Forced Magnetic Reconnection Modeling with NIMROD

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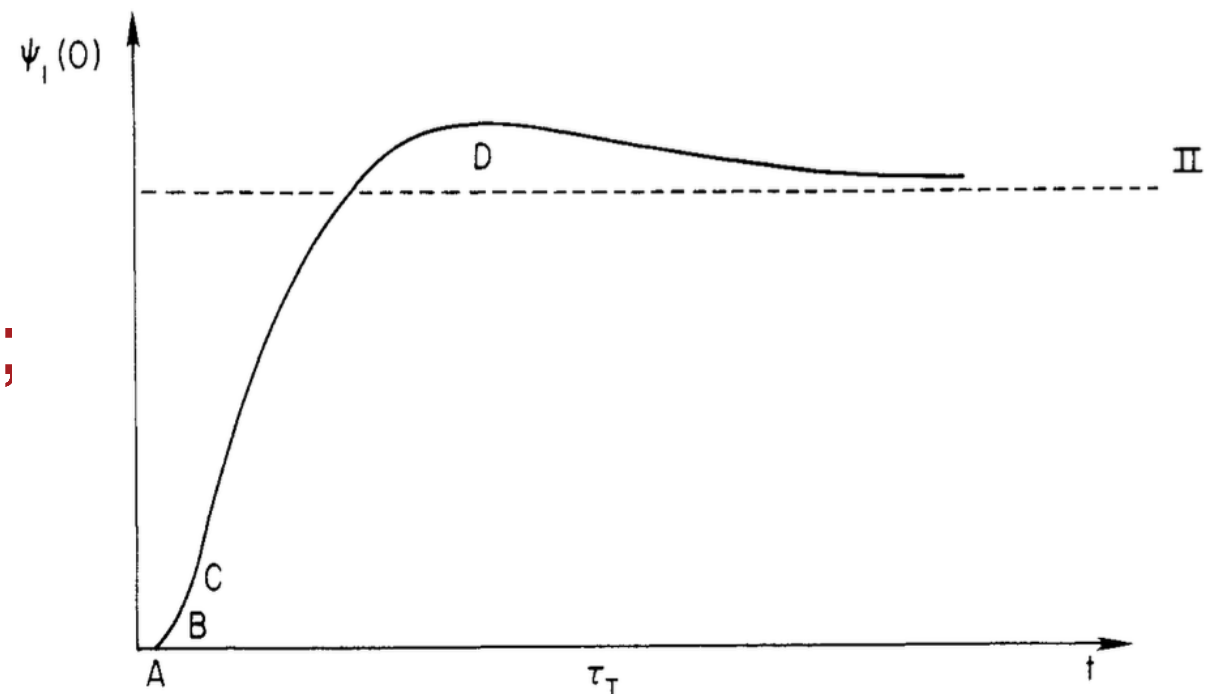
Motivation is Verification of Forced Magnetic Reconnection Physics

- RMPs modify the magnetic field at the top of the H-mode pedestal; additional work is needed to understand their role in suppressing or mitigating ELMs
- Benchmarking of **forced magnetic reconnection** (FMR) with NIMROD and M3D-C[1] is needed to understand general linear and nonlinear responses to applied fields
- FMR in slab and cylindrical geometry is well understood analytically [e.g. Hahm and Kulsrud (1985)] and numerically; verification exercise with NIMROD is necessary before moving on to toroidal problem



We Use the NIMROD Code to Solve the Resistive-MHD Equations

- NIMROD capable of solving extended-MHD equations
  - Assume $\beta = 0$ in the following
- Semi-implicit leapfrog time evolution is used:
  - Hold equilibrium fields constant and evolve perturbation fields

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E} + \kappa_{divb} \nabla \nabla \cdot \mathbf{B}$$

$$\mathbf{E} = -\mathbf{V} \times \mathbf{B} + \eta \mathbf{J}$$

$$\mu_0 \mathbf{J} = \nabla \times \mathbf{B}$$

$$\frac{\partial n}{\partial t} + \nabla \cdot (n\mathbf{V}) = \nabla \cdot D \nabla n$$

$$\rho \left(\frac{\partial \mathbf{V}}{\partial t} + \mathbf{V} \cdot \nabla \mathbf{V} \right) = \mathbf{J} \times \mathbf{B} + \nabla \cdot \nu \rho \nabla \mathbf{V}$$

- Uses 2-D C^0 finite elements with Fourier decomposition in third dimension: $\mathbf{A}(R, Z, \phi) = \mathbf{A}_0 + \sum_{n=1} \mathbf{A}_n(R, Z) e^{in\phi} + \mathbf{A}_n^*(R, Z) e^{-in\phi}$
  - Expansion coefficients of perturbation fields are complex

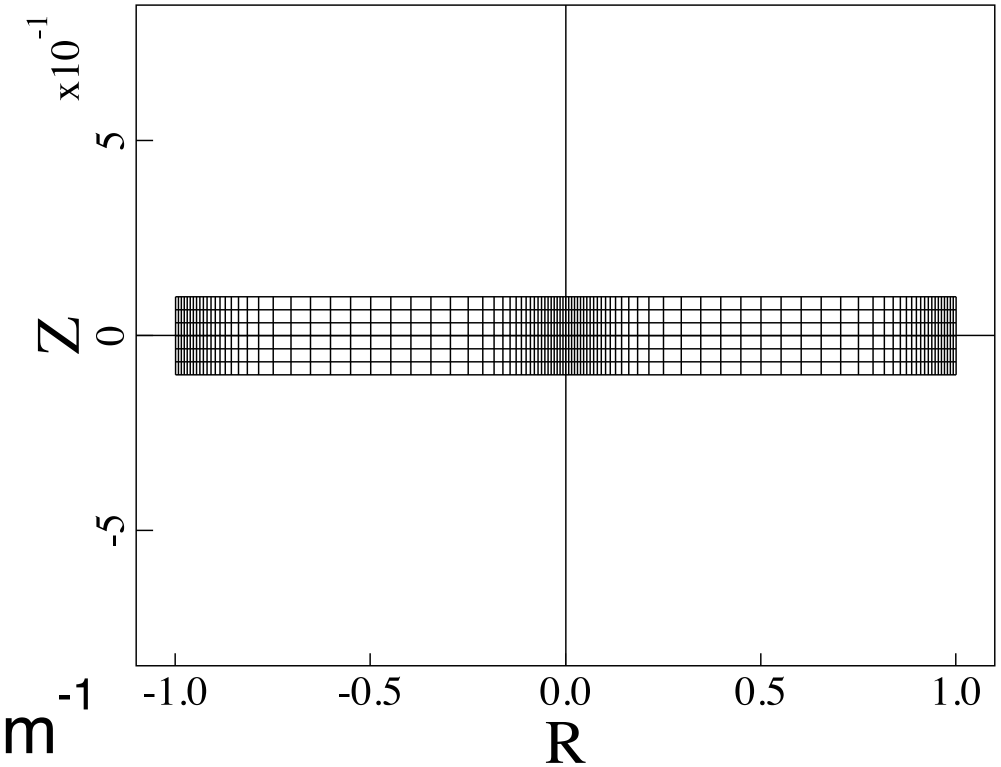
Many Parameters Are Specified For NIMROD Modeling

- R, Z, ϕ coordinates with $(L_R, L_Z) = (2{*}a = 2\text{ m}, 0.2\text{ m})$, $(n_R, n_Z) = (96,6)$, and FEs with a polynomial degree of 4
    - 2-D reconnection with symmetric Z direction
    - Grid packing at boundary and rational surface with minimum grid (node spacing) $\approx 7.5\times10^{-3}$ m (1.9×10^{-3} m)
- Periodic length of $L_\phi = 2$ m
    - $n_\phi = 1$ linear calculations have $k_\phi \equiv 2\pi n_\phi/L_\phi = \pi\ \text{m}^{-1}$
- $B_{z,0} = 10$ T, $B_{\phi,0} = 0.1$ T, $\boxed{\Delta' = -2\pi}$
    - $\tau_A \equiv a^*(B_{\phi,0}{}^2/\mu_0\rho)^{1/2} = 1.45\times10^{-6}$ s ➛ $dt = 10^{-6}$ s
- Initial simulation uses $\eta/\mu_0 = 2\ \text{m}^2/\text{s}$, $\nu = 0.002\ \text{m}^2/\text{s}$

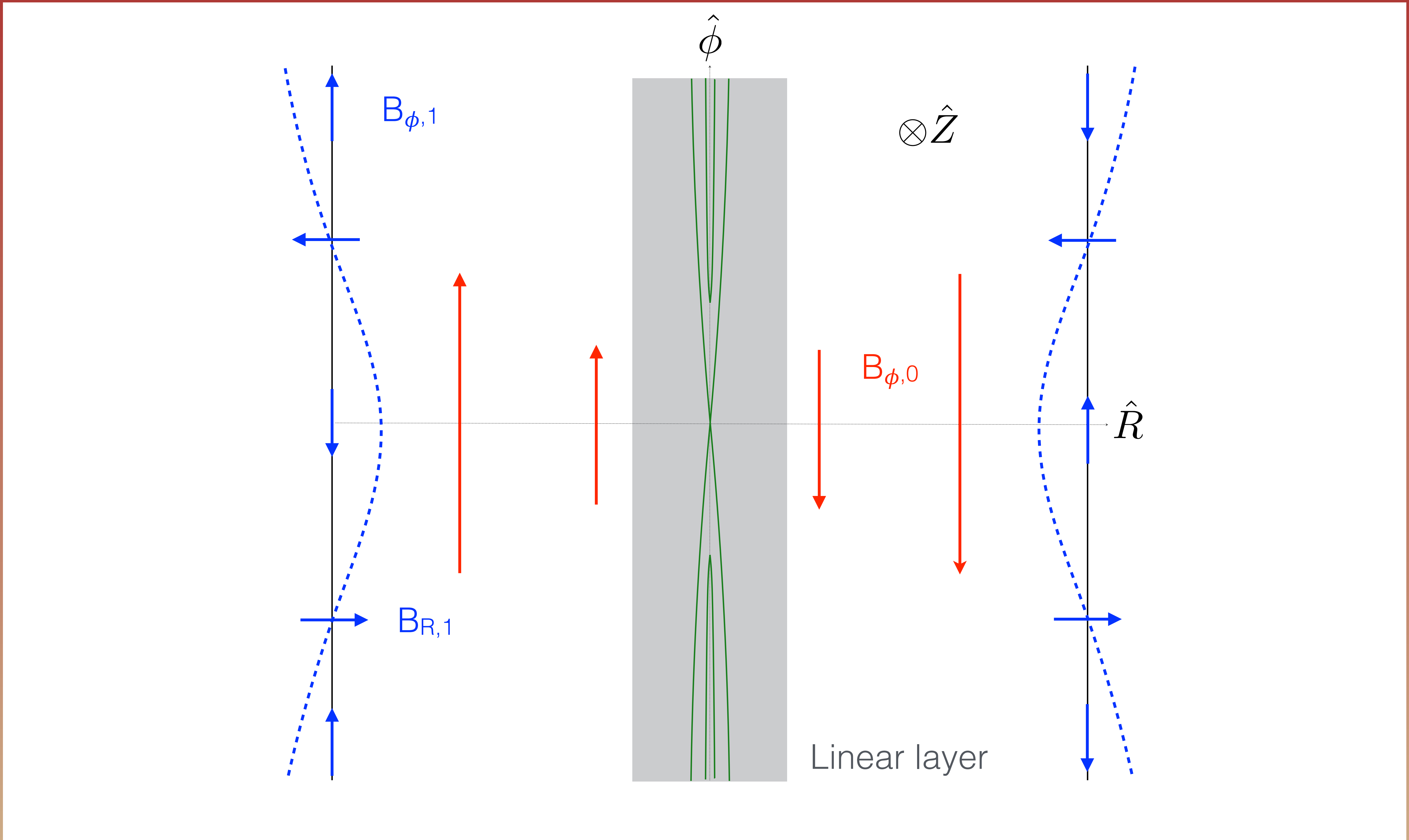
➛ $P \equiv \tau_R/\tau_V = 10^{-3}$ $[\tau_R \equiv a^2/(\eta/\mu_0),\ \tau_V \equiv a^2/\nu]$, $S \equiv \tau_R/\tau_A = 3.45\times10^5$

