Computation of Resistive Inner Region Solutions with the DELTAC Code

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Outline

- Ideal DCON uses 2D Newcomb equation to determine ideal MHD stability of axisymmetric toroidal plasmas.
- Resistive DCON solves the same equations in the outer region, using a different algorithm, a singular Galerkin method, à la Pletzer and Dewar. Determines outer region matching data.
- Greatly improved convergence due to advanced basis functions and grid packing scheme.
- Inner region equations of GGJ solved with DELTAR (adaptive integration in Fourier space z) or INNER (4^{th}-order finite difference in configuration space x), vacuum region with VACUUM, matched with MATCH code, dispersion relation.
- Compare to straight-through MARS code. Good agreement with one singular surface, poor agreement with two or more.
- Explanation: difficult boundary conditions for realistic parameters, especially for odd-Ψ solution.
- New inner region code DELTAC, Galerkin method in configuration space, Hermite cubic basis functions, packed grid, better treatment of boundary conditions.
- Result: excellent agreement with MARS with multiple singular surfaces.

Outer Region, Singular Galerkin Method

Euler-Lagrange Equation

$$\mathbf{L}\Xi = -(\mathbf{F}\Xi' + \mathbf{K}\Xi)' + (\mathbf{K}^\dagger\Xi' + \mathbf{G}\Xi) = 0$$

Galerkin Expansion

$$(u, v) \equiv \int_0^1 u^\dagger(\psi)v(\psi)d\psi$$

$$\Xi(\psi) = \sum_{i=0}^{N} \Xi_i \alpha_l(\psi)$$

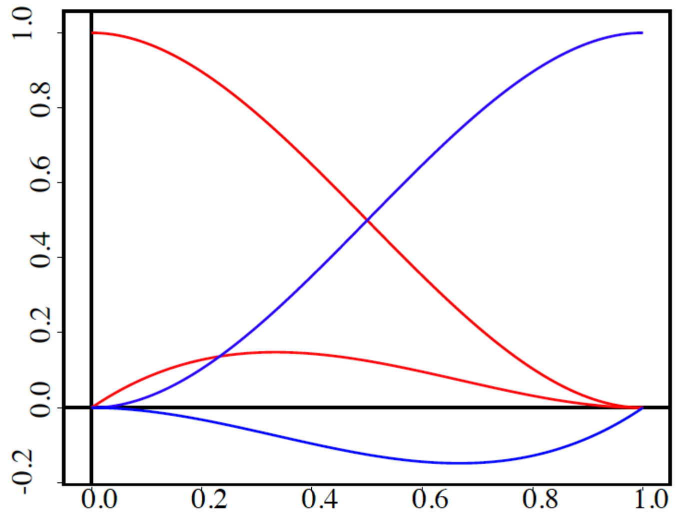
$$(\alpha_i, \mathbf{L}\Xi) = (\alpha_i, \mathbf{L}\alpha_j)\Xi_j = 0$$

$$\mathbf{L}_{ij} = (\alpha_i', \mathbf{F}\alpha_j') + (\alpha_i', \mathbf{K}\alpha_j) + (\alpha_i, \mathbf{K}^\dagger\alpha_j') + (\alpha_i, \mathbf{G}\alpha_j)$$

Finite-Energy Response Driven by Large Solution

$$L_{ij}\check{\Xi}_j = -(\alpha_i, L\hat{\Xi})$$

Better Basis Functions:
C^1 Hermite Cubics



- Cubic polynomials on (0,1), within each grid cell.
- C^1 continuity of function values and first derivatives across grid cells.
- Imposes boundary conditions on nonresonant solutions across the singular surface.

Better Basis Functions: Singular Elements

- Weierstrass Convergence Theorem:
  Polynomial approximation uniformly convergent for analytic functions.

- Large and small resonant solutions are non-analytic near the singular surface.

