Simulating Plasma Flows In Tokamaks

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Theses:

1) Fluid moment equations for n, T and $\vec{V}$ with neoclassical-type closure for $\langle \vec{B} \cdot \vec{\nabla} \cdot \overleftrightarrow{\pi} \rangle$ provide framework for tokamak plasma transport equations.

2) New approach: a) starts from kinetics not Braginskii; b) solves radial, parallel, toroidal force balances; & c) uses E_r for ambipolar particle fluxes.

3) Next step issue for M3D and NIMROD codes is to explore dissipative parallel/poloidal flow damping effects to obtain trapped-particle effects and bootstrap current in neoclassical $\|$ Ohm's law, and poloidal ion flow.

Outline:

Motivation and multi-stage strategy

Faster time scale constraint from ion radial force balance

Simulating $\|$ viscous stress, force — for neo $\|$ Ohm's law, poloidal ion flow

Toroidal rotation equation (and E_r) from condition for no radial current

Net flux-surface-average density equation — and some consequences

Summary

Motivation: Develop Xport Equations For Low Collisionality

- **Tokamak plasma transport equations for modeling codes (e.g., ONETWO, TRANSP) are usually obtained from n, $\vec{V}$, T fluid moment equations with collisional Braginskii closures, and then *ad hoc* terms are added for**

  neoclassical effects on $\|$ Ohm's law (trapped particle effects on $\eta_\|$, bootstrap current),

  fluctuation-induced transport induced by micro-turbulence,

  heating, current-drive and flow sources & sinks,

  effects of small 3D magnetic field asymmetries, etc.

- **But tokamak plasmas are not in a collisional regime! — And we should develop transport equations that naturally include all these other effects.**

- **Here, we develop[1] self-consistent fluid-moment-based radial transport equations that include all these effects for nearly axisymmetric single-ion-species tokamak plasmas using neoclassical-based closures instead of Braginskii's.**

- **The procedures (solve for flows in flux surfaces first) and net plasma transport equations are analogous to those developed for stellarator transport.**

[1] J.D. Callen, A.J. Cole, C.C. Hegna, "Toroidal flow and particle flux in tokamak plasmas," UW-CPTC 08-7, April 2009 (`www.cptc.wisc.edu`); Monday afternoon poster S1.00047 at upcoming Sherwood (APS) meeting.

Multi-Stage Strategy Is Used To Determine Xport Equations

- **I. Average the density, momentum and energy equations over fluctuations (average over toroidal angle ζ) and then flux-surface-average (FSA) them.**

- **II. Consider sequentially specific components of the momentum equation:**

  IIA. *Radial* **($\sim$ μs): Use zeroth order radial force balance enforced by comp. Alfvén waves to obtain relation between toroidal, poloidal flows & electric field E_r, dp_i/dr.**

  IIB. *Parallel* **($\sim$ ms): Use FSA parallel viscous forces to obtain the parallel neoclassical Ohm's law and and ion poloidal flow from $\|$ equilibrium momentum, heat flux equations.**

  IIC. *Toroidal* **($\sim$ s): Require ambipolar radial particle fluxes ($\langle \vec{J} \cdot \vec{\nabla} \rho \rangle = 0$) at second order to obtain FSA toroidal momentum equation, and hence toroidal rotation and E_r.**

  KEY POINT is to solve first for poloidal flows, then toroidal flows on longer time scale.

- **III. Substitute net second order ambipolar fluxes back into FSA transport equations to obtain final comprehensive "radial" transport equations.**

I. Average Moment Equations Over Fluctuations, Then FSA

- **First, use a small gyroradius expansion to order various terms.**

- **Next, "ensemble (ζ-) average" over fluctuations (overbar) & flux-surface-average (FSA, $\langle\cdots\rangle$) density, energy equations $[V' \equiv dV(\rho)/d\rho]$:**

$$\text{density:}\quad \left.\frac{\partial n_0}{\partial t}\right|_{\vec{x}} + \frac{1}{V'}\frac{\partial}{\partial\rho}\left(V'\,\Gamma_0\right) = \langle \bar{S}_n\rangle, \qquad \boxed{\Gamma_0 \equiv \langle\,(n_0\bar{\vec{V}}_2 + \overline{\tilde{n}_1\tilde{\vec{V}}_1})\cdot\vec{\nabla}\rho\rangle,}$$

$$\text{energy:}\quad \frac{3}{2}\left.\frac{\partial p_0}{\partial t}\right|_{\vec{x}} + \frac{1}{V'}\frac{\partial}{\partial\rho}\left[V'\left\langle\left(\bar{\vec{q}}_2 + \frac{5}{2}\left(p_0\vec{V}_2 + \overline{\tilde{p}_1\tilde{\vec{V}}_1}\right)\right)\cdot\vec{\nabla}\rho\right\rangle\right]$$
$$= \langle\bar{Q}_\Delta\rangle - \left\langle \bar{\vec{R}}_1\cdot\bar{\vec{V}}_1 + \overline{\tilde{\vec{R}}_1\cdot\tilde{\vec{V}}_1}\right\rangle + \left\langle \bar{\vec{V}}_2\cdot\vec{\nabla}p_0 + \overline{\tilde{\vec{V}}_1\cdot\vec{\nabla}\tilde{p}_1}\right\rangle - \left\langle \bar{\overleftrightarrow{\pi}} : \vec{\nabla}\bar{\vec{V}}_1\right\rangle + \langle\bar{S}_E\rangle.$$