➛ Linear layer width $\sim a^*S^{-1/3}P^{1/6} = 4.5\times10^{-3}$ m, calculated island half-width $\sim 1.05\times10^{-3}$ m

Finite Element Mesh

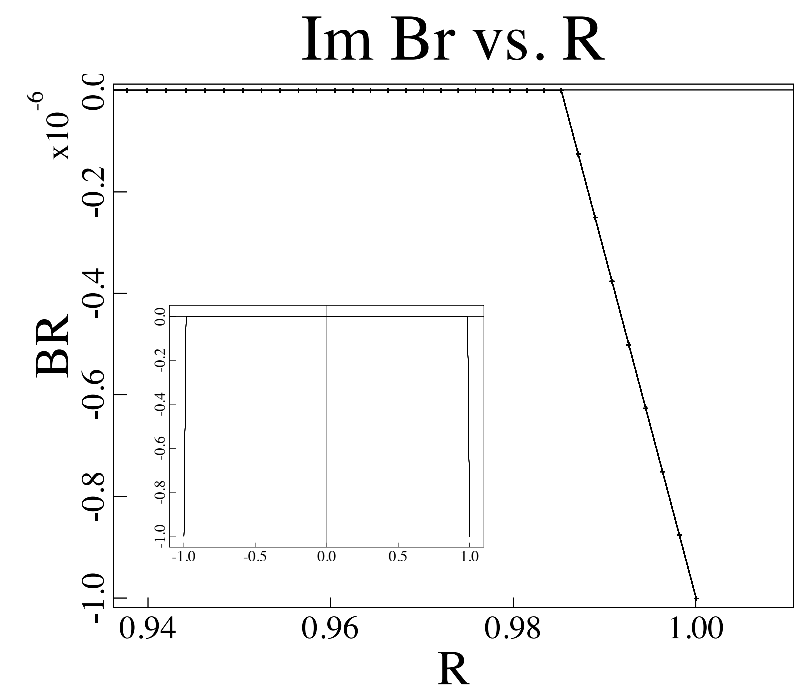


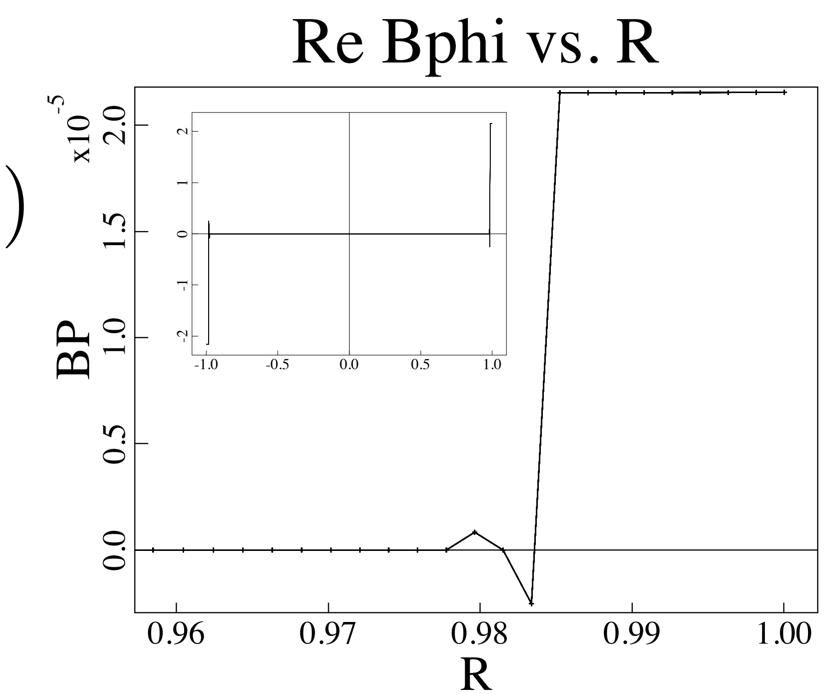
Paradigm Is Taylor's Slab Model Problem: Apply Edge Magnetic Field Perturbation



Boundary Perturbation Is Implemented In Two Edge Grid Points

- Solve vacuum (no plasma, currents) boundary-value problem in *edge region*
  - $\nabla^2 \varphi_1(\mathrm{R},\phi) = 0$ for perturbed $\mathbf{B}_{\perp,1} = -\nabla\varphi_1$
  - Two outermost grid cells of width Δ
  - $\boxed{\mathrm{B}_{\mathrm{R},1}(\mathrm{a},\phi) = -\mathrm{i}\mathrm{B}_{\mathrm{nw}}\xi(\phi)}$, $\mathrm{B}_{\mathrm{R},1}(\mathrm{a}-\Delta,\phi) = 0$
    - B_{nw} is the normal component of the magnetic field at the radial boundary
- $$\varphi_1(R,\phi) = \frac{B_{nw}}{k_\phi}\frac{\cosh[k_\phi(R-a+\Delta)]}{\sinh(k_\phi\Delta)}\left(ie^{ik_\phi\phi} - ie^{-ik_\phi\phi}\right)$$
- HK δ is consistent with $\boxed{\delta = \mathrm{B}_{\mathrm{nw}}/\mathrm{B}_{\phi,0}\mathrm{k}_\phi}$
  - Initial linear simulation uses $\mathrm{B}_{\mathrm{nw}} = 10^{-6}$ T ($\delta \sim 3.2\times10^{-6}$ m), which avoids nonlinear forcing



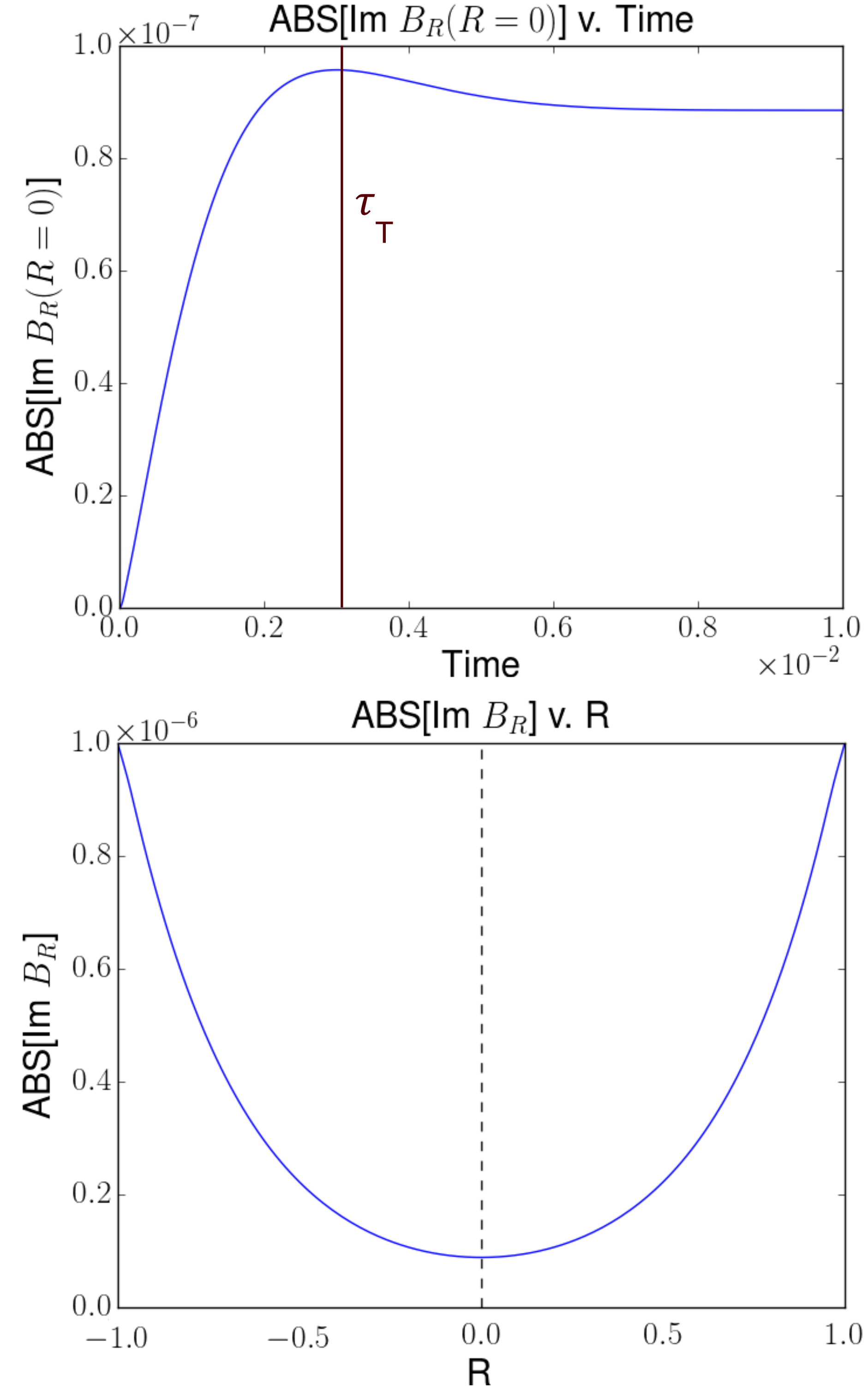


NIMROD Linear Results for Small Perturbation ($B_{nw} = 10^{-6}$ T) Are Close To HK Predictions

- Clearly observe overshoot in $B_{R,1}(R=0)$ evolution in top plot at $\tau_T = 3.05\times10^{-3}$ s
  - $B_{R,1}(R=0,t) \sim t^{1.22}$ (HK $\sim t^{5/4}$) for $t \ll \tau_T$
- For sheet pinch with constant current, HK showed that force balance reduces to vacuum boundary-value problem
  - $\mathbf{B}_\perp = \hat{\mathbf{z}}\times\nabla\psi$ and BC $\psi_1(R=a) = B_{nw}/k_\phi$
  - Solution for system with $\psi_1(R=0) \neq 0$ (i.e. resistive evolution):

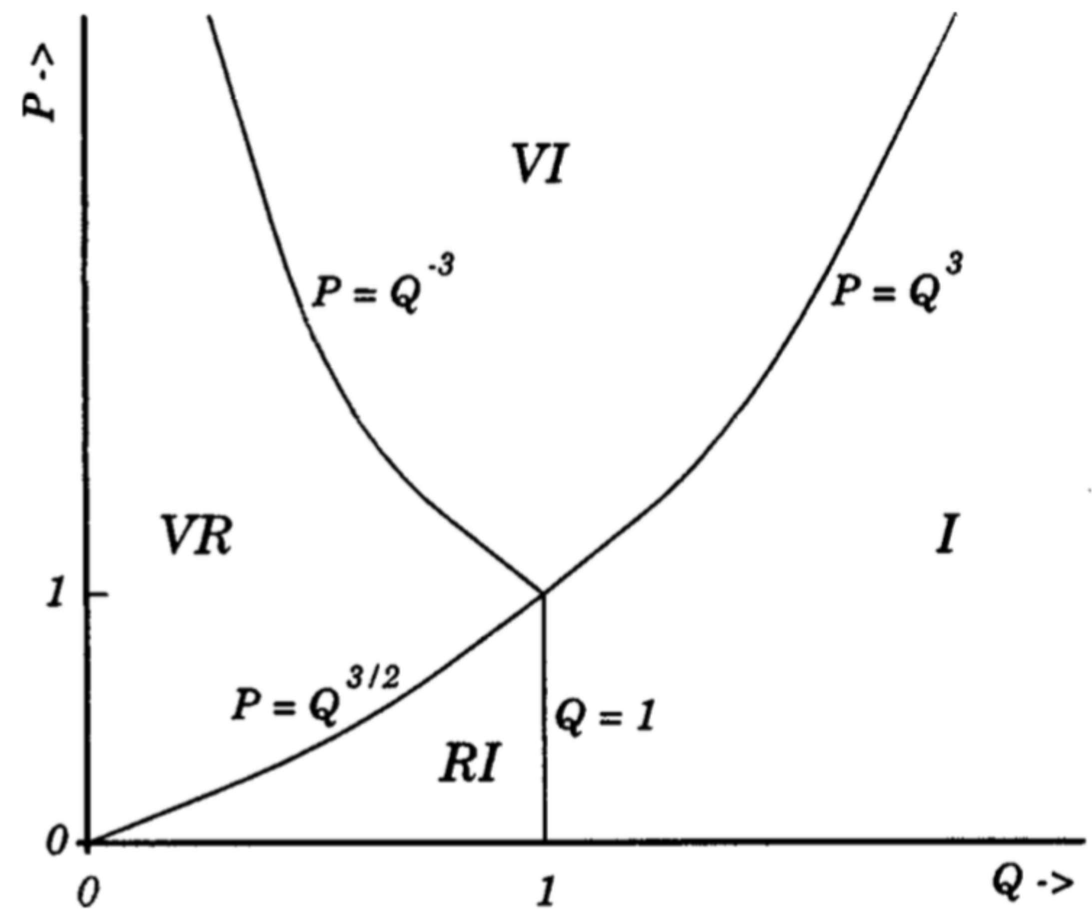
$$\psi_1(R,\phi) = \frac{-B_{nw}}{k_\phi}\frac{\cosh(k_\phi R)}{\cosh(k_\phi a)}\left(e^{ik_\phi\phi} + e^{-ik_\phi\phi}\right)$$