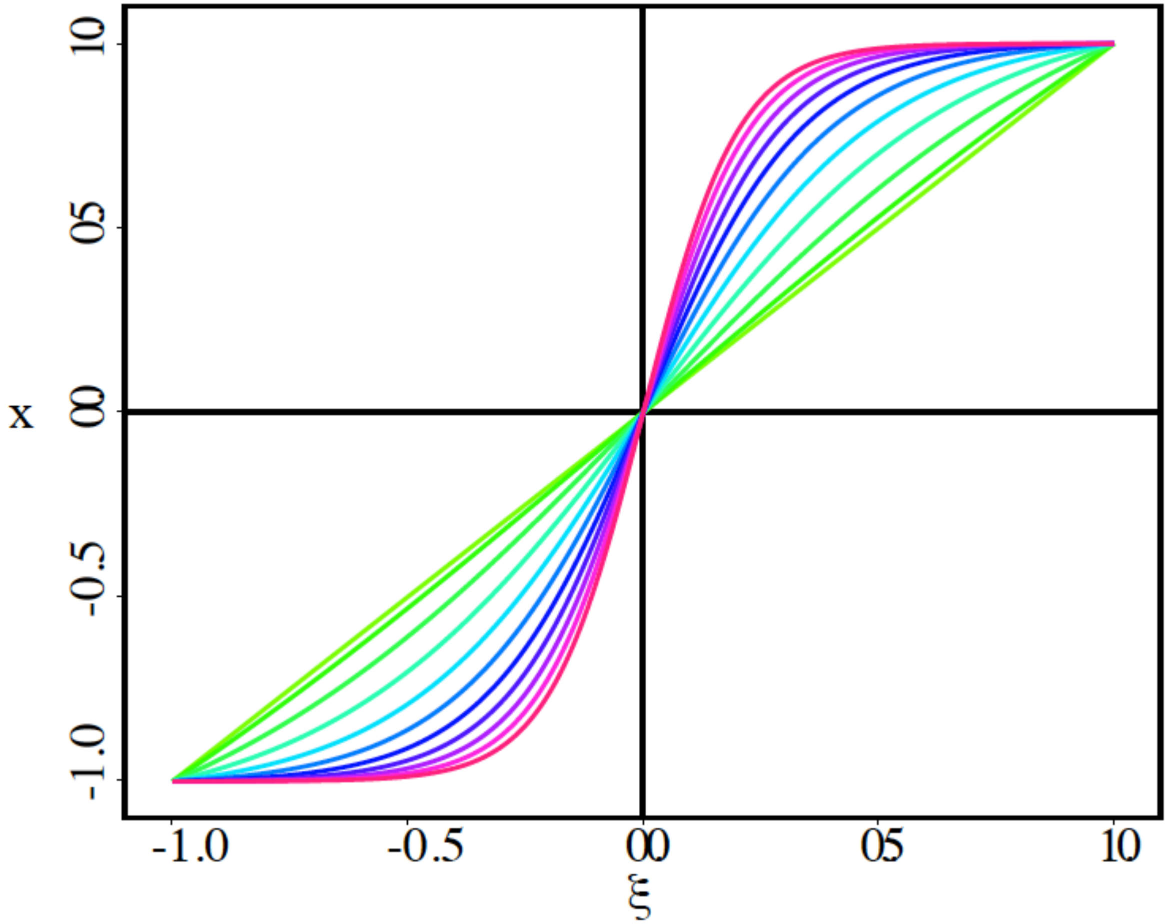
- Supplement Hermite basis with power series for resonant solution near singular surface.

- Evaluation of singular element quadratures with LSODE.

- DCON fits equilibrium data to Fourier series and cubic splines, computes resonant power series to arbitrarily high order. Recent work extends this to the degenerate zero-β limit.

- Convergence requires that the large solution be computed to at least $n = 2\sqrt{(-D_I)}$ terms. PEST 3 is limited to $n = 1$. Higher n required for small shear and high β.

Better Basis Functions: Adjustable Grid Packing Between Singular Surfaces



Layout of Basis Functions

$$\mathbf{L}\bar{\mathbf{u}} = -\left(\mathbf{F}\bar{\mathbf{u}}' + \mathbf{K}\bar{\mathbf{u}}\right)' + \left(\mathbf{K}^{\dagger}\bar{\mathbf{u}}' + \mathbf{G}\bar{\mathbf{u}}\right) = \mathbf{r}$$

Variational Principle

$$W = \frac{1}{2}\left(\bar{\mathbf{u}}, \mathbf{L}\bar{\mathbf{u}}\right) - \left(\bar{\mathbf{u}}, \mathbf{r}\right)$$

$$\delta W = \left(\delta\bar{\mathbf{u}}, \mathbf{L}\bar{\mathbf{u}}\right) - \left(\delta\bar{\mathbf{u}}, \mathbf{r}\right) = 0$$