- **Finally, similarly average the momentum (force balance) equation and determine its radial ($\vec{\nabla}\rho\,\cdot$) component and the FSA of its parallel ($\vec{B}_0\,\cdot$) and toroidal angular ($\vec{e}_\zeta\,\cdot = R\,\hat{\vec{e}}_\zeta\,\cdot$) components (minus some terms in $\|$, t eqns):**

$$\text{radial } \mathcal{O}\{\delta^0\}:\quad mn_0\frac{\partial\vec{V}}{\partial t} = nq(\vec{E} + \vec{V}\times\vec{B}) - \vec{\nabla}p \quad \overset{\sum_s}{\Longrightarrow} \quad \rho_m\frac{\partial\vec{V}}{\partial t} = \vec{J}\times\vec{B} - \vec{\nabla}P,$$

$$\text{parallel } \mathcal{O}\{\delta\}:\quad mn_0\frac{\partial\langle\vec{B}_0\cdot\bar{\vec{V}}\rangle}{\partial t} = n_0q\langle\vec{B}_0\cdot\bar{\vec{E}}^A\rangle - \langle\vec{B}_0\cdot\vec{\nabla}\cdot\bar{\overleftrightarrow{\pi}}\rangle + \langle\vec{B}_0\cdot\bar{\vec{R}}\rangle + \langle\vec{B}_0\cdot\bar{\vec{S}}_m\rangle - mn_0\langle\vec{B}_0\cdot\overline{\tilde{\vec{V}}\cdot\vec{\nabla}\tilde{\vec{V}}}\,\rangle,$$

$$\text{toroidal } \mathcal{O}\{\delta^2\}:\quad \left.\frac{\partial}{\partial t}\right|_{\vec{x}}\langle\vec{e}_\zeta\cdot mn_0\bar{\vec{V}}\rangle = \boxed{q\,\Gamma_0} + q\langle\vec{e}_\zeta\cdot\overline{\widetilde{n\vec{V}}\times\tilde{\vec{B}}}\rangle - \langle\vec{e}_\zeta\cdot\vec{\nabla}\cdot\bar{\overleftrightarrow{\pi}}\rangle - \langle\vec{\nabla}\cdot\overline{mn(\vec{e}_\zeta\cdot\tilde{\vec{V}})\tilde{\vec{V}}}\rangle + \cdots.$$

II. *Order $\delta^0, \delta^1, \delta^2$ Force Balance Equations Have Consequences*

- **δ^0: Zeroth order fluid moment equations yield ideal MHD model.**

- **IIA. Compressional Alfvén waves $\perp$ to $\vec{B}_0$ enforce $\vec{J}_0 \times \vec{B}_0 = \vec{\nabla} P_0$ plus Ohm's law $\vec{E}_0 + \vec{V} \times \vec{B}_0 = (\vec{J}_0 \times \vec{B}_0 - \vec{\nabla} p_e)/n_e e$ yields radial force balance:**

$$0 = \vec{e}_\rho \cdot [n_i q_i (\vec{E} + \vec{V} \times \vec{B}) - \vec{\nabla} p_i] \quad \Longrightarrow \quad \Omega_t \equiv \vec{V} \cdot \vec{\nabla} \zeta = -\left(\frac{d\Phi}{d\psi_p} + \frac{1}{n_i q_i} \frac{dp_i}{d\psi_p} - q \, \vec{V} \cdot \vec{\nabla} \theta \right)$$

$$\Longrightarrow \quad \boxed{V_t \simeq \frac{E_r}{B_p} - \frac{1}{n_i q_i} \frac{dp_i}{dr} + \frac{B_t}{B_p} V_p \,,} \quad \text{relation between toroidal, pol. flows and } E_r,\ dp_i/dr.$$

- **Maxwellianization of electron, ion distributions on their collision times of $1/\nu_e, 1/\nu_i$ cause n, T to be constant over collision lengths λ_e, λ_i and hence on flux surfaces, and flows $\vec{V}$ to become physically meaningful.**

- **δ^1: First order flows are on magnetic flux surfaces (θ, ζ or $\wedge, \|$ directions):**

$$\vec{\bar{V}}_1 \equiv \underbrace{\vec{e}_\theta (\vec{\bar{V}} \cdot \vec{\nabla} \theta)}_{\text{poloidal}} + \underbrace{\vec{e}_\zeta (\vec{\bar{V}} \cdot \vec{\nabla} \zeta)}_{\text{toroidal}} = \underbrace{\bar{V}_\| \vec{B}_0 / B_0}_{\text{parallel}} + \underbrace{\vec{\bar{V}}_\wedge}_{\text{cross}} \,, \qquad \vec{\bar{V}}_{s\wedge} \equiv \underbrace{\frac{\vec{B}_0 \times \vec{\nabla} \psi_p}{B_0^2} \left(\frac{d\Phi_0}{d\psi_p} + \frac{1}{n_{s0} q_s} \frac{dp_{s0}}{d\psi_p} \right)}_{\vec{E} \times \vec{B} \text{ and diamagnetic}} .$$

- **δ^2: Radial flows $\perp$ to flux surfaces are second order: $\vec{V}_2 \cdot \vec{\nabla} \psi_p \neq 0$ — to calculate, need to determine flows in surface first, as in stellarators.**

IIB. Electron Parallel Force Balance Yields $\parallel$ Ohm's Law

- Flux surface average of $\parallel$ component of fluctuation-averaged momentum equation yields first order parallel force balances ($\vec{B} = \vec{B}_0$, $\langle \bar{S}_n \rangle = 0$ here):

$$m_s n_{s0} \frac{\partial \langle B \bar{V}_{s\parallel} \rangle}{\partial t} = n_{s0} q_s \langle \vec{B} \cdot \bar{\vec{E}}^A \rangle - \langle \vec{B} \cdot \vec{\nabla} \cdot \overset{\leftrightarrow}{\pi}_s \rangle + \langle \vec{B} \cdot \bar{\vec{R}}_s \rangle + \langle \vec{B} \cdot \bar{\vec{S}}_m \rangle - m_s n_{s0} \langle \vec{B} \cdot \overline{\tilde{\vec{V}} \cdot \vec{\nabla} \tilde{\vec{V}}} \rangle + n_{s0} q_s \langle \vec{B} \cdot \overline{\tilde{\vec{V}}_\wedge \times \tilde{\vec{B}}_\perp} \rangle .$$

- For times $t > 1/\nu_e \sim 10\,\mu$s, equilibrium electron $\parallel$ force balance becomes

$$0 = -\, n_e e \langle \vec{B} \cdot \vec{E}^A \rangle - \langle \vec{B} \cdot \vec{\nabla} \cdot \overset{\leftrightarrow}{\pi}_e \rangle + \langle \vec{B} \cdot \vec{R}_e \rangle + \langle \vec{B} \cdot \vec{S}_{em} \rangle - m_e n_{e0} \langle \vec{B} \cdot \overline{\tilde{\vec{V}}_e \cdot \vec{\nabla} \tilde{\vec{V}}_e} \rangle - n_{e0} e \langle \vec{B} \cdot \overline{\tilde{\vec{V}}_e \times \tilde{\vec{B}}_\perp} \rangle .$$