  - $B_{R,1}(\phi=0) = -iB_{nw}*[\cosh(k_\phi R)/\cosh(k_\phi a)]$
    - Predict $\mathrm{Im}[B_{R,1}(R=0)] = -8.63\times10^{-8}$ T, measure -8.85×10^{-8} T



Equilibrium Flow Introduces Field Screening Physics

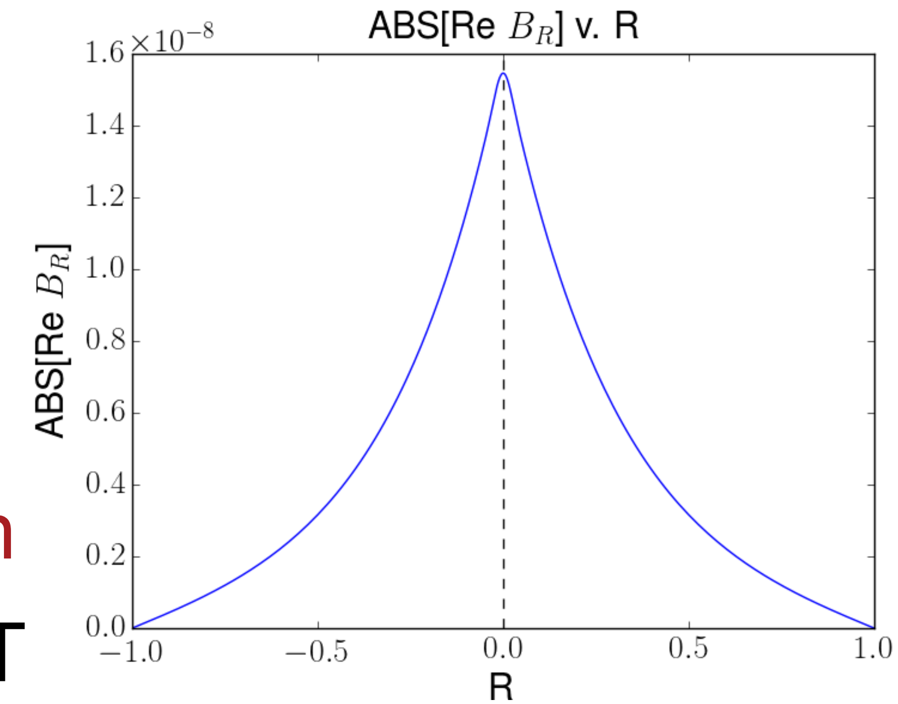
- Flows generate eddy currents that suppress tearing process
- Scaling with magnitude of flow is highly dependent upon layer physic [e.g. Fitzpatrick, POP (1998)]
  - $$\boxed{B_{norm} = \frac{1}{-\Delta' + \Delta(\omega)} \frac{2k_\phi}{\cosh(k_\phi a)} B_{nw}}$$
  - B_{norm} is $B_{R,1}$ at rational surface

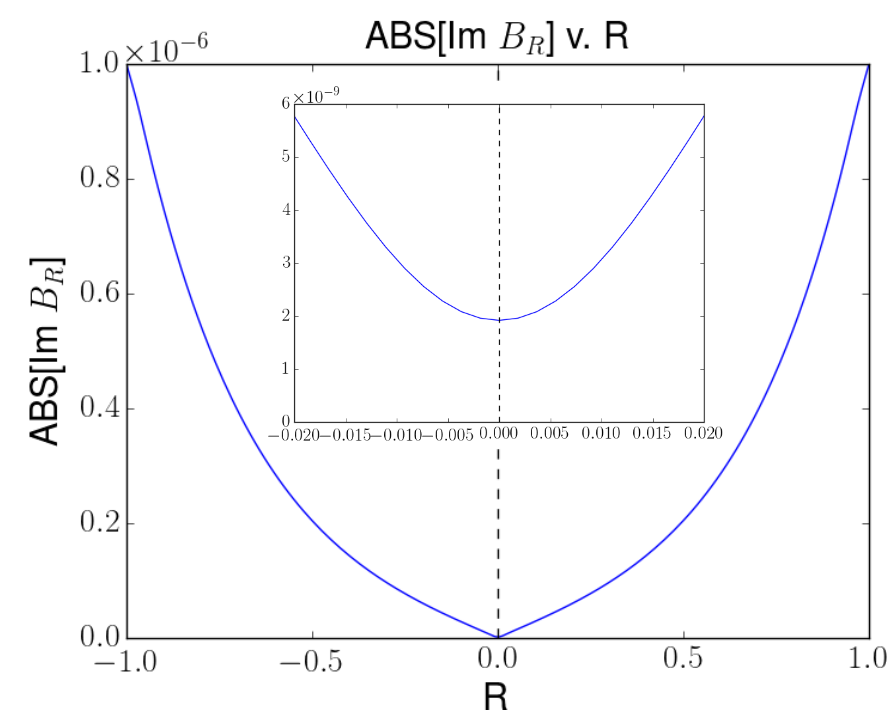


- Form of $\Delta(\omega)$ depends on relative values of Prandtl number $P = \tau_R/\tau_V$ and normalized "slip frequency" $Q = \tau_A S^{1/3}\omega$, where $\omega = k_\phi v_\phi$
  - P-Q space splits into 4 regimes: Resistive-Inertial (RI), **Visco-Resistive (VR)**, Visco-Inertial (VI), and Inertial (I)

VR Regime Asymptotic State Agrees Well With Fitzpatrick Predictions

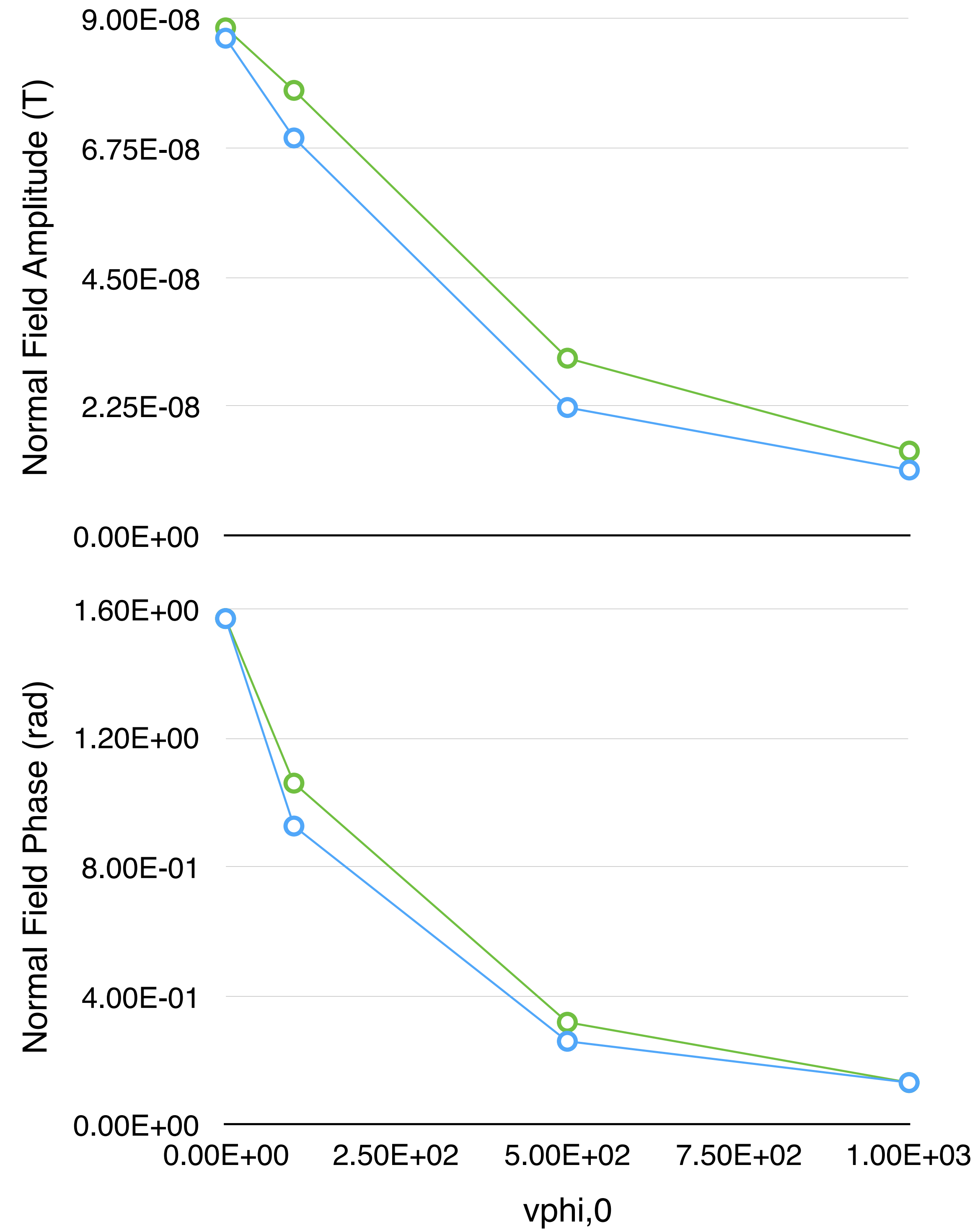
- With $P = 1$ and $v_{\phi,0} = 10^3$ m/s : $\mathrm{Re}(Q) = 0.319$
  - ➢ In the VR regime
- VR regime has $\boxed{\Delta(\omega) = i\,2.104\,\omega\,\tau_A\,a^{-1} S^{2/3} P^{1/6}}$
  - $\omega \sim v_\phi$ is complex, with imaginary component due to Fourier decomposition
- Theory predicts $\mathrm{Re}[B_{R,1}(R = 0)] = -1.16\times10^{-8}$ T and $\mathrm{Im}[B_{R,1}(R=0)] = -1.54\times10^{-9}$ T
  - Numerically within ~ 25% of predictions
- Increasing $v_{\phi,0}$ parallel to $B_{\phi,0}$ decreases the magnitude of the normal field at the rational surface (*smaller islands*)
  - Also *shifts island* along direction of flow