Resonant-Galerkin Expansion

$$\mathbf{u}(\psi) = \sum_{k=0}^{p} \bar{\mathbf{u}}_k \alpha_k(\psi) + \sum_{j,m=L,R} \mathbf{u}_m^{j(s)}(\psi)\Delta_{lm}^{ij} \qquad \delta\mathbf{u}(\psi) = \sum_{k=0}^{p} \alpha_k(\psi)\delta\bar{\mathbf{u}}_k + \sum_{j,m=L,R} \mathbf{u}_m^{j(s)}(\psi)\delta\Delta_{lm}^{ij}$$

Hermite cubic **Small solution**

N	...	N	E	R	R	E	N	...	N

ψ_r^i

Extension element (E) connecting Resonant element (R) and Normal element (N) allows the resonant small solution smoothly vanishes.

Adjustable grid packing is applied to the interval between each two adjacent resonant surfaces.

Inner Region Equations

Inner Region Equations

$$\Psi_{XX} - H\Upsilon_X = Q(\Psi - X\Xi)$$
$$Q^2\Xi_{xx} - QX^2\Xi + QX\Psi + (E+F)\Upsilon + H\Psi_X = 0$$
$$Q\Upsilon_{XX} - X^2\Upsilon + X\Psi + Q^2\left[G(\Xi - \Upsilon) - K(E\Xi + F\Upsilon + H\Psi_X)\right] = 0$$

Matrix Form

$$\boldsymbol{\Psi} \equiv \begin{pmatrix} \Psi \\ \Xi \\ \Upsilon \end{pmatrix}, \quad \boldsymbol{\Psi}'' - \mathbf{V}\boldsymbol{\Psi}' - \mathbf{U}\boldsymbol{\Psi} = 0$$

Galerkin Expansion

$$\boldsymbol{\Psi}(X) = \sum_{i=0}^{N} \boldsymbol{\psi}_i \alpha_i(X)$$

Discretized Equations

$$(u, v) \equiv \int_0^{x_{max}} u(x)v(x)dx$$

$$\mathbf{L}_{i,j} \equiv (\alpha_i', \alpha_j') - (\alpha_i, \mathbf{V}\alpha_j') - (\alpha_i, \mathbf{U}\alpha_j), \quad \mathbf{L}_{i,j}\boldsymbol{\psi}_j^{\text{small}} = -\mathbf{L}_{i,j}\boldsymbol{\psi}_j^{\text{big}}$$

Details

Use Hermite cubics on a packed grid, 4th-order accurate, C^1 continuous.
LAPACK complex banded routines ZGBTRF and ZGBTRS.
Δ_{inner} = coefficient of small power-like solution at x_{max}.

Boundary Conditions

Behavior at Large X

$$\Psi(X) = \sum_{j=1}^{6} c_j \exp(\beta_j X^2/2) X_j^p \sum_{k=0}^{K} X^{-k} s_{j,k}$$

Exponentially Large and Small Solutions

$$j = 1, 2, 5, 6, \quad \beta_j = \sigma_j Q^2, \quad \sigma_{1,2} = +1, \quad \sigma_{5,6} = -1$$

$$\begin{aligned} p_j = & -\tfrac{1}{2} + \tfrac{1}{4}\sigma_j Q^{3/2}\left[1 + G + K(F + H^2)\right] \\ & \pm Q^{3/2}\left\{(G + KF - 1)^2 + KH^2\left[KH^2 + 2(G + KF + 1)\right]\right. \\ & \left. +4\left[(G - KE - 1)D_R + H^2\right]\right\} \end{aligned}$$

Power-Like Solutions

$$j = 3, 4, \quad \beta_j = 0, \quad p_j = -\tfrac{1}{2} \pm (-D_I)^{1/2}$$

Boundary Conditions

Exclude exponentially large solutions, $c_1 = c_2 = 0$.
Match ratio power-like coefficients to outer region, $c_4/c_3 = \Delta$.
Exponentially small solutions arbitrary, solve for c_5 and c_6.

Problem

For large scaled growth rate $|Q| \gg 1$, power-like terms dominate exponential solutions, making it difficult to impose boundary conditions. Both DELTAR and INNER fail in this limit.

