–Simulation





- $\beta = 0.064$, $k_\parallel \rho_i = 0.0284$, $\kappa_{\rm Te}\rho_i = 0.1$
- $16 \times 32 \times 32$ grids, $1,048,576$ particles per species, $\triangle t \Omega_i = 0.1$.
- For small γ $(k_\perp \rho_i = 1.7,\ 2.0)$ smaller time step needed to reduce numerical damping of implicit scheme
- Ions are strongly stabilizing. Even unstable, the modes are not whistler.

The MHD equations for the shear Alfven wave

- **Quasi-neutrality**

$$\boxed{-\frac{n_0 m_i}{B^2}\nabla_\perp{}^2\phi = \delta n_i - \delta n_e}$$

- **Continuity equations**

$$\left.\begin{aligned}\frac{\partial \delta n_i}{\partial t} + n_0\hat{b}\cdot\nabla u_{\|i} + \vec{E}\times\hat{b}\cdot\nabla n_0 &= 0\\ \frac{\partial \delta n_e}{\partial t} + n_0\hat{b}\cdot\nabla u_{\|e} + \vec{E}\times\hat{b}\cdot\nabla n_0 &= 0\end{aligned}\right\}$$

$$\boxed{\frac{\partial}{\partial t}(\delta n_i - \delta n_e) + n_0\hat{b}\cdot\nabla(u_{\|i} - u_{\|e}) = 0}$$

- **Ampere's law**

$$\boxed{-\nabla_\perp A_\| = \mu_0 q n_0 (u_{\|i} - u_{\|e})}$$

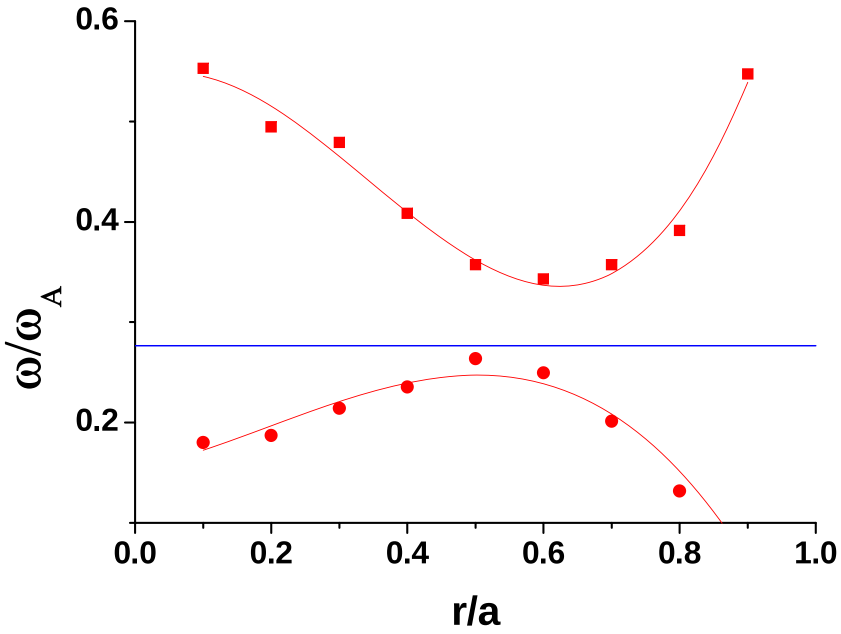
- **Faraday's law**

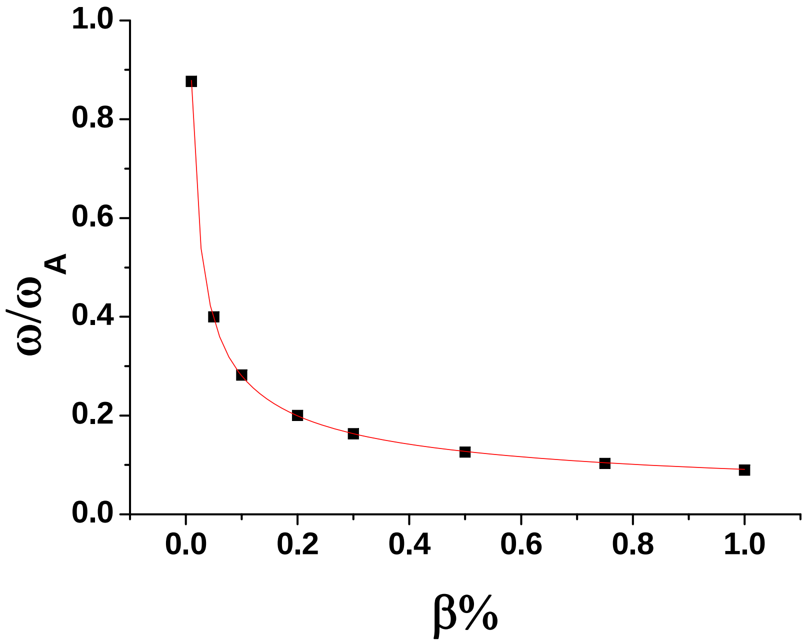
$$\boxed{\frac{\partial A_\|}{\partial t} + \hat{b}\cdot\nabla_\perp\phi = 0}$$

- **MHD TAE equation**

$$\boxed{\frac{\partial^2}{\partial t^2}\frac{1}{V_A{}^2}\nabla_\perp{}^2\phi = \hat{b}\cdot\nabla\nabla_\perp{}^2\hat{b}\cdot\nabla\phi}$$

The simulation results





- Field-line-following coordinates are employed in the simulations
- Global geometry is applied
- The gap spectra are observed
- There is a global Alfven wave with the frequency fallen in the gap

- BUT, the frequency of the global Alfven wave does not agree with analytical MHD result

SUMMARY

- We proposed a kinetic simulation model with Vlasov ions/Drift kinetic electrons which is
  - Quasi-neutral and fully electromagnetic
  - suitable for MHD scales and plasmas with strong $\mathbf{E} \times \mathbf{B}$ flows
- The time step for explicit integration limited by the compressional Alfvén wave
- Semi-implicit scheme allows $\Omega_i \triangle t \geq 0.1$
  - Treat Faraday's law and $\mathbf{E}_\perp \cdot \mathbf{v}_\perp$ in the ion weight equation implicitly
- Demonstrated 3-D shearless slab simulation for compressional and shear Alfvén waves, and whistler instabilities driven by electron temperature gradient