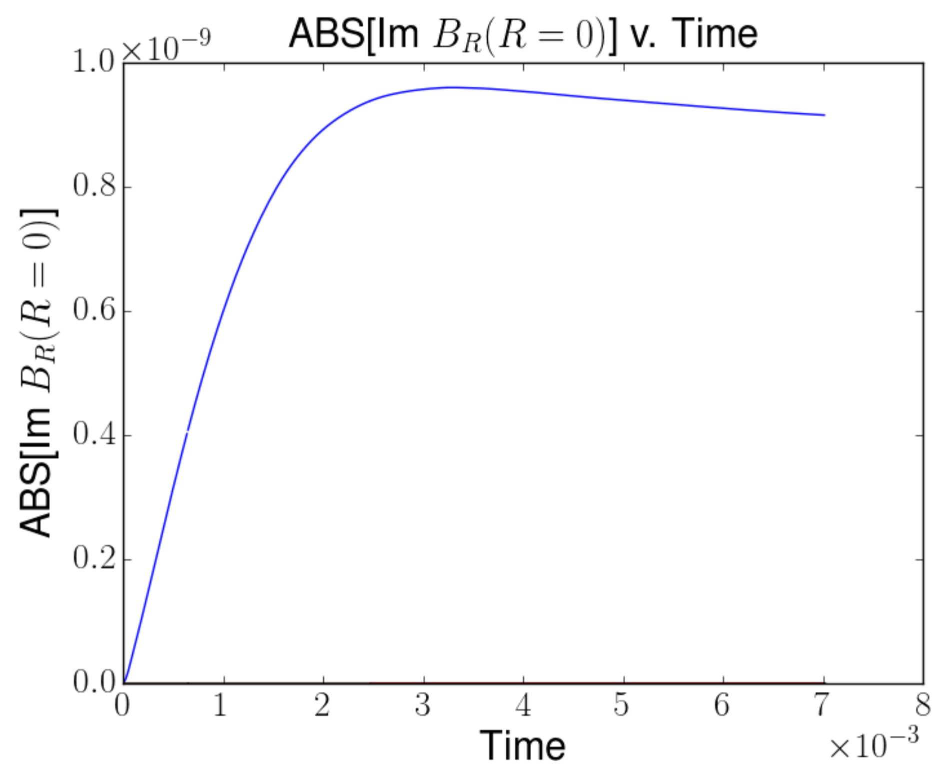
Scaling with Flow Follows Fitzpatrick's Results

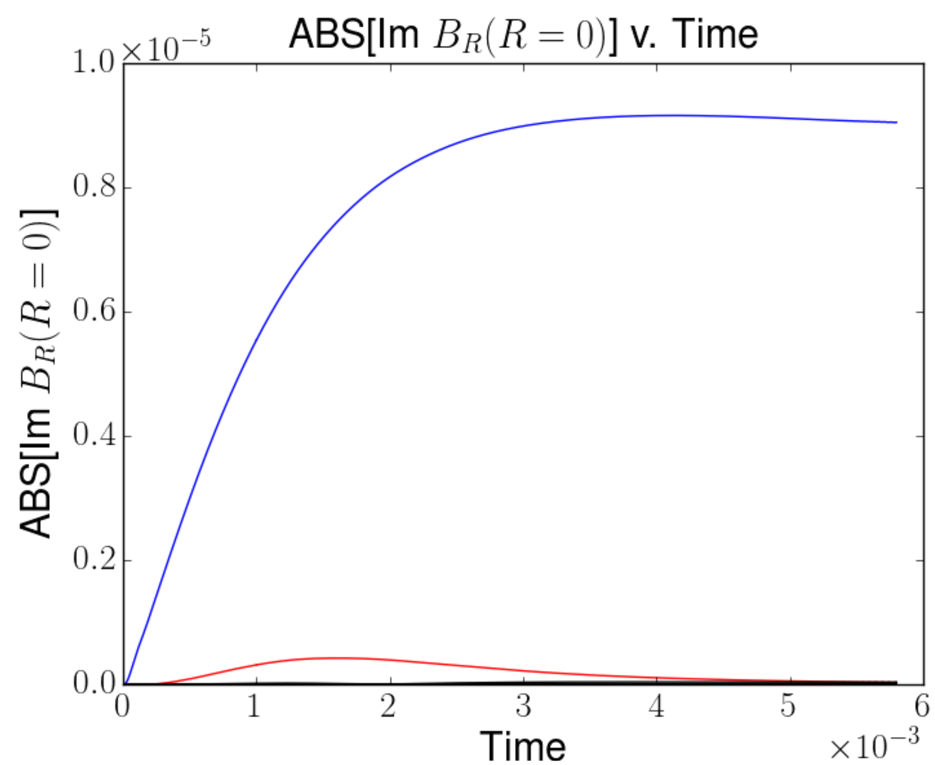
- Magnitude (top) and phase (bottom) of normal field as equilibrium flow is varied in the VR regime
  - Blue data correspond to analytically predicted values from Fitzpatrick POP (1998)
  - Green data correspond to numerically observed values
- NIMROD slightly over predicts the magnitude and under predicts the phase shift as the flow is increased



Larger Amplitude Boundary Perturbation ($B_{nw} = 10^{-4}$ T) Causes Nonlinear Forcing

- Nonlinear simulations keep $n_\phi = 0 - 5$
  - Increased polynomial degree and edge viscosity to improve convergence
  - Turned off density evolution
- Top: $B_{nw} = 10^{-8}$ T
  - Resembles linear simulations
  - Early growth scales as $\sim t^{1.15}$ (HK $\sim t^{5/4}$)
- Bottom: $B_{nw} = 10^{-4}$ T
  - $n_\phi = 1$ mode evolution (blue trace) consistent with boundary-driven Rutherford evolution:
    - Early growth as $\sim t^{0.845}$ (HK $\sim t^{2/3}$)
  - Higher order modes (e.g. $n_\phi = 2$ red trace) observed on same scale



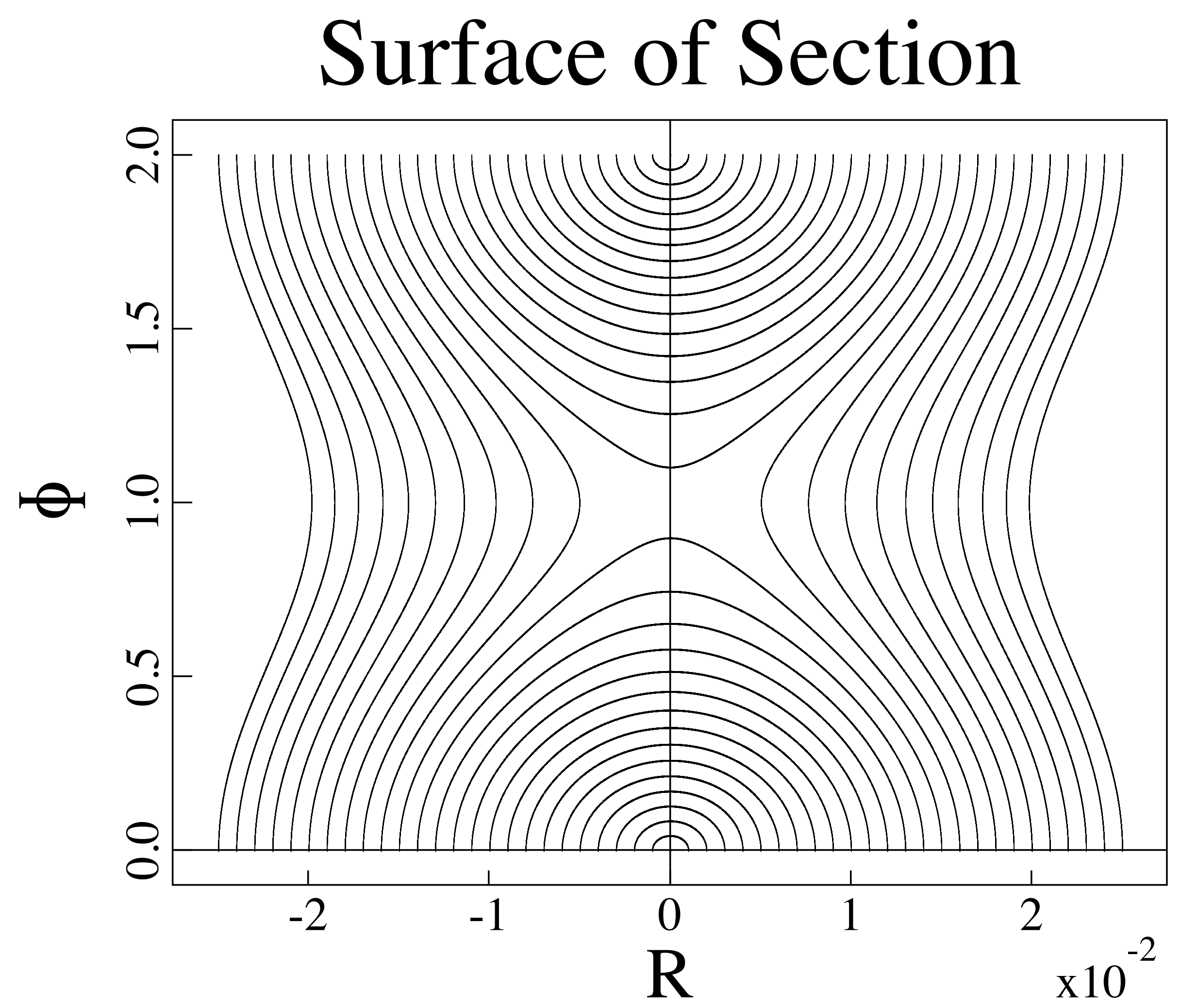


Nonlinear Driving Forms Large Island

- For $B_{nw} = 10^{-4}$ T and $v_{\phi,0} = 0$, HK predicts a saturated island half-width of

$$\sqrt{\frac{4aB_{norm}}{B_{\phi,0}k_{\phi}}} = 1.05 \times 10^{-2}\text{m}$$

  - Appropriate for a visco-resistive linear layer width of $a^*S^{-1/3}P^{1/6} = 4.51\times10^{-3}$ m



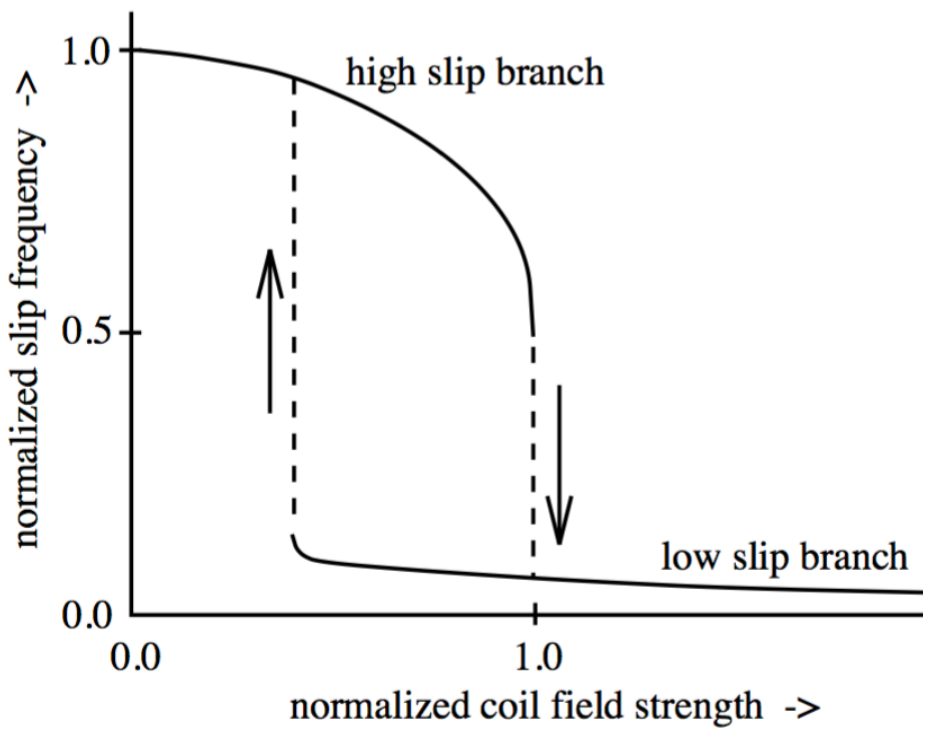
- Observed island is ~50% wider than predicted
- Note the different scales of the R, Z axis distort the X-null and island structure appearance