Solution

At x_{max} impose power-like and small functions *and derivatives.*

Comparison of Inner Region Codes

- INNER
  - Authors: Jardin & Tesauro
  - Space: Configuration
  - Method: 4^{th}-order finite difference
- DELTAR
  - Author: Glasser
  - Space: Fourier
  - Method: Adaptive Integration
- DELTAC
  - Authors: Glasser & Wang
  - Space: Configuration
  - Method: Galerkin, Hermite Cubic, 4^{th}-order, C^1 continuous, grid packing

Parameters

Published Test Cases

$-.1 < E < 1,\ F = G = H = K = 0,\ 1e\text{-}5 < Q < 10$

MARS Benchmark Parameters

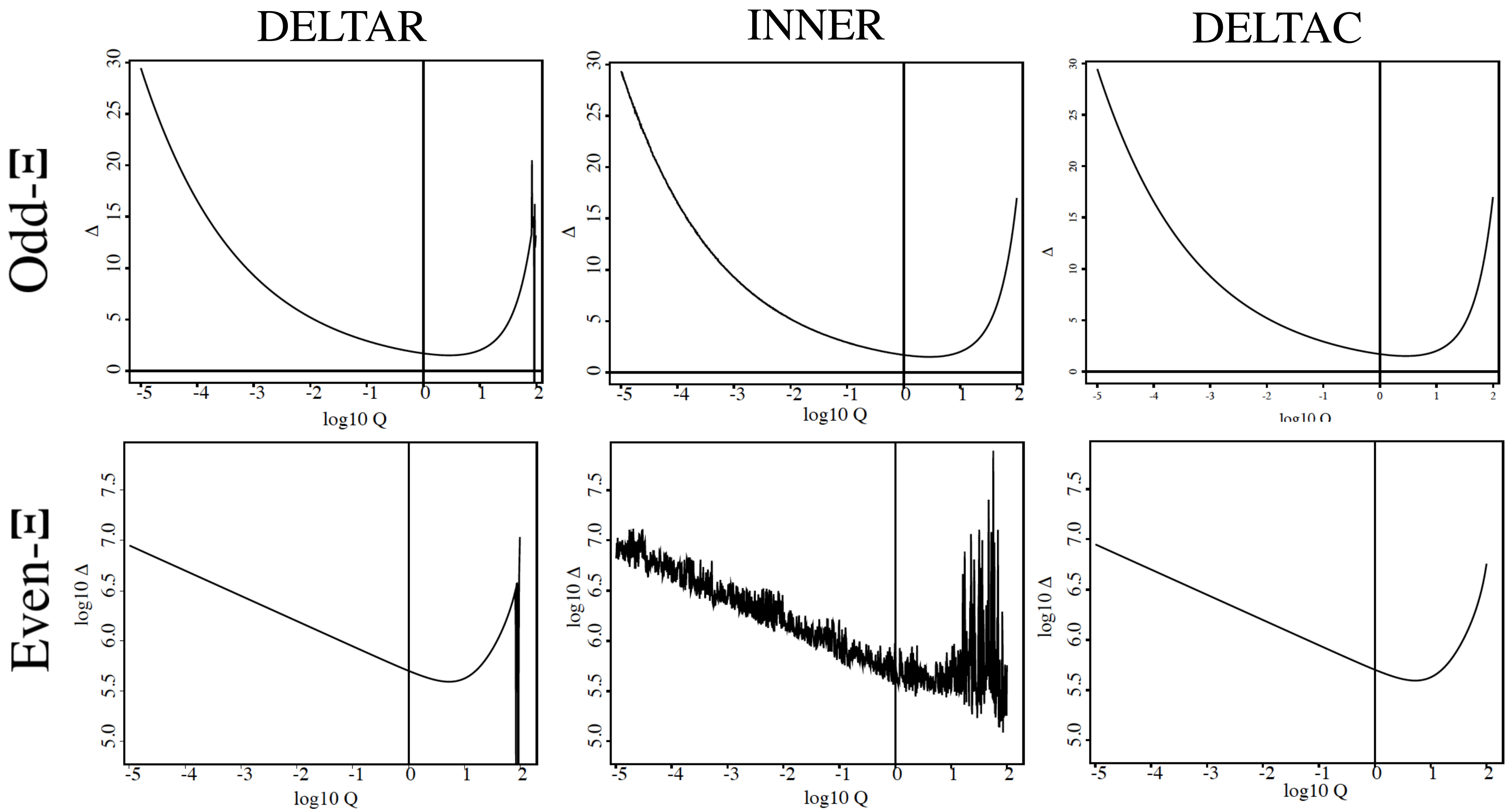
E = -5.912E-003
F = 5.480E-004
H = 3.721E-003