- Using the collisional friction relation $\vec{B}_0 \cdot \bar{\vec{R}}_e = -\, \bar{\vec{B}}_0 \cdot \vec{R}_i \simeq n_{e0}\, e\, B_0 J_\parallel / \sigma_\parallel$, this equation yields the neoclassical parallel Ohm's law:

$$\boxed{\langle B_0 J_\parallel \rangle \;=\; \underbrace{\sigma_\parallel \langle \vec{B}_0 \cdot \bar{\vec{E}}^A \rangle}_{\text{ohmic current}} + \underbrace{(\sigma_\parallel / n_{e0} e) \langle \vec{B}_0 \cdot \vec{\nabla} \cdot \overset{\leftrightarrow}{\pi}_{e\parallel} \rangle}_{\text{tp, bootstrap current}} + \underbrace{\langle \vec{B}_0 \cdot \vec{J}_{\rm CD} \rangle}_{\text{current drive}} + \underbrace{\langle \vec{B}_0 \cdot \vec{J}_{\rm dyn} \rangle}_{\text{dynamo}} .}$$

- Parallel currents are driven by $\parallel$ electron mom. sources and fluctuations:

$$\langle \vec{B}_0 \cdot \vec{J}_{\rm CD} \rangle \equiv -\, (\sigma_\parallel / n_{e0} e) \langle \vec{B}_0 \cdot \left(\bar{\vec{S}}_{em} - m_e \bar{\vec{V}}_e \bar{S}_{en} \right) \rangle \quad \text{— non-inductive current drive,}$$

$$\langle \vec{B}_0 \cdot \vec{J}_{\rm dyn} \rangle = \underbrace{(m_e \sigma_\parallel / e) \langle \vec{B}_0 \cdot \overline{(\tilde{\vec{V}}_e \cdot \vec{\nabla} \tilde{\vec{V}}_e + \vec{\nabla} \cdot \overset{\leftrightarrow}{\pi}_{e\wedge})} \rangle}_{\parallel \text{ Reynolds stress}} + \underbrace{\sigma_\parallel \langle \vec{B}_0 \cdot \overline{\tilde{\vec{V}}_e \times \tilde{\vec{B}}_\perp} \rangle}_{\text{Maxwell stress}} \quad \text{— fluctuation-driven.}$$

IIB. Poloidal Flow Is Obtained From Plasma $\parallel$ Force Balance

- Summing the parallel force balances over species yields (for $\bar{S}_n = 0$)

$$m_i n_{i0} \frac{\partial \langle B_0 V_{i\|} \rangle}{\partial t} \simeq -\langle \vec{B}_0 \cdot \vec{\nabla} \cdot \overset{\leftrightarrow}{\pi}_i \rangle - m_i n_0 \langle \vec{B}_0 \cdot \overline{\tilde{\vec{V}}_i \cdot \vec{\nabla} \tilde{\vec{V}}_i} \rangle + \langle \vec{B}_0 \cdot \overline{\tilde{\vec{J}}_\wedge \times \tilde{\vec{B}}_\perp} \rangle + \langle \vec{B}_0 \cdot \textstyle\sum_s \vec{S}_{sm} \rangle.$$

- The poloidal flow is determined mainly from parallel ion viscous force:

$$\langle \vec{B} \cdot \vec{\nabla} \cdot \overset{\leftrightarrow}{\pi}_{i\|} \rangle \simeq m_i n_{i0} \left[\mu_{i00} U_{i\theta} + \mu_{i01} \frac{-2}{5 n_i T_i} Q_{i\theta} + \cdots \right] \langle B^2 \rangle, \qquad \mu_{i00}, \mu_{i01} \sim \sqrt{\epsilon}\, \nu_i.$$

- For $t > 1/\nu_i \sim 1$ ms, poloidal flow is usually obtained from $\langle \vec{B} \cdot \vec{\nabla} \cdot \overset{\leftrightarrow}{\pi}_{i\|} \rangle \simeq 0$:

$$U^0_{i\theta}(\psi_p) \equiv \frac{\vec{V} \cdot \vec{\nabla} \theta}{\vec{B} \cdot \vec{\nabla} \theta} \simeq -\frac{\mu_{i01}}{\mu_{i00}} \frac{-2}{5 n_i T_i} Q_{i\theta} \simeq \frac{c_p I}{q_i \langle B^2 \rangle} \frac{dT_{i0}}{d\psi_p} \quad \Longrightarrow \quad \boxed{V_p \simeq \frac{1.17}{q_i B} \frac{dT_{i0}}{dr} + \mathcal{O}\{\delta^2\}.}$$

- Including all the drives in the parallel plasma force balance above yields

$$\boxed{U_{i\theta}(\psi_p) \simeq \underbrace{U^0_{i\theta}(\psi_p)}_{\text{neoclassical}} - \underbrace{\frac{\langle \vec{B}_0 \cdot (\overline{\tilde{\vec{V}}_i \cdot \vec{\nabla} \tilde{\vec{V}}_i + \vec{\nabla} \cdot \overset{\leftrightarrow}{\pi}_{i\wedge})} \rangle}{\mu_{i00} \langle B_0^2 \rangle}}_{\|\ \text{Reynolds stress}} + \underbrace{\frac{\langle \vec{B}_0 \cdot \overline{\tilde{\vec{J}}_\wedge \times \tilde{\vec{B}}_\perp} \rangle + \langle \vec{B}_0 \cdot \sum_s \vec{S}_{sm} \rangle}{m_i n_{i0} \mu_{i00} \langle B_0^2 \rangle}}_{\|\ \text{Maxwell stress + flow sources}} .}$$

- Having determined the poloidal flow, the toroidal flow is ($\Omega_{*p} \equiv I\, U_{i\theta}/R^2$):

$$\Omega_t \equiv \vec{V} \cdot \vec{\nabla} \zeta = -\left(\frac{d\Phi}{d\psi_p} + \frac{1}{n_i q_i} \frac{dp_i}{d\psi_p} \right) + \Omega_{*p} \quad \Longrightarrow \quad \boxed{V_t \simeq \frac{E_r}{B_p} - \frac{1}{n_i q_i B_p} \frac{dp_i}{dr} + \frac{1.17}{q_i B_p} \frac{dT_i}{dr}.}$$

How Can We Add These $\|$ Flow Damping Effects To Codes?