Nonlinear Electromagnetic and Viscous Force Balance Gives Rise to Bifurcation

- Integrating J×B and $\rho\nu\nabla^2$v over ϕ and radially about the rational surface gives n = 0 electromagnetic and viscous forces [e.g. Fitzpatrick NF (1993)]

$$\hat{F}_{\phi,EM} = \frac{2k_\phi L_\phi}{\mu_0 \cosh^2(k_\phi a)} \frac{\mathrm{Im}[\Delta(\omega)]}{(-\Delta')^2 + |\Delta(\omega)|^2} B_{nw}^2$$

$$\hat{F}_{\phi,VS} = -\frac{L_\phi^2 \rho\nu}{n\pi a}(\omega_0 - \omega)$$



- Force balance gives cubic relation in ω

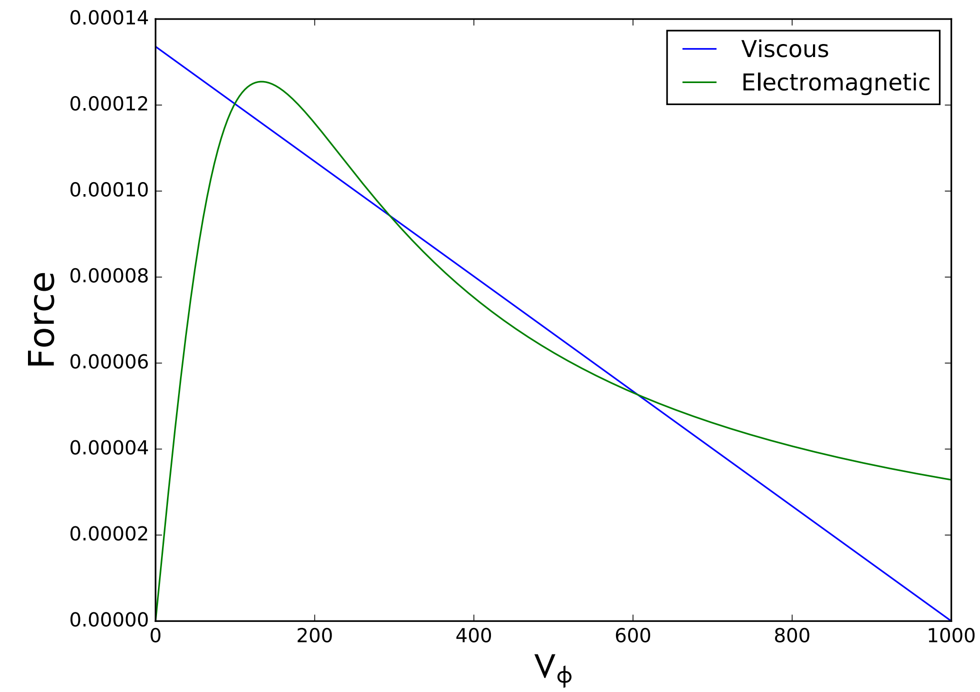
$$\frac{\omega_0}{\omega} - 1 + \omega_0\omega\tau_L^{*2} - 2\omega^2\tau_L^{*2} = \frac{\tau_L}{(-\Delta')\rho\nu} \frac{k_\phi^2}{\mu_0 \cosh^2(k_\phi a)} B_{nw}^2$$

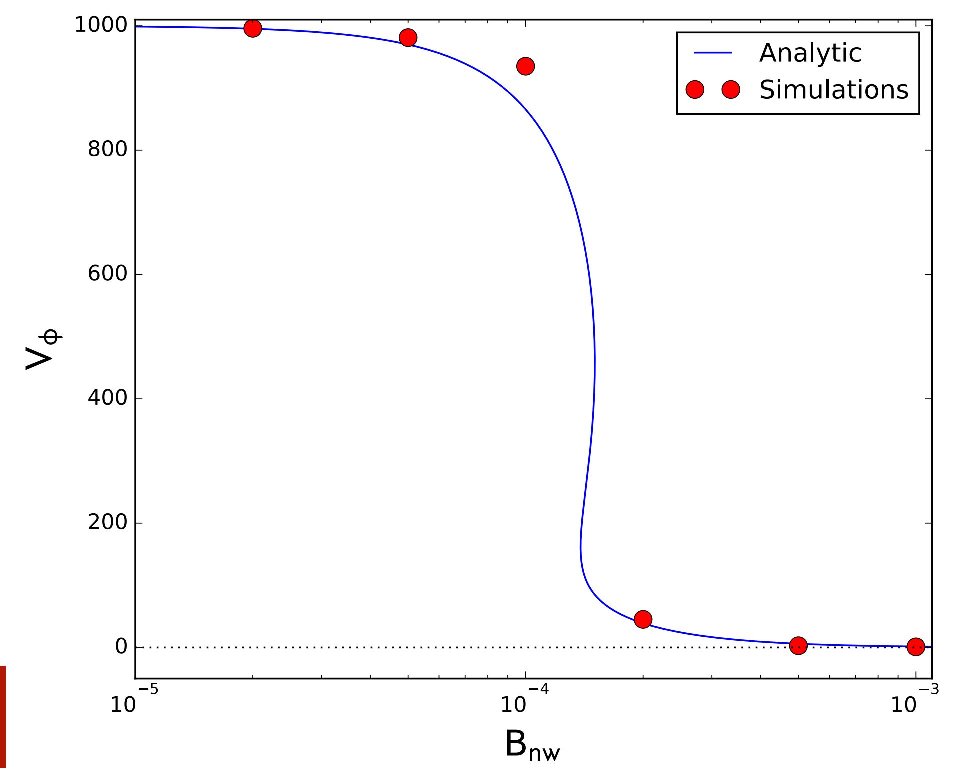
where $\tau_L = 2.104\tau_A S^{2/3} P^{1/6}$ and $\tau_L^* = \frac{\tau_L}{a(-\Delta')}$

- Bifurcation when initial angular frequency exceeds $\omega_{0,crit} = \frac{3\sqrt{3}}{\tau_L^*}$

Force Balance Bifurcation is Numerically Observed in Asymptotic States

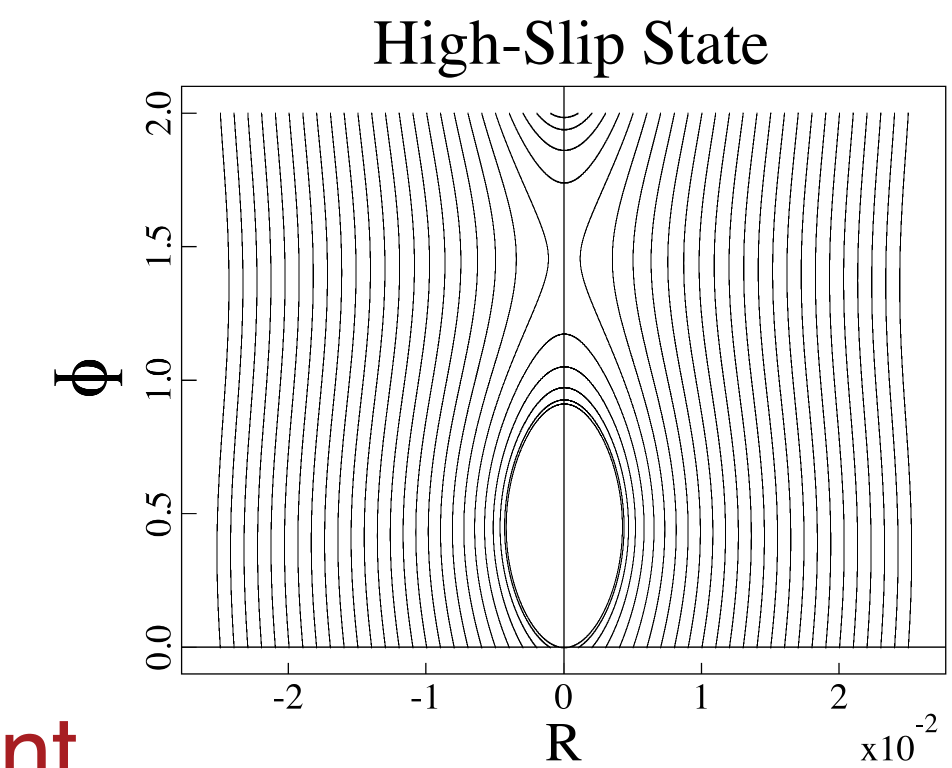
- Analytic, integrated, nonlinear EM (green) and viscous (blue) forces plotted in top figure
  - Both figures shown for $v_{\phi,0} = 10^3\ \text{m/s} > v_{0,crit} = 6.92\times10^2\ \text{m/s}$
  - Two stable (and one metastable) solutions exist, **high and low slip**
- Bottom figure shows cubic force balance equation in blue overlaid with asymptotic NIMROD results in red
  - Simulations clearly show bifurcation

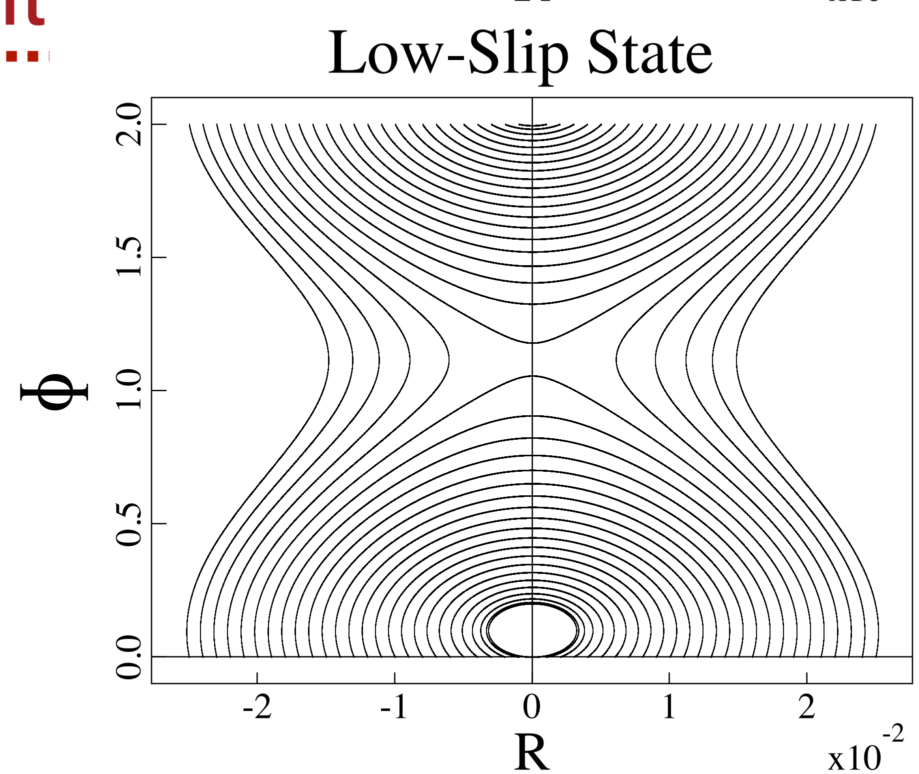




High/Low Slip Solutions Have Reduced/Typical Islands

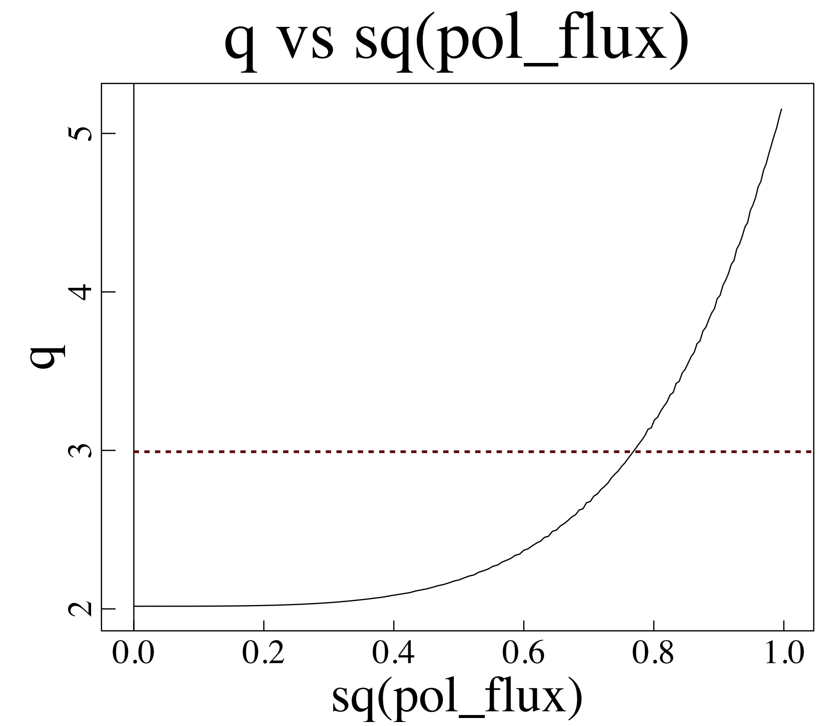
- High slip solution (top plot) has $B_{nw} = 10^{-4}$ T and low slip solution (bottom plot) has $B_{nw} = 2 \times 10^{-4}$ T
  - Predicted island half-widths given by $\sqrt{\dfrac{4aB_{norm}}{k_\phi B_{\phi,0}}}$
    - Takes flow screening into account
    - Predict 4.54×10^{-3} m and 1.44×10^{-2} m, respectively
  - Measured island half-widths of 6.17×10^{-3} m and 2.07×10^{-2} m
- Also observe flow shifting of island