G = 3.200E+003
K = 1.716E+003
M = 1.943E+001

Q = 5.390E+002

The benchmark parameters are much more challenging

Inner Region Code Comparison, Benchmark Case



Inner and Outer Region Solutions

Outer Region Basis Functions and Linear Combination

$$\mathbf{u}_{i,k}(\psi) \equiv \sum_{j=1}^{n} \sum_{l=L}^{R} \left[\delta_{i,j}\delta_{k,l}\mathbf{u}^{b}_{j,l}(\psi) + \Delta'_{i,k;j,l}\mathbf{u}^{s}_{j,l}(\psi)\right]$$

$$\mathbf{u}(\psi) = \sum_{i=1}^{n} \sum_{k=L}^{R} c_{i,k}\mathbf{u}_{i,k}(\psi)$$

Inner Region Basis Functions and Linear Combination

$$\mathbf{v}_{i,\pm}(x) \equiv \mathbf{v}^{b}_{i,\pm}(x) + \Delta_{i,\pm}(Q)\mathbf{v}^{s}_{i,\pm}(x) = \pm\mathbf{v}_{i,\pm}(-x)$$

$$\mathbf{v}_{i}(x) = d_{i,+}\mathbf{v}_{i,+}(x) + d_{i,-}\mathbf{v}_{i,-}(x)$$

Inner region solutions computed with DELTAC.

Matching Conditions

Matching Conditions

$$c_{j,L} = d_{j,+} - d_{j,-}, \quad c_{j,R} = d_{j,+} + d_{j,-}$$

$$\sum_{i=1}^{n} \sum_{k=L}^{R} c_{i,k} \Delta'_{i,k;j,L} = d_{j,+} \Delta_{j,+}(Q) - d_{j,-} \Delta_{j,-}(Q)$$

$$\sum_{i=1}^{n} \sum_{k=L}^{R} c_{i,k} \Delta'_{i,k;j,R} = c_{j,+} \Delta_{j,+}(Q) + c_{j,-} \Delta_{j,-}(Q)$$

Matrix Form and Dispersion Relation

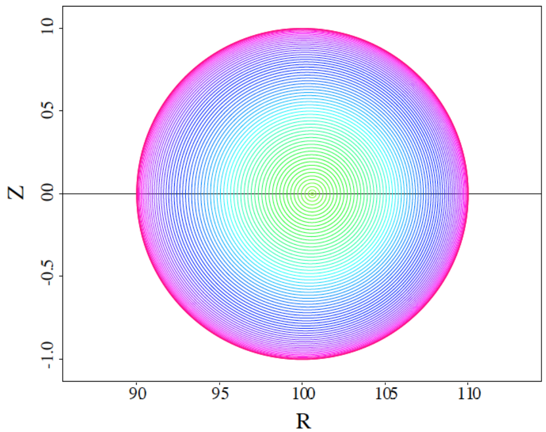
$$\mathbf{c} \equiv (c_{1L}, d_{1+}, d_{1-}, c_{1R}, c_{2L}, d_{2+}, d_{2-}, c_{2R}, \cdots)^T$$

$$\mathbf{M}(Q) \cdot \mathbf{c} = 0, \quad \det \mathbf{M}(Q) = 0$$

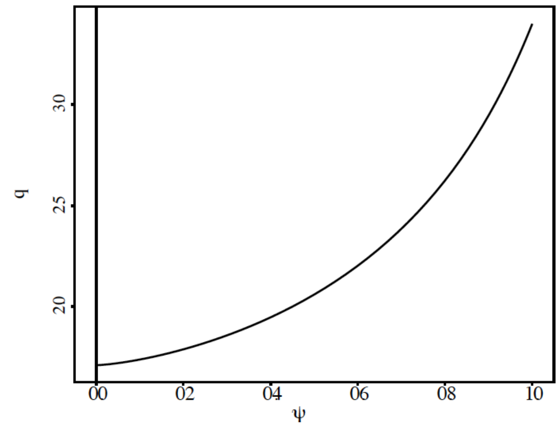
Outer region solved once in < 10 seconds.
Inner region solved many times, 20,000 per second.