- **Braginskii viscous force due to CGL form for parallel stresses is**

$$\overset{\leftrightarrow}{\pi}_\| \equiv \pi_\| \left(\frac{\vec{B}\vec{B}}{B^2} - \frac{\overset{\leftrightarrow}{\mathbf{I}}}{3} \right), \qquad \pi_\| \equiv -\frac{3}{2}\eta_0 \frac{\vec{B}\cdot\overset{\leftrightarrow}{\mathbf{W}}\cdot\vec{B}}{B^2}, \qquad \overset{\leftrightarrow}{\mathbf{W}} \equiv \vec{\nabla}\vec{V} + (\vec{\nabla}\vec{V})^{\mathsf{T}} - \frac{2}{3}\overset{\leftrightarrow}{\mathbf{I}}(\vec{\nabla}\cdot\vec{V}).$$

- **Parallel component of parallel rate of strain has a couple of forms:**

$$\boxed{\vec{B}\cdot\overset{\leftrightarrow}{\mathbf{W}}\cdot\vec{B}/2} = B(\vec{B}\cdot\vec{\nabla})(\vec{V}\cdot\vec{B}/B) + [\vec{B}\times(\vec{B}\times\vec{V})]\cdot\vec{\kappa} - (B^2/3)\vec{\nabla}\cdot\vec{V}$$

$$\boxed{= B^2\vec{V}\cdot\vec{\nabla}\ln B + \vec{B}\cdot\vec{\nabla}\times(\vec{V}\times\vec{B}) + (2B^2/3)\vec{\nabla}\cdot\vec{V} - (\vec{B}\cdot\vec{V})(\vec{\nabla}\cdot\vec{B}).}$$

- **For $\vec{\nabla}\cdot\vec{B} = 0$, $\vec{\nabla}\cdot\vec{V} = 0$ and $\vec{V}_\perp = (1/B^2)\vec{B}\times\vec{\nabla}f$, the last form yields**

$$\pi_\| = -3\eta_0\,(\vec{V}\cdot\vec{\nabla}\ln B) + \Delta\pi_\|, \quad \text{where } \Delta\pi_\| \equiv -(3\eta_0/B^3)(\vec{B}\cdot\vec{\nabla}f)[\vec{B}\cdot\vec{\nabla}\times(\vec{B}/B)] \text{ is small.}$$

- **Viscous force for the Braginskii viscous stress is ($\vec{\kappa}$ is curvature vector)**

$$\vec{\nabla}\cdot\overset{\leftrightarrow}{\pi}_\| = \pi_\|\,[\vec{\kappa} - \vec{B}(\vec{B}\cdot\vec{\nabla}\ln B)/B^2] + (1/B^2)\vec{B}(\vec{B}\cdot\vec{\nabla})\pi_\| - (1/3)\vec{\nabla}\pi_\|$$

$$\Longrightarrow \qquad \vec{B}\cdot\vec{\nabla}\cdot\overset{\leftrightarrow}{\pi}_\| = -\pi_\|\,(\vec{B}\cdot\vec{\nabla}\ln B) + (2/3)(\vec{B}\cdot\vec{\nabla})\pi_\|.$$

- **FSA of this neglecting $\Delta\pi_\|$ and using $\vec{V}\cdot\vec{\nabla}\ln B = (\vec{B}\cdot\vec{\nabla}\ln B)\,U_\theta(\psi_p)$ is**

$$\boxed{\langle\vec{B}\cdot\vec{\nabla}\cdot\overset{\leftrightarrow}{\pi}_\|\rangle = 3\eta_0\,\langle(\vec{B}\cdot\vec{\nabla}\ln B)^2\rangle U_\theta,} \text{ with } U_\theta(\psi) \equiv \frac{\vec{V}\cdot\vec{\nabla}\theta}{\vec{B}\cdot\vec{\nabla}\theta} \text{ from } 0 = \vec{\nabla}\cdot\vec{V} = (\vec{B}\cdot\vec{\nabla}\theta)\frac{\partial}{\partial\theta}\left(\frac{\vec{V}\cdot\vec{\nabla}\theta}{\vec{B}\cdot\vec{\nabla}\theta}\right).$$

Adding Parallel Flow Damping Effects To Codes (continued)

- **The neolassical parallel viscous force is given by the same form:**

  $\langle \vec{B} \cdot \vec{\nabla} \cdot \overleftrightarrow{\pi}_{\parallel} \rangle = mn\mu \langle B^2 \rangle U_\theta$, which relates to Braginskii via $\mu \equiv 3\eta_0 \dfrac{\langle (\vec{B} \cdot \vec{\nabla} \ln B)^2 \rangle}{mn \langle B^2 \rangle}$.

- **The neoclassical poloidal flow damping frequency μ for electrons is**

  $\mu_e \simeq \dfrac{2.3\sqrt{\epsilon}\,\nu_e}{(1+\nu_{*e}^{1/2}+\nu_{*e})(1+\epsilon^{3/2}\nu_{*e})}$, for collisionality parameter $\nu_{*e} \equiv \dfrac{\nu_e}{\epsilon^{3/2}\omega_{te}} = \dfrac{Rq}{\epsilon^{3/2}\lambda_e}$.

  $\Longrightarrow$ banana regime for $\nu_{*e} \ll 1$, plateau for $1 \ll \nu_{*e} \ll \epsilon^{-3/2}$, Braginskii for $\nu_{*e} \gg \epsilon^{-3/2}$.

- **PROPOSAL: Implement Braginskii operator with neo. viscous damping frequency μ in M3D and NIMROD[2]? Some issues for such a proposition:**

  Best form of $\pi_\parallel$ to use? $\vec{V}_e \to -\vec{J}/n_e e \sim \nabla^2 \vec{B}$ yields 4th order operator in $\partial \vec{B}/\partial t$ eqn.

  Poloidal variation of viscous force for $\nu_* \ll 1$ not properly captured — but do we care?

  Long parallel scale variations should still be relaxed with Braginskii coefficient η_0?

  Heat flow offsets $[U_\theta \to U_\theta - (c_p I/q_i \langle B^2 \rangle)(dT_{i0}/d\psi_p)]$ to damp flows to nonzero values.

  Need Z_{eff} effects on μ for realistic tokamak plasma situations.

- **Ultimate test of procedure is via Held et al. kinetic-based approach.**

[2] C.R. Sovinec, `www.cptc.wisc.edu/sovinec_research/notes/e_viscosity2.pdf`; C.R. Sovinec et al., 2007 Sherwood Conf., Annapolis, MD.