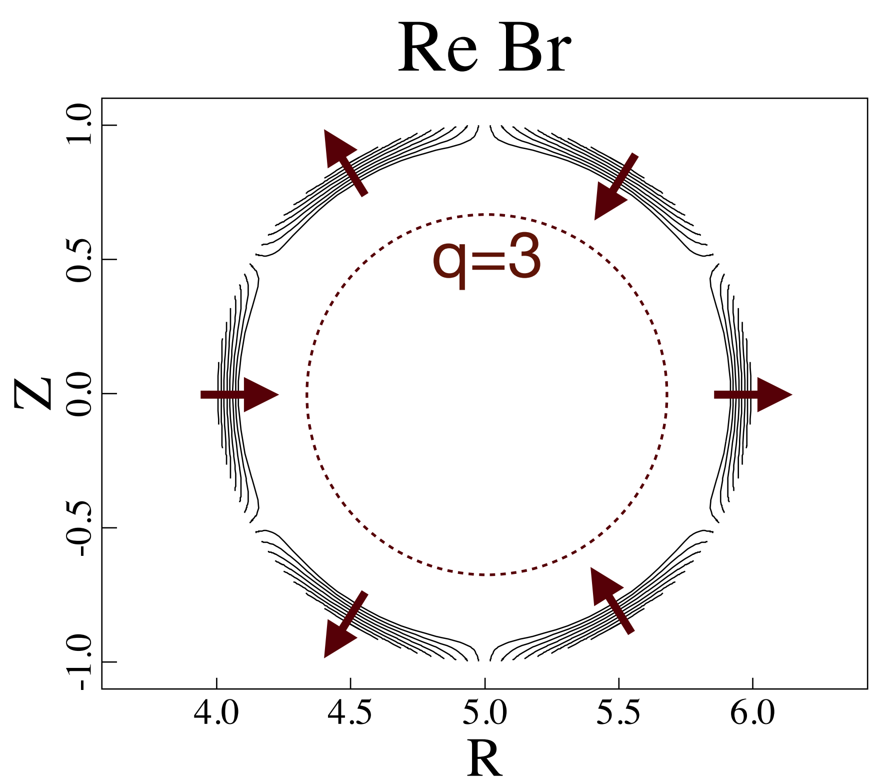




FMR Simulations are Underway in the Cylindrical Geometry

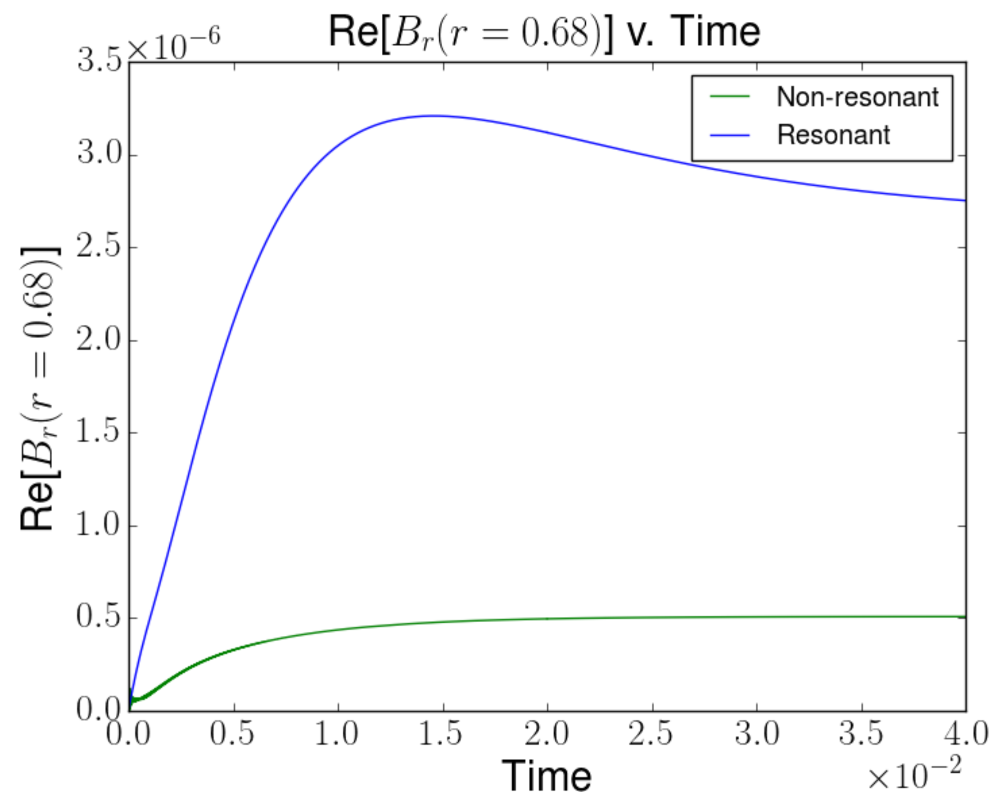
- Specify q-profile according to Furth, Rutherford, and Selberg (1973) form
  - $q(r) = q_0 \left[1 + \left(\frac{r}{r_0}\right)^{2\lambda}\right]^{1/\lambda}$
  - $q_0 = 2.02$, $r_0 = 0.65$, $\lambda = 2$
  - $r(q{=}3) = 0.681$ m, $\rho(q{=}3) = 0.796$
- P = 1 and similar S to slab case
- Follow same procedure as in slab geometry to prescribe edge boundary perturbation
  - Initialize (m,n) = (-3,1) perturbation in linear n=1 simulations

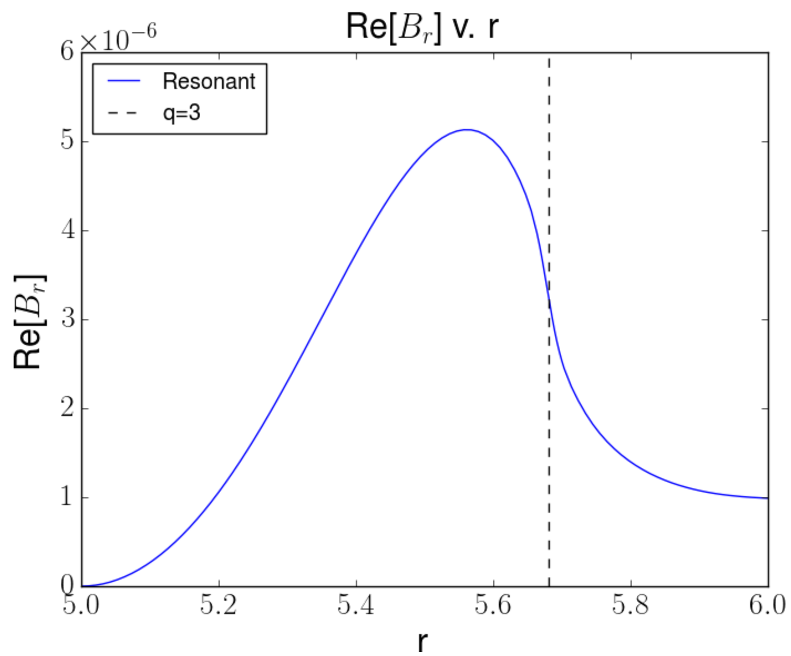


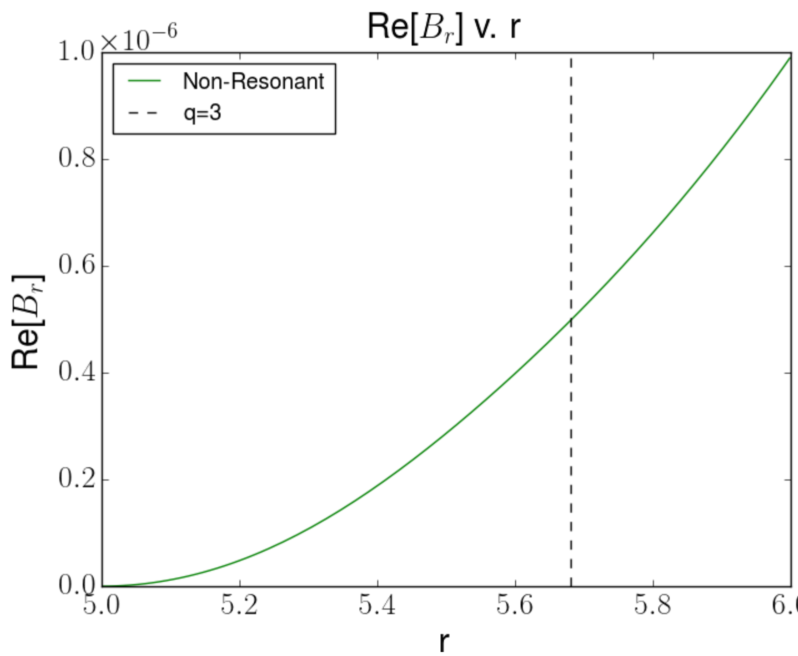


Resonant Perturbation Field in Cylinder Evolves Similar to HK Case

- $B_{nw} = 10^{-6}$ T edge perturbation triggers FMR when resonant with q = 3 surface (blue trace)
  - Plotting field at the rational surface on the outboard midplane
  - Cross-helicity case with m = 3 (green trace) diffuses inward on longer timescale
- Resonant field is amplified at rational surface with zero equilibrium flow
  - $B_{norm} = \dfrac{2m}{-\Delta' + \Delta(\omega)} B_{nw}$