Chease Equilibrium, 2 Singular Surfaces, $\beta_N = 0.5$

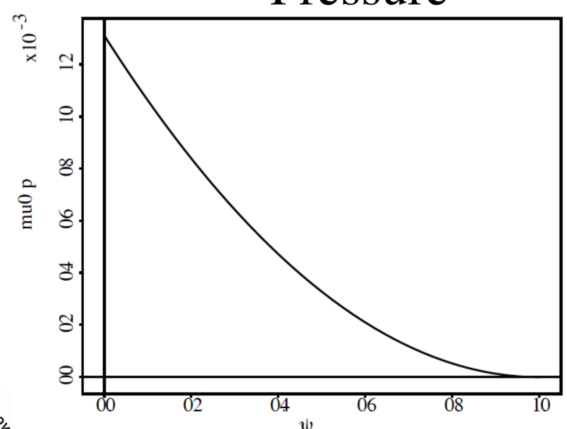
Flux Surfaces



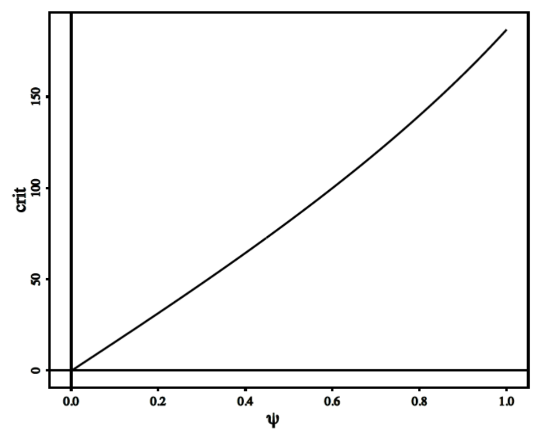
Safety Factor



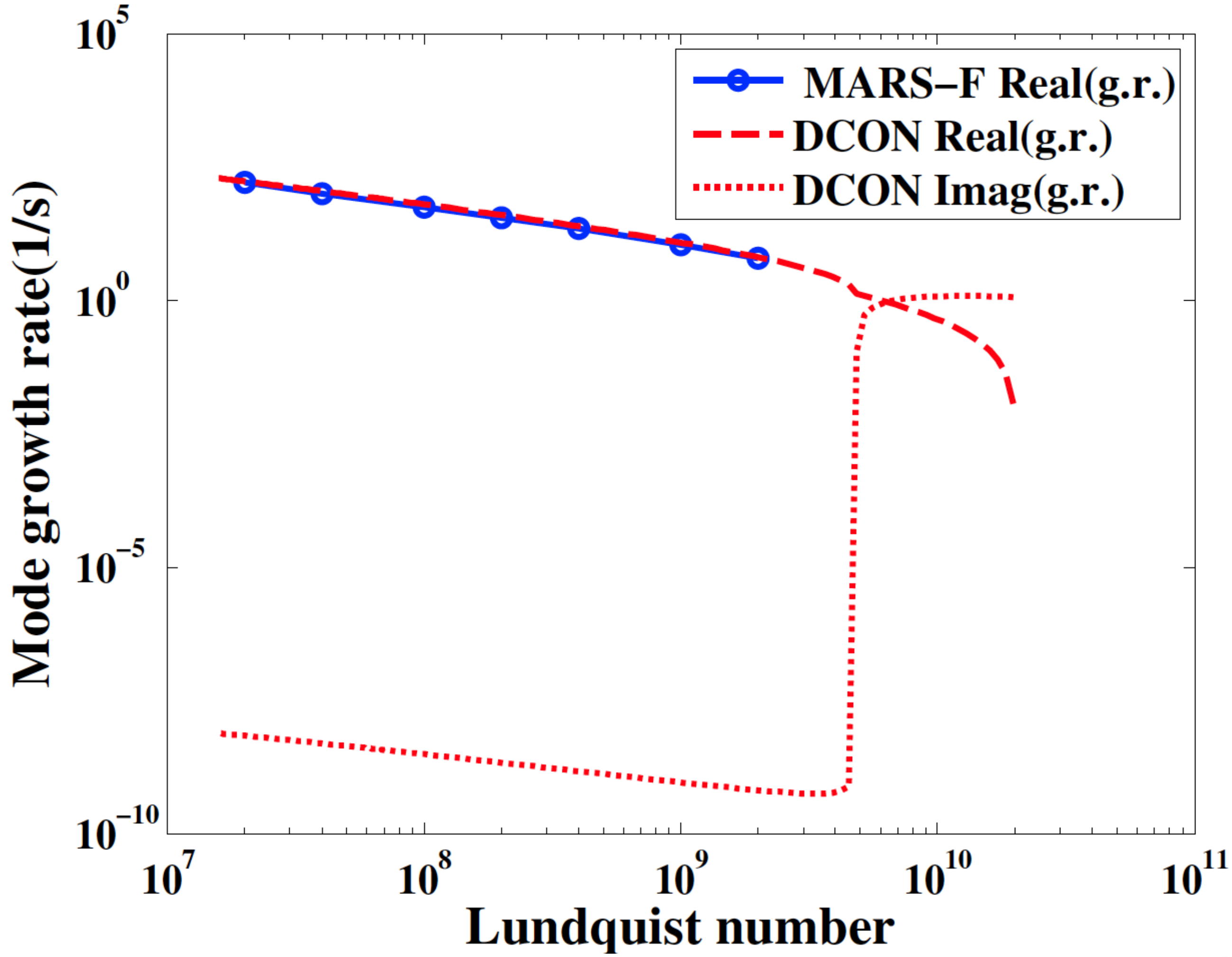
Pressure



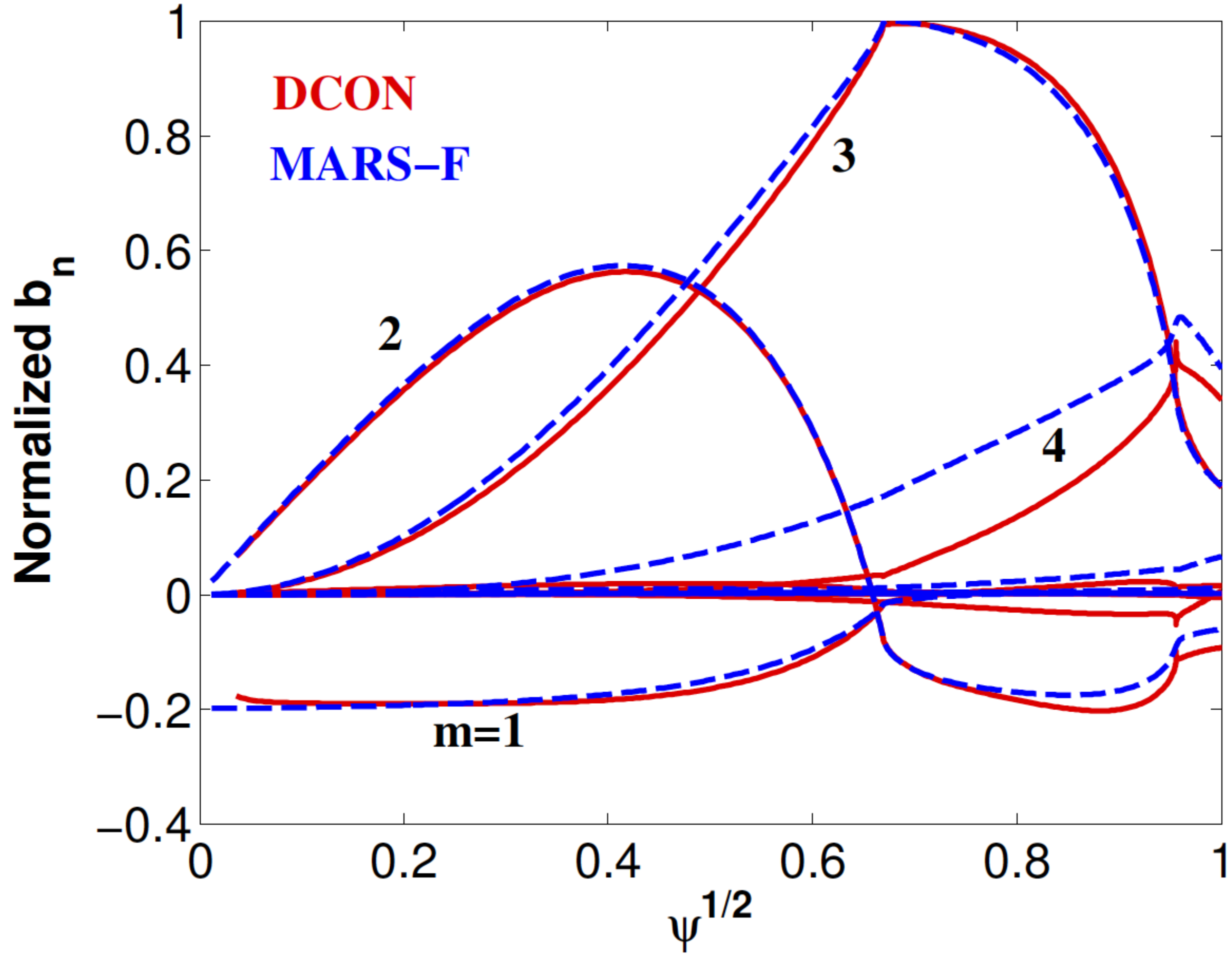
Newcomb Criterion



MARS Benchmark, Growth Rate



MARS Benchmark, Eigenfunction



More Complete Inner Region

Fields

$$\mathbf{E} = -\partial_t \mathbf{A} - \nabla\varphi, \quad \mathbf{b} = \nabla \times \mathbf{A}$$
$$\nabla \cdot \mathbf{j} = \nabla \cdot \mathbf{A} = 0, \quad \mathbf{j} = -\nabla^2 \mathbf{A}$$

Density and Pressure

$$\partial_t \rho + \nabla \cdot (\rho \mathbf{v}) = 0$$
$$\partial_t p + \mathbf{v} \cdot \nabla p + \gamma P \nabla \cdot \mathbf{v} + \frac{2}{3}\left(\nabla \cdot \mathbf{q} + \mathbf{R} \cdot \mathbf{u} + \pi : \nabla \mathbf{v}\right) = 0$$

Momentum Conservation and Ohm's Law