IIC. Toroidal Torques From Force Balance Give Radial Flows

- A key vector identity for determining radial flows is ($\vec{e}_\zeta \equiv R^2\vec{\nabla}\zeta = R\hat{\vec{e}}_\zeta$)

$\vec{e}_\zeta \cdot \vec{V}\times\vec{B}_0 = -\vec{V}\cdot\vec{e}_\zeta\times\vec{B}_0 = \vec{V}\cdot\vec{\nabla}\psi_p$ — toroidal component of $\vec{V}\times\vec{B}_0$ gives radial flow.

- Thus, taking toroidal angular ($\vec{e}_\zeta\,\cdot$) component of species force balance and averaging over fluctuations and a flux surface yields particle flux:

$$\langle n_0\vec{\bar{V}}_2\cdot\vec{\nabla}\psi_p\rangle + \langle\overline{\tilde{n}_1\vec{\tilde{V}}_1}\cdot\vec{\nabla}\psi_p\rangle \qquad \text{average plus fluctuation-induced radial particle flux,}$$

$$= \frac{1}{q}\Big[-\langle\vec{e}_\zeta\cdot\vec{\bar{R}}\rangle + \langle\vec{e}_\zeta\cdot\vec{\nabla}\cdot\overleftrightarrow{\bar{\pi}}\rangle\Big] - n_0\langle\vec{e}_\zeta\cdot\vec{\bar{E}}^A\rangle \qquad \text{collision-induced particle fluxes } \Gamma_\nu,\ \Gamma_\pi,$$

$$-\langle\vec{e}_\zeta\cdot n_0\overline{\vec{\tilde{V}}_1\times\vec{\tilde{B}}}\rangle - \frac{1}{q}\langle\vec{e}_\zeta\cdot\vec{\bar{S}}_m\rangle + \frac{1}{q}\left(\frac{\partial}{\partial t}\Big|_{\vec{x}}[mn_0\langle\vec{e}_\zeta\cdot\vec{\bar{V}}_1\rangle] + \langle\vec{\nabla}\cdot\overline{mn(\vec{e}_\zeta\cdot\vec{\tilde{V}}_1)\vec{\tilde{V}}_1}\rangle\right), \text{ fluct., inertia.}$$

- This equation must also be transformed from $\vec{x}$ to ψ_p coordinates using

$$\langle\vec{e}_\zeta\cdot\mathcal{D}\{mn_0\vec{\bar{V}}_1\}\rangle \simeq -\dot{\rho}_{\psi_p}\frac{\partial}{\partial\rho}[mn_0\langle\vec{e}_\zeta\cdot\vec{\bar{V}}_1\rangle] + \langle\vec{\nabla}\cdot[mn_0(\vec{e}_\zeta\cdot\vec{\bar{V}}_1)\vec{u}_G]\rangle + \frac{1}{V'}\frac{\partial^2}{\partial\rho^2}[V'\bar{D}_\eta mn_0\langle\vec{e}_\zeta\cdot\vec{\bar{V}}_1\rangle].$$

- Using $\vec{e}_\zeta \equiv R^2\vec{\nabla}\zeta = I\vec{B}_0/B_0^2 - \vec{B}_0\times\vec{\nabla}\psi_p/B_0^2$ and $\vec{\bar{R}}_e \simeq n_{e0}e(\vec{J}_\parallel/\sigma_\parallel + \vec{J}_\wedge/\sigma_\perp)$:

$$\frac{1}{q_s}\langle\vec{e}_\zeta\cdot\vec{\bar{R}}_s\rangle = \frac{I}{q_s}\left\langle\frac{\vec{B}_0\cdot\vec{\bar{R}}_s}{B_0^2}\right\rangle - \frac{1}{q_s}\left\langle\frac{\vec{B}_0\times\vec{\nabla}\psi_p}{B_0^2}\cdot\vec{\bar{R}}_s\right\rangle = -\frac{n_{e0}I}{\sigma_\parallel}\left\langle\frac{J_\parallel B_0}{B_0^2}\right\rangle + \frac{n_{e0}}{\sigma_\perp}\left\langle\frac{|\vec{\nabla}\psi_p|^2}{B_0^2}\right\rangle\frac{dP_0}{d\psi_p}.$$

Particle Flux Has Many Contributions I: 7 Ambipolar

- **The radial particle flux can be written in terms of its various components:**

$$\Gamma \equiv \langle \vec{\Gamma}\cdot\vec{\nabla}\rho\rangle \equiv \langle n_{s0}(\bar{\vec{V}}_2 - \vec{u}_G)\cdot\vec{\nabla}\rho\rangle + \langle\overline{\tilde{n}_1\tilde{\vec{V}}_1}\cdot\vec{\nabla}\rho\rangle - (\partial/\partial\rho)[\,V'\,\bar{D}_\eta\, n_{s0}] \equiv \Gamma_\nu^a + \Gamma^{na} + \Gamma_{\rm pc}^a$$

$$= \underbrace{\Gamma_{\rm cl} + \Gamma_{\rm PS} + \Gamma_{\rm bp} + \Gamma_{\rm pc} + \Gamma_{\rm CD} + \Gamma_{\rm dyn} + \Gamma_{\rm E^A}}_{\Gamma_\nu^a + \Gamma_{\rm pc}^a,\ \text{ambipolar (superscript } a)} + \underbrace{\Gamma_{\pi\|}^{\rm NA} + \Gamma_{\pi\perp} + \Gamma_{\rm pol} + \Gamma_{\rm Rey} + \Gamma_{\rm Max} + \Gamma_{\rm JxB} + \Gamma_{\dot{\psi}_p} + \Gamma_{\rm S}}_{\Gamma^{na},\ \text{non-ambipolar (superscript } na)}.$$