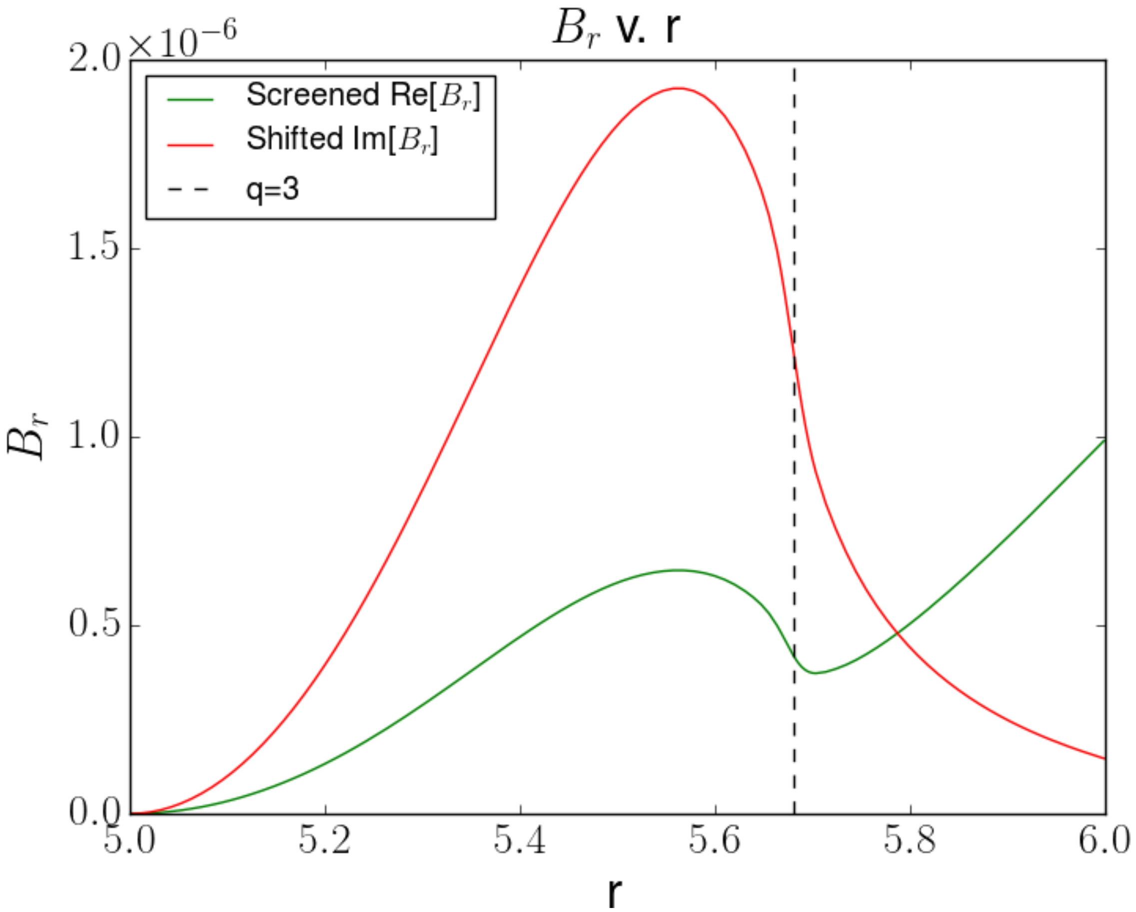






Flow Screening Qualitatively Agrees with Fitzpatrick Predictions

- With $v_{\phi,0} = 10^3$ m/s, observe $\mathrm{Re}[B_r(\theta=0)]$ is decreased (green trace) and $\mathrm{Im}[B_r(\theta=0)]$ is now nonzero (red trace)
  - Flow screens and shifts B_r
- As in slab, we are in VR regime
  - Theory predicts $\mathrm{Re}[B_r(\theta=0)] = 1.03 \times 10^{-6}$ T and $\mathrm{Im}[B_r(\theta=0)] = -1.53 \times 10^{-6}$ T
    - Note figure has $\mathrm{ABS}[\mathrm{Im}(B_r)]$
  - Hypothesize discrepancy due to value of Δ' used in calculations



Future Direction Of This Research

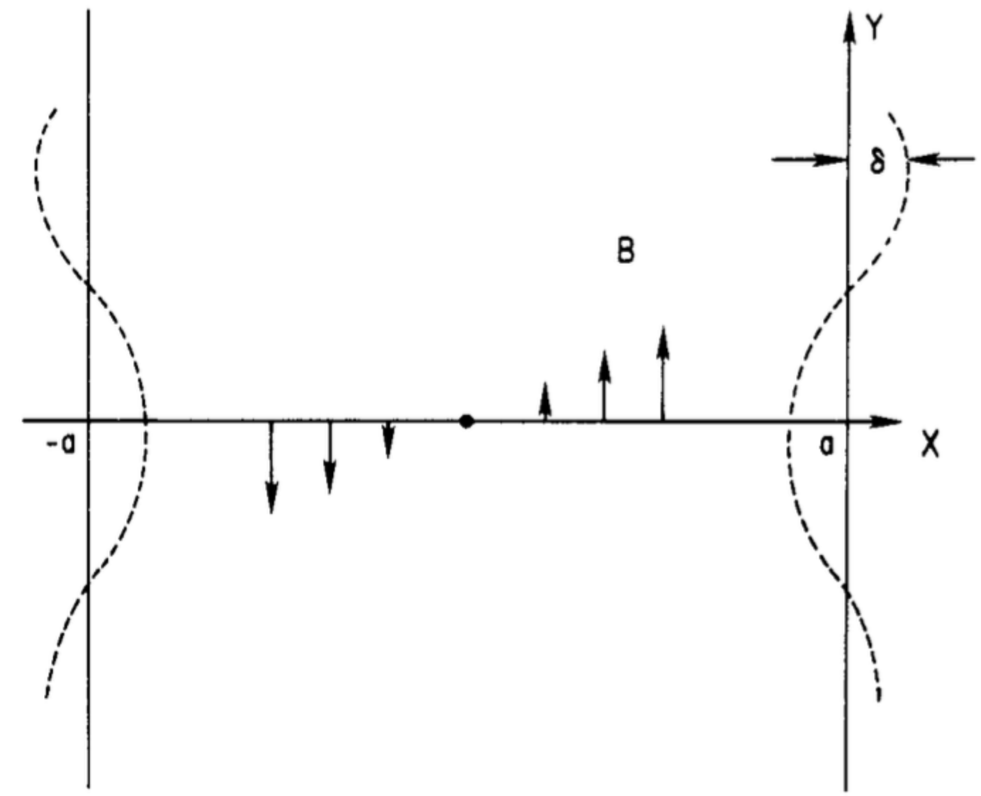
- ***Trigger bifurcation*** in slab by increasing edge field magnitude slowly compared to system evolution
  - Observe hysteresis by slowly decreasing edge field
- Linear and nonlinear flow-scaling in a cylinder
  - Torque balance bifurcation
- Linear and nonlinear flow-scaling in a torus with the addition of two-fluid effects
- **Benchmarks between NIMROD and M3D-C[1]**

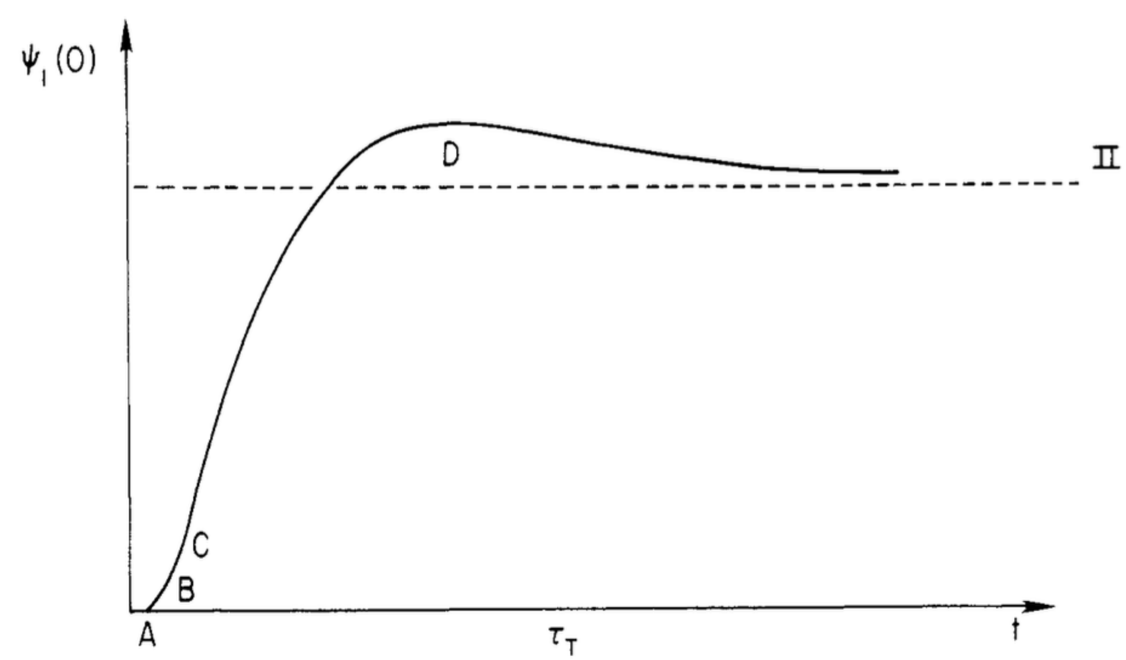
Conclusions

- Verified Hahm and Kulsrud analytical model of linear and nonlinear evolution of Taylor's problem by applying spatially-varying, boundary-normal magnetic field in NIMROD simulations
- Verified flow-screening effects consistent with Fitzpatrick model for linear slab, notably in the visco-resistive regime typical for tokamak H-mode pedestals
- Verified nonlinear force balance bifurcation consistent with Fitzpatrick model, that gave rise to high and low slip solutions
- Preliminary numerical study of FMR in cylindrical geometry is underway

Hahm & Kulsrud (HK) Slab Model Provides Paradigm Problem

- Taylor's problem: resistive sheet pinch with $\mathbf{B} = B_T\hat{\mathbf{z}} + (B_0/a)x\hat{\mathbf{y}}$
  - Boundaries perturbed by $x = \pm[a - \delta\cos(k_y y)]$
- Tearing creates magnetic islands [$\psi(x{=}0) \neq 0$] with finite resistivity
  - For resistive phase $t \ll \tau_{Res} \equiv \tau_A S^{1/3}$, $\psi(x{=}0,t) \sim t^2$
  - For tearing phase $t \sim \tau_T \equiv \tau_A S^{3/5}$, $\psi(x{=}0,t) \sim t^{5/4}$ for $t \ll \tau_T$ and $\psi(x{=}0,t) \sim t^{-5/4}$ for $t \gg \tau_T$
- Nonlinear driving when $\delta/a \geq S^{-4/5}$
  - For $t \ll \tau_{NL} \sim (\delta/a)^{1/2}\tau_R$, $\psi(x{=}0,t) \sim t^{2/3}$
  - For $t \gg \tau_{NL}$, $\psi(x{=}0,t) \sim \tanh^2(t/\tau_{NL})$