$$\partial_t(\rho \mathbf{v}) = \mathbf{j} \times \mathbf{B} + \mathbf{J} \times \mathbf{b} - \nabla p - \nabla \cdot \boldsymbol{\pi}$$
$$\mathbf{E} + \mathbf{v} \times \mathbf{B} = \mathbf{R} + \frac{1}{N_e e}\left(\mathbf{j} \times \mathbf{B} + \mathbf{J} \times \mathbf{b} - \nabla p_e - \nabla \cdot \boldsymbol{\pi}_e\right)$$

Ions and Electrons

$$N_e = Z N_i, \quad T_e = T_i, \quad P_e = N_e T_e, \quad P_i = N_i T_i$$
$$P = P_e + P_i, \quad \boldsymbol{\pi} = \boldsymbol{\pi}_e + \boldsymbol{\pi}_i, \quad \mathbf{q} = \mathbf{q}_e + \mathbf{q}_i, \quad \mathbf{u} \equiv -\frac{1}{N_e e}\mathbf{j}$$

A Future Role for Matched Asymptotic Expansions

- The method of matched asymptotic expansions was introduced by Furth, Killeen, and Rosenbluth in order to obtain analytical results.
- Most recent work uses straight-through methods, such as MARS, M3D and NIMROD, using packed grids to resolve singular layers.
- Thermonuclear plasmas are in a regime where conditions for the validity of matched asymptotic expansion are very well satisfied.
- Resistive DCON and DELTAC provide numerical methods to do the full matching problem numerically and *very* efficiently.
- Inner region dynamics can be extended to include full fluid and kinetic treatments.
- Nonlinear effects are localized to the neighborhood of the singular layers and can be solved with the 2D HiFi code, exploiting helical symmetry, matched through ideal outer regions.
- Asymptotic matching and straight-through methods can complement and verify each other.

Future Work

- Further efforts to extend agreement with MARS code.
- More complete fluid regime model of linear inner region; Braginskii.
- Neoclassical inner region model, drift kinetic equation; Ramos.
- Nonlinear model, NTM, with nonlinear effects localized to inner regions, coupled through ideal linear outer region. 2D HiFi code, helical symmetry.
- Nonlinear verification with straight-through nonlinear codes: NIMROD, M3D-C1.