- **Ambipolar Fluxes**[3] **($\psi_p' \equiv d\psi_p/d\rho \simeq B_p R\, a$):**

$$\Gamma_{\rm cl} = \left\langle \frac{\vec{B}_0\times\vec{\nabla}\rho}{B_0^2}\cdot\frac{\bar{\vec{R}}_{s\perp}}{q_s}\right\rangle = -\frac{n_{e0}}{\sigma_\perp}\left\langle\frac{|\vec{\nabla}\rho|^2}{B_0^2}\right\rangle\frac{dP_0}{d\rho}, \qquad D_{\rm cl} \simeq \frac{T_e+T_i}{\sigma_\perp\langle B_0^2\rangle} \simeq \nu_e\varrho_e^2, \quad \text{classical,}$$

$$\Gamma_{\rm PS} = -\frac{n_{e0}I^2}{\sigma_\|\,\psi_p'^2}\left\langle\frac{1}{B_0^2}\left(1-\frac{B_0^2}{\langle B_0^2\rangle}\right)^2\right\rangle\frac{dP_0}{d\rho}, \quad D_{\rm PS} \simeq \frac{2\sigma_\perp}{\sigma_\|}q^2 D_{\rm cl} \sim q^2 D_{\rm cl}, \quad \text{Pfirsch-Schlüter,}$$

$$\Gamma_{\rm bp} = \frac{I}{e\psi_p'\langle B_0^2\rangle}\langle\vec{B}_0\cdot\vec{\nabla}\cdot\overleftrightarrow{\bar{\pi}}_{e\|}\rangle, \qquad D_{\rm bp} \simeq \mu_e\varrho_{ep}^2 \sim \frac{q^2}{\epsilon^{3/2}}D_{\rm cl}, \quad \text{banana-plateau,}$$

$$\Gamma_{\rm pc} = -\left(\bar{D}_\eta\frac{dn_{e0}}{d\rho} + n_{e0}V_{\rm pc}\right), \quad V_{\rm pc} \equiv \frac{1}{V'}\frac{\partial}{\partial\rho}(V'\bar{D}_\eta), \quad D_\eta \equiv \frac{\eta_\|^{\rm nc}}{\mu_0} \sim \frac{D_{\rm cl}}{\beta_e}, \quad \text{paleoclassical,}$$

$$\Gamma_{\rm CD} + \Gamma_{\rm dyn} = [(n_{e0}I)/(\sigma_\|\,\psi_p'\langle B_0^2\rangle)]\langle\vec{B}_0\cdot(\vec{J}_{\rm CD}+\vec{J}_{\rm dyn})\rangle, \quad \text{current drive, dynamo effects,}$$

$$\Gamma_{\rm E^A} = -\,n_{e0}\langle\vec{e}_\zeta\cdot\bar{\vec{E}}^A\rangle\left(1 - I^2\langle 1/R^2\rangle/\langle B_0^2\rangle\right)/\psi_p', \qquad \bar{\vec{E}}^A\times\vec{B}_p/B_0^2 \ \text{radial pinch.}$$

[3]K.C. Shaing, S.P. Hirshman, and J.D. Callen, Phys. Fluids **29**, 521 (1986); K.C. Shaing, Phys. Fluids **29**, 2231 (1986).

Particle Flux Has Many Contributions II: 8 Non-ambipolar

- **Non-ambipolar fluxes (Γ's here are multiplied by $\psi_p' \equiv d\psi_p/d\rho \simeq B_p R\, a$):**

$$\Gamma_{\pi\|}^{\rm NA} = \frac{1}{q_s}\langle \vec{e}_\zeta\cdot\vec{\nabla}\cdot\overleftrightarrow{\pi}^{\rm NA}_{s\|}\rangle \simeq \frac{m_i n_{i0}\mu_{i\,t}}{q_i}\left(\frac{\tilde{B}_{\rm eff}}{B_0}\right)^2(\langle R^2\Omega_t\rangle - \langle R^2\Omega_*\rangle), \;\; \Omega_* \simeq \frac{c_p+c_t}{q_i}\,\frac{dT_i}{d\psi_p}, \;\; \text{NA } \| \text{ visc.},$$

$$\Gamma_{\pi\perp} = \frac{1}{q_s}\langle\vec{e}_\zeta\cdot\vec{\nabla}\cdot\overleftrightarrow{\pi}_{s\perp}\rangle \simeq \frac{1}{q_i}\left\langle \vec{e}_\zeta\cdot\vec{\nabla}\cdot(\overleftrightarrow{\pi}^{\rm cl}_{i\perp}+\overleftrightarrow{\pi}^{\rm nc}_{i\perp}+\overleftrightarrow{\pi}^{\rm pc}_{i\perp})\right\rangle \sim -\chi_t\nabla^2\Omega_t, \;\; \chi_t\sim(1+q^2)\nu_i\varrho_i^2 + D_\eta,$$

$$\Gamma_{\rm pol} = \frac{1}{q_s}\,\frac{\partial}{\partial t}\Big|_{\psi_p} m_s n_{s0}\langle\, \vec{e}_\zeta\cdot\vec{\bar{V}}_s\rangle \simeq \frac{1}{q_i}\,\frac{\partial}{\partial t}\Big|_{\psi_p} m_i n_{i0}\langle R^2\Omega_t\rangle, \quad \text{ion polarization flow for } \frac{\partial\Omega_t}{\partial t}\neq 0,$$

$$\Gamma_{\rm Rey} = \frac{1}{q_s V'}\frac{\partial}{\partial\rho}(V'\Pi_{s\rho\zeta}), \;\; \boxed{\Pi_{s\rho\zeta}\equiv m_s n_{s0}\langle\overline{(\vec{\nabla}\rho\cdot\vec{\tilde{V}}_s)(\vec{\tilde{V}}_s\cdot\vec{e}_\zeta)}\rangle + \langle\vec{\nabla}\rho\cdot\overline{\overleftrightarrow{\pi}_{s\wedge}}\cdot\vec{e}_\zeta\rangle,} \;\; \text{Reynolds stress},$$

$$\Gamma_{\rm Max} = -\,\langle\vec{e}_\zeta\cdot\overline{\widetilde{n_1\vec{V}_1}\times\vec{\tilde{B}}}\rangle \simeq \frac{1}{e}\langle\vec{e}_\zeta\cdot\overline{\vec{\tilde{J}}\times\vec{\tilde{B}}}\rangle = \frac{1}{e\mu_0}\langle\vec{e}_\zeta\cdot\overline{\vec{\tilde{B}}\cdot\vec{\nabla}\vec{\tilde{B}}}\rangle, \qquad \text{Maxwell stress},$$

$$\Gamma_{\rm JxB} \simeq \frac{1}{e}\langle\vec{e}_\zeta\cdot\overline{\vec{\tilde{J}}_{\|mn}\times\vec{\tilde{B}}_{\perp mn}}\rangle \simeq \delta(\rho-\rho_{mn})\,\frac{c_{A\theta}}{e}\,\frac{\omega\; m_i n_{i0}R}{\Delta'^2+(\omega\tau_\delta)^2}\,\frac{\tilde{B}^2_{rmn}}{B_0^2}, \;\; \text{FE-induced res. layer},$$

$$\Gamma_{\dot{\psi}_p} = \frac{\dot{\rho}_{\psi_p}}{q_s}\,\frac{\partial}{\partial\rho}(m_s n_{s0}\langle\vec{e}_\zeta\cdot\vec{\bar{V}}_s\rangle), \qquad \psi_p \text{ transients},$$

$$\Gamma_{s\rm S} = -\,\frac{1}{q_s}\langle\vec{e}_\zeta\cdot\vec{S}_{sm}\rangle, \qquad \text{momentum sources (e.g., NBI, CD).}$$