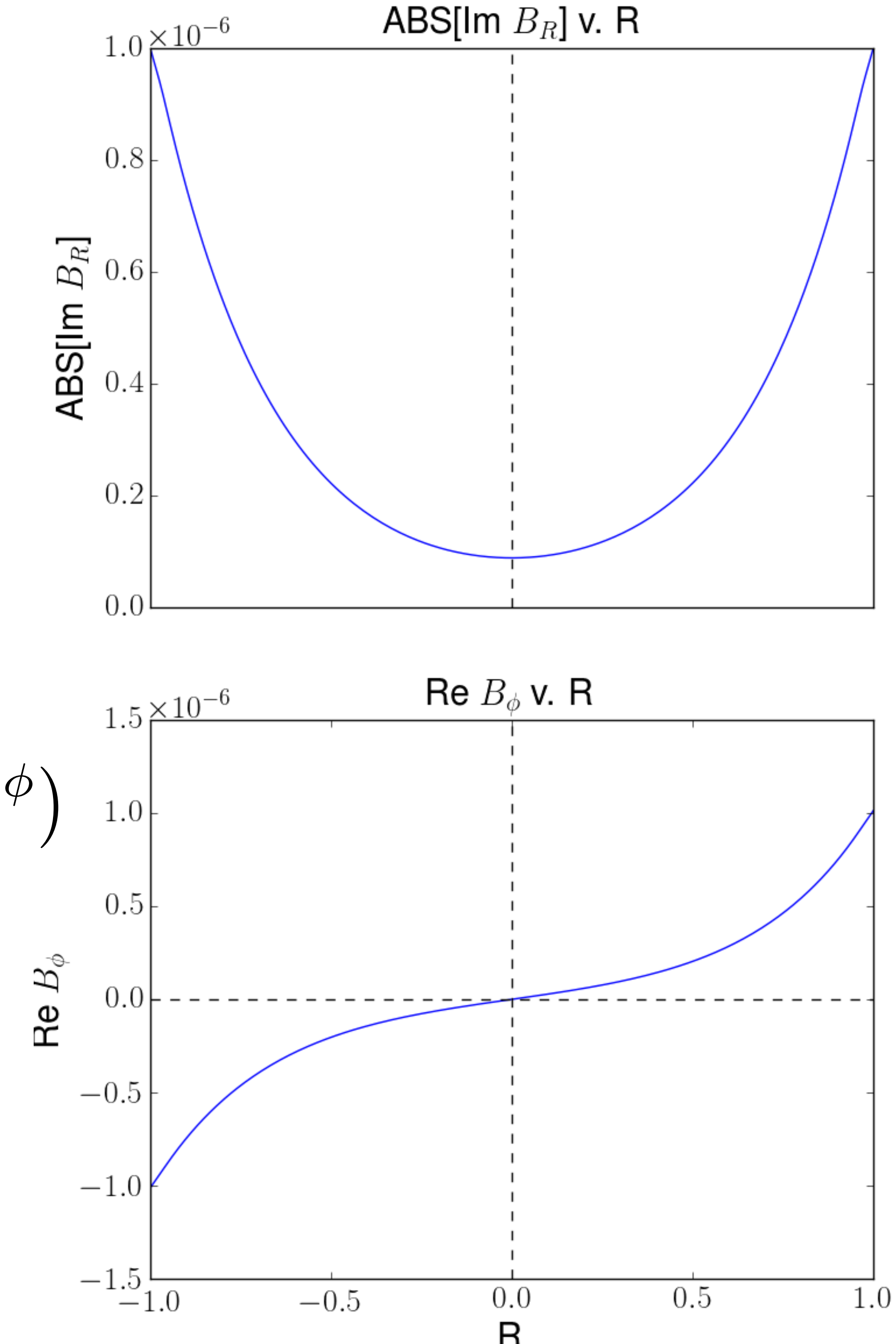


Time Asymptotic State Of Linear Results Agree with HK

- For sheet pinch with constant current in the symmetry direction, HK showed that force balance reduces to vacuum boundary-value problem
  - $\mathbf{B}_\perp = \hat{\mathbf{z}} \times \nabla\psi$ and BC $\psi_1(R=a) = B_{nw}/k_\phi$
  - Solution for system with $\psi_1(R=0) \neq 0$ (i.e. resistive evolution):

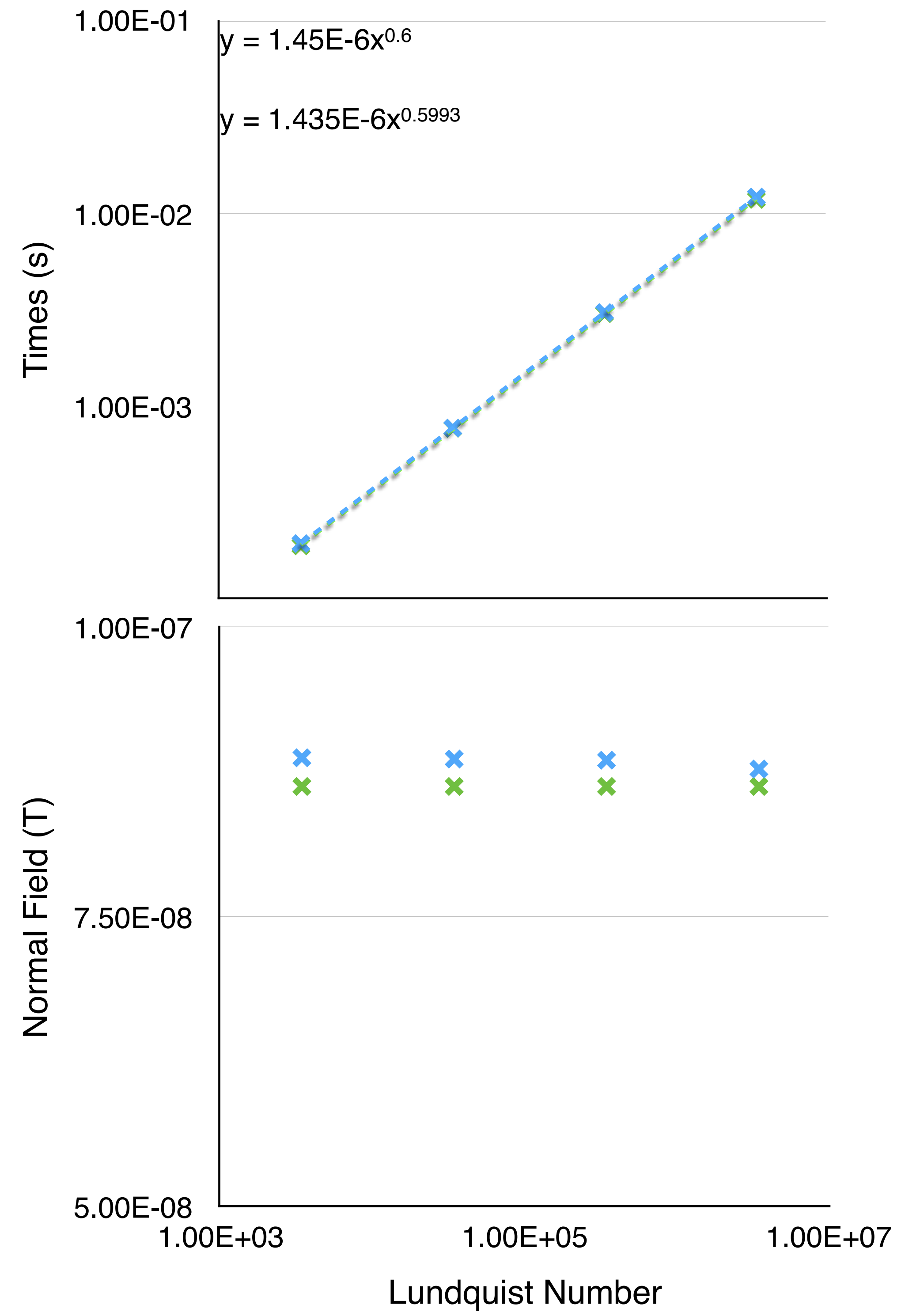
$$\psi_1(R,\phi) = \frac{-B_{nw}}{k_\phi}\frac{\cosh(k_\phi R)}{\cosh(k_\phi a)}\left(e^{ik_\phi\phi} + e^{-ik_\phi\phi}\right)$$

- $B_{R,1}(\phi=0) = -iB_{nw}*[\cosh(k_\phi R)/\cosh(k_\phi a)]$
  - $\mathrm{Re}[B_{R,1}] \sim 0$
- $B_{\phi,1}(\phi=0) = B_{nw}*[\sinh(k_\phi R)/\cosh(k_\phi a)]$



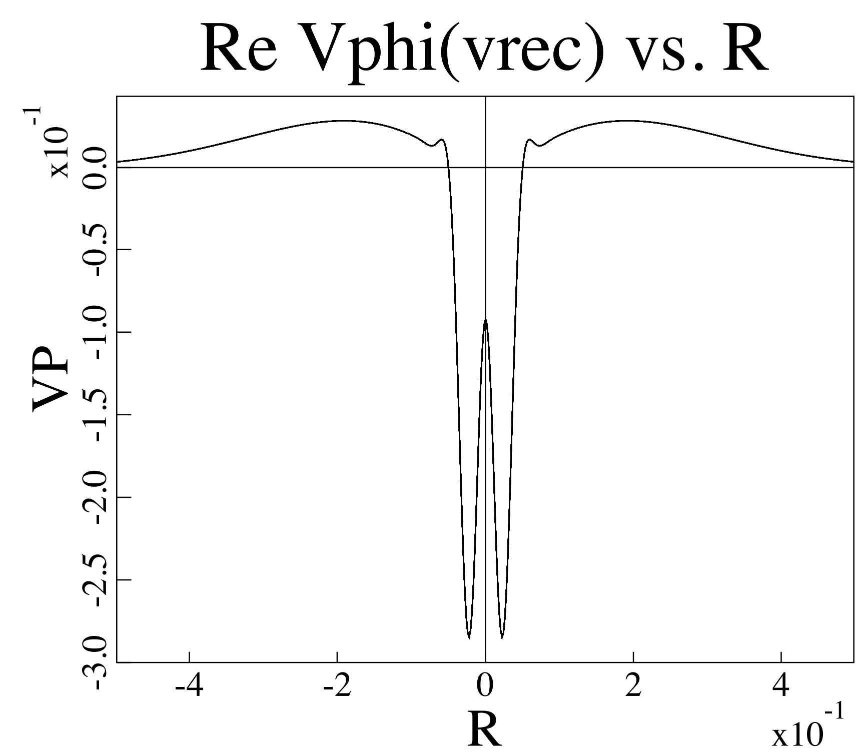
Parametric Results Agree with HK

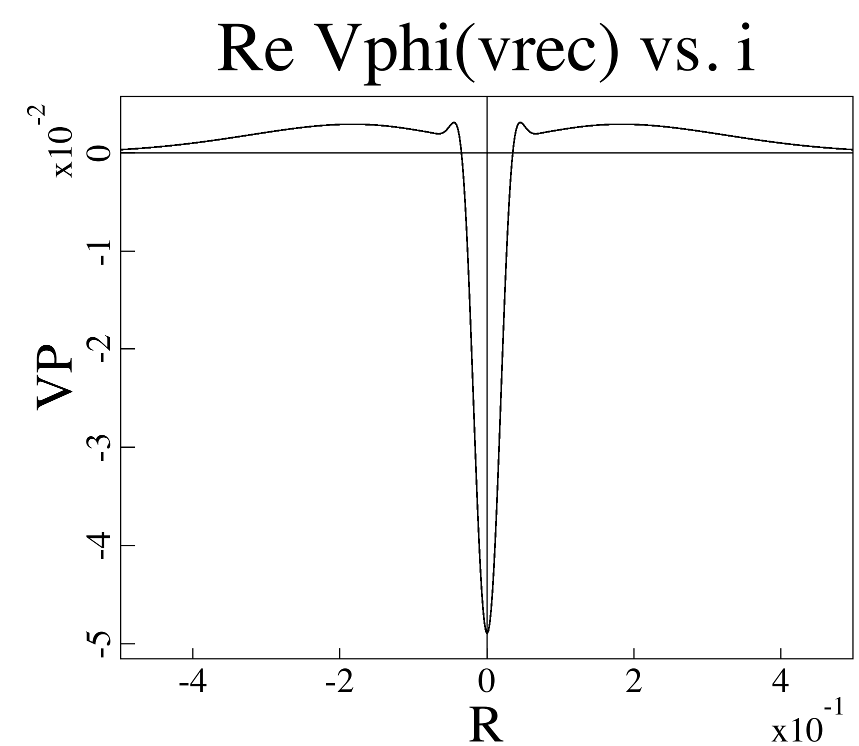
- Varying Lundquist number gives outstanding agreement with analytical predictions
  - Time at maximum amplitude of $B_{R,1}$ (blue data) coincides with τ_T (green data)
    - *Traces lie on each other*
  - Varying B_{nw} has no effect on time of maximum amplitude of $B_{R,1}$
- Asymptotic value of $B_{R,1}(R=0)$ is independent of S as expected
  - Green data are HK predictions, blue data from NIMROD simulations



Alfvén Resonances Emerge for Increased Flows

- With $v_{\phi,0} = 10^4$, Re(Q) ~ 3.19 and we are in the I regime (shown in the top figure)
  - Compared to the case with $v_{\phi,0} = 10^3$ (bottom figure), there are now two negative peaks in $\text{Re}(v_{\phi,1})$
- Due to an Alfvén Resonance
  - Where $(\mathbf{k} \cdot \mathbf{v})^2 = (k_\phi v_{A,\phi})^2$





Inertial Regime Has a Different Asymptotic State From the VR Regime

- Alfvén resonances creates smaller scale structure within the layer
- While top plot shows $Re[B_{R,1}(R=0)] \neq 0$, as $v_{\phi,0}$ is increased the normal field at the rational surface asymptotes to zero
  - Bottom plot shows that the imaginary component of the normal field changes sign at the rational surface
- **Essentially no tearing (reconnection) in this regime**