IIC. Radial Current Is Obtained By Summing Particle Fluxes Over Species And Yields Toroidal Torque Balance

- **Sum radial species currents to obtain net radial plasma current:**

$$\langle \vec{J}\cdot\vec{\nabla}\rho\rangle \equiv \sum_s q_s\left(\Gamma^a_{s\nu}+\Gamma^a_{\rm spc}+\Gamma^{na}_s\right) = \sum_s q_s\,\Gamma^{na}_s \quad \text{— sum of non-ambipolar currents.}$$

- **Charge continuity equation on a ψ_p surface obtained by summing q_s times density equations over species is ($\dot{\rho}_{\psi_p}=0$ and $\sum_s q_s\langle\bar{S}_{sn}\rangle=0$ for simplicity)**

$$\left.\frac{\partial}{\partial t}\right|_{\psi_p}(V'\langle\rho_q\rangle)+\frac{1}{V'}\frac{\partial}{\partial\rho}(V'\langle\vec{J}\cdot\vec{\nabla}\rho\rangle)=0 \overset{\epsilon_0\vec{\nabla}\cdot\vec{E}=\rho_q}{\Longrightarrow} \frac{1}{V'}\frac{\partial}{\partial\rho}\left[V'\left(\epsilon_0\left.\frac{\partial}{\partial t}\right|_{\psi_p}\langle\vec{E}\cdot\vec{\nabla}\rho\rangle+\langle\vec{J}\cdot\vec{\nabla}\rho\rangle\right)\right]=0.$$

- **"Vacuum" term $\partial\langle\vec{E}\cdot\vec{\nabla}\rho\rangle/\partial t$ is $\sim c^2_{Ap}/c^2\sim 10^{-5}\ll 1$ smaller than the neoclassical polarization flow from $\partial\Omega_t/\partial t\sim\partial\langle\vec{E}\cdot\vec{\nabla}\rho\rangle/\partial t$ in $\langle\vec{J}\cdot\vec{\nabla}\rho\rangle\sim q_i\Gamma_{i\rm pol}$; thus, this quasineutral charge continuity equation requires $\langle\vec{J}\cdot\vec{\nabla}\rho\rangle=0$.**

- **Setting $\langle\vec{J}\cdot\vec{\nabla}\rho\rangle$ to zero yields comprehensive toroidal torque balance equation for the toroidal angular momentum density $L_t\equiv m_i n_{i0}\langle R^2\Omega_t\rangle$:**

$$\boxed{\underbrace{\frac{1}{V'}\left.\frac{\partial}{\partial t}\right|_{\psi_p}(V'L_t)}_{\text{inertia}} \simeq -\underbrace{\langle\vec{e}_\zeta\cdot\vec{\nabla}\cdot\overset{\leftrightarrow}{\pi}{}^{\rm NA}_{i\parallel}\rangle}_{\text{NTV from }\tilde{B}_\parallel} - \underbrace{\langle\vec{e}_\zeta\cdot\vec{\nabla}\cdot\overset{\leftrightarrow}{\pi}_{i\perp}\rangle}_{\text{cl, neo, paleo}} - \underbrace{\frac{1}{V'}\frac{\partial}{\partial\rho}(V'\Pi_{i\rho\zeta})}_{\text{Reynolds stress}} + \underbrace{\langle\vec{e}_\zeta\cdot\overline{\tilde{\vec{J}}\times\tilde{\vec{B}}}\rangle}_{\text{res. FE, Max}} - \underbrace{\dot{\rho}_{\psi_p}\frac{\partial L_t}{\partial\rho}}_{\psi_p\text{ motion}} + \underbrace{\langle\vec{e}_\zeta\cdot\textstyle\sum_s\bar{\vec{S}}_{sm}\rangle}_{\text{sources}}.}$$

IIC. ***Toroidal Rotation Determines Radial Electric Field Required For Net Ambipolar Radial Particle Flux***

- **From toroidal rotation** $\langle\Omega_t\rangle \equiv L_t/(m_i n_{i0}\langle R^2\rangle)$**, radial electric field** E_ρ **is:**

$$\boxed{E_\rho \;\equiv\; -\,|\vec{\nabla}\rho\,|\,\frac{d\Phi_0}{d\rho} \;\simeq\; |\vec{\nabla}\rho\,|\left(\langle\Omega_t\rangle\psi_p' + \frac{1}{n_{i0}q_i}\,\frac{dp_{i0}}{d\rho} - \frac{c_p}{q_i}\,\frac{dT_{i0}}{d\rho}\right),} \qquad |\vec{\nabla}\rho\,| \text{ varies with } \theta.$$

- **The resultant** E_ρ **(or** Ω_t**) causes the electron and ion non-ambipolar radial particle fluxes to become equal (i.e., ambipolar):**

$$\Gamma_e^{na}(E_\rho) \;=\; Z_i\,\Gamma_i^{na}(E_\rho) \quad\Longrightarrow\quad \langle\vec{J}\cdot\vec{\nabla}\rho\rangle \;=\; 0 \quad\Longrightarrow\quad \Omega_t \text{ (or } E_\rho\text{) equation.}$$

- **Hence, net ambipolar radial particle flux is sum of** Γ^a **and** $\Gamma^{na}(E_\rho)$**, which is easiest to evaluate for electrons since** $\langle\vec{J}\cdot\vec{\nabla}\rho\rangle \simeq \Gamma_i^{na}(E_\rho) \simeq 0$ **("ion root"):**

$$\boxed{\Gamma_e^{\rm net} \;\equiv\; \underbrace{\Gamma_{e\nu}^a \;+\; \Gamma_{e\rm pc}^a}_{\substack{\text{intrinsically}\\ \text{ambipolar}}} \;+\; \underbrace{\Gamma_e^{na}(E_\rho)}_{\substack{\text{non-ambipolar}\\ \xrightarrow{E_\rho}\ \text{ambipolar}}} \;=\; \Gamma_i^{\rm net}.}$$

- **Dominant electron contributions to** Γ_e^{na} **are usually from electron Reynolds and Maxwell stresses:** $\Gamma_e^{na}(E_\rho) \simeq \Gamma_{e\,\rm Rey}(E_\rho) + \Gamma_{e\,\rm Max}(E_\rho)$**.**

III. Resultant Transport Equations Can Now Be Specified

- Density (assuming for simplicity the particle source $\langle \bar{S}_n \rangle$ is ambipolar):

$$\boxed{\frac{1}{V'}\frac{\partial}{\partial t}\bigg|_{\psi_p}(V' n_0) + \dot{\rho}_{\psi_p}\frac{\partial n_0}{\partial \rho} + \frac{1}{V'}\frac{\partial}{\partial \rho}\left[V'\,\Gamma_e^{\rm net}(E_\rho)\right] = \langle \bar{S}_n \rangle,} \qquad n_{e0} = n_{i0}, \qquad \dot{\rho}_{\psi_p} \equiv \frac{\dot{\psi}_p}{\psi'_p},$$

$$\Gamma_e^{\rm net}(E_\rho) \equiv \Gamma_e^a + \Gamma_e^{na}(E_\rho) \simeq \underbrace{\Gamma_{\rm bp} + \Gamma_{\rm pc}}_{\text{collision-induced}} + \underbrace{\Gamma_{e\,\rm Rey}(E_\rho) + \Gamma_{e\,\rm Max}(E_\rho)}_{\text{fluctuations}}$$

$$\simeq \Gamma_{\rm bp} - \underbrace{\bar{D}_\eta\frac{\partial n_0}{\partial \rho} - n_0 V_{\rm pc}}_{\text{paleo diffusion, pinch}} - \underbrace{\frac{1}{e}\frac{1}{V'\psi'_p}\frac{\partial}{\partial \rho}(V'\Pi_{e\rho\zeta})}_{e\text{ Reynolds stress}} - \underbrace{\frac{1}{\psi'_p}\langle \vec{e}_\zeta\cdot n_{e0}\overline{\tilde{\vec{V}}_e\times\tilde{\vec{B}}}\,\rangle}_{e\text{ Maxwell stress}}.$$

- Toroidal rotation $\langle \Omega_t \rangle \equiv L_t/(m_i n_{i0}\langle R^2 \rangle)$ (and radial electric field E_ρ):

see bottom of p 12 viewgraph for $L_t \equiv m_i n_{i0}\langle R^2 \Omega_t \rangle$ (and preceding viewgraph for E_ρ).

- Energy (similar contributions as density but without ambipolar constraint) — equations still being worked on.

This Approach Is Different And Has Some Consequences

- **Key differences in this new approach for plasma transport equations:**

  First solve for flows of electrons, ions in flux surfaces $\rightarrow \parallel$ Ohm's law, poloidal ion flow.

  Simultaneously solve transport equations for Ω_t ($\rightarrow E_\rho$) and ψ_p, as well as usual n, T.

  Effects of micro-turbulence on parallel Ohm's law (p 6), poloidal ion flow (p 7), particle fluxes (p 11), mom. flux (p 11) and Ω_t (p 12) are all included self-consistently.

  Fluctuation-induced particle flux is determined from e Rey, Max stresses, not $\langle \overline{\tilde{n}\vec{\tilde{V}}} \cdot \vec{\nabla}\rho \rangle$.

  Source effects (e.g., NBI momentum input and $\vec{J}_{\rm CD}$) are included self-consistently.

  Poloidal field transients ($\dot{\psi}_p \neq 0$) and current diffusion time scale effects are included.

  Net transport equations follow naturally from extended two-fluid moment equations;
  hence they are consistent with M3D, NIMROD code frameworks $\rightarrow$ basis for FSP?

- **Some consequences that result from this new approach are:**

  Radial electric field is determined self-consistently & forces ambipolar particle transport.

  Paleoclassical n, Ω_t, T diffusion and pinch effects are included naturally, important?

  Poloidal flux transients ($\dot{\psi}_p \neq 0$) induce radial motion of n, Ω_t, T which could lead to:
  density "pump-out" due to ECH causing $\dot{\psi}_p < 0$?,
  Ω_t decrease with co-current ECCD due to its momentum input and to $\dot{\psi}_p < 0$?,
  Increased "transient transport" of T_e when $\dot{\psi}_p \neq 0$?

Summary

- **Comprehensive transport equations for n, Ω_t have been derived:**

  A systematic gyroradius expansion procedure is used.

  Fluid moment eqns. are averaged over fluctuations, then flux surface averaged (FSA).

- **Radial, parallel and toroidal components of force balance are considered:**

  δ^0: **Radial force balance leads to relation between poloidal, toroidal flows &** $\vec{E}_0$, $\vec{\nabla} p_{i0}$.

  δ: **Parallel viscous damping determines neoclassical $\|$ Ohm's law and poloidal flow.**

  δ^2: **Radial particle fluxes (7 ambipolar, 8 non-ambipolar) from FSA of average ζ torques.**

- **M3D, NIMROD issue is how to best include parallel viscosity dissipation:**

  Maybe we can just use Braginskii operator — but use neoclassical viscosity coefficient.

  But this needs to be tested by comparisons with Held et al. kinetic-based procedure.

- **Requiring an ambipolar radial particle flux (i.e., $\langle \vec{J} \cdot \vec{\nabla} \rho \rangle = 0$) etc. yields:**

  Evolution equation for toroidal angular mom. density $L_t \equiv m_i n_{i0} \langle R^2 \Omega_t \rangle \Longrightarrow \langle \Omega_t \rangle$, E_ρ,

  Net radial particle flux with collisional ($\Gamma_\nu + \Gamma_{\rm pc}$) and electron $\Gamma_e^{na}(E_\rho)$ contributions,

  Radial motion ($\dot{\rho}_{\psi_p} \partial / \partial \rho$) in n, Ω_t, T equations due to poloidal flux motion ($\dot{\psi}_p \neq 0$).