Two-Fluid Benchmarking of the 1/1 Internal Kink

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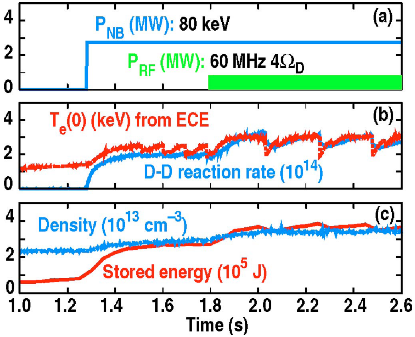
A verification effort is underway as part of a larger effort to model the nonlinear evolution of the giant sawtooth.

Sawteeth are periodic relaxation events of the core plasma.

- In tokamaks sawteeth result from the nonlinear evolution of a $n = 1$ mode.
- The sawtooth cycle is characterized by a slow build-up of the core n and T_e followed by a rapid crash.
- Two fluid drifts and kinetic effects temporarily stabilize the kink leading to larger but less frequent giant sawteeth.
- Giant sawteeth are a concern for modern tokamaks and ITER.



[Choi et al., PoP 14, 2007]

Modeling of the giant sawtooth requires an accurate representation of multiple two fluid effects.

- Different two fluid effects modify the stability of the internal kink in opposing ways.
  - Diamagnetic drifts reduce the linear growth rate when the diamagnetic frequency is comparable to the MHD growth rate [Ara et al., 1978].
  - Finite electron compressibility allows the electrons and ions to decouple and increases the linear growth rate in the semi-collisional and collisionless regimes [Zakharov and Rogers, Phys Fluids B. 4, 1992].
  - Electron inertia increases the growth rate in in collisionless regime.
- We present the results of a verification effort to test NIMROD's ability to capture the different two fluid effects for the 1/1 kink in a screw pinch equilibrium.
  - NIMROD accurately captures the transition from ideal to resistive kink in resistive MHD.
  - NIMROD accurately captures the increased growth rate in the semi-collisional regime in the absence of diamagnetic drifts.
  - There is a significant discrepancy between NIMROD and the analytic theory of Zakharov and Rogers when drifts are included.

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The ideal MHD 1/1 kink is characterized by a "top hat" radial displacement.

- The Euler-Lagrange equation for a screw pinch is:

$$\frac{\delta W}{2\pi} = \frac{\pi}{\mu_0} \int \left(f \left(\frac{d\xi}{dr} \right)^2 + g\xi^2 \right) dr$$

$$f \sim rB_\theta^2 \left(1 - q\right)^2$$

$$g \sim \frac{B_\theta^2}{r} \left(\left(1 - q\right)^2 - 2\left(1 + q\right)\left(1 - q\right) \right) + 2\frac{r^2}{R^2} \frac{d\mu_0 p}{dr}$$

- The quantity g is negative for $|q| < 1$.
- δW is negative indicating instability for the top hat trial function:

$$\xi = \begin{cases} \xi_\infty & \text{for} \quad r < r_s \\ 0 & \text{for} \quad r > r_s \end{cases}$$

The dynamics in a thin layer around the discontinuity at the rational surface determines the linear growth rate.

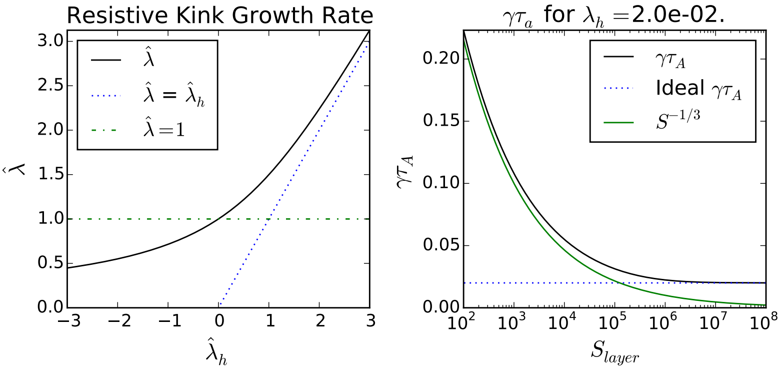
- Coppi et al. worked out the resistive MHD kink dispersion relation in the limit of a thin layer [Sov J. Plasma Phys. 2, 1976]

Resistive MHD kink dispersion relation

$$\hat{\lambda} = \hat{\lambda}_h \left(\frac{\hat{\lambda}^{9/4}}{8} \frac{\Gamma\left(\frac{\hat{\lambda}^{3/2}-1}{4}\right)}{\Gamma\left(\frac{\hat{\lambda}^{3/2}+5}{4}\right)} \right)$$

- $\hat{\lambda} = \gamma \tau_A S^{1/3}$ and $\hat{\lambda}_h = \lambda_h S^{1/3}$
- $\lambda_h = \frac{-\pi}{B_\theta^2 q'^2 r_s^2} \int_0^{r_s} g dr$ is the normalized ideal MHD growth rate.
- The resistive kink growth rate is $\gamma_r \tau_A = S^{-1/3}$ $(\lambda_h = 0)$.
- The Alfven time and Lundquist number are defined relative to the layer: $\tau_A^2 = \frac{\mu_0 \rho_0}{B_z^2} \frac{R^2}{q'^2 r_s^2}$, $\tau_R = \mu_0 r_s^2 / \eta$, and $S = \tau_R / \tau_A$.

The resistive MHD dispersion relation captures the transition from tearing mode to ideal kink.



- The ideal MHD growth rate $\hat{\lambda} = \hat{\lambda}_h$ is recovered in the limit of large $\hat{\lambda}_h$ (large S).
- The resistive kink growth rate $\hat{\lambda} = 1$ is recover in the limit of $\hat{\lambda}_h \approx 0$ (small S).
- Tearing behavior is recovered when $\hat{\lambda}_h \ll 0$.

Two-fluid modifications to the internal kink are described by Zakharov and Rogers inner layer equation [Phys. Fluids B 4, 1992].

The two-fluid kink growth rate, Γ, is an eigenvalue of the inner layer equation

$$\left[\lambda_s^2 + \frac{\lambda_e^2 \Gamma\left(\Gamma - i\Omega_{*i}\right)}{q'^2 x^2} Z'\right]' = \left(1 + \frac{\Gamma\left(\Gamma - i\Omega_{*i}\right)}{q'^2 x^2}\right) Z - \frac{2L_h}{\pi x^2}\int_0^\infty Z dx$$

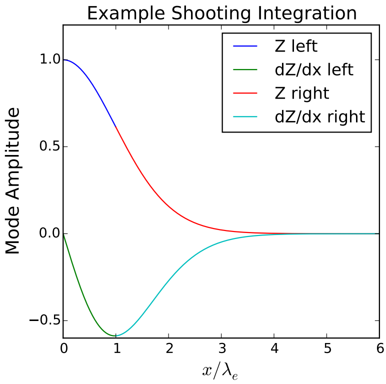
that satisfy the boundary condition

$$\lim_{|x|\to\infty} Z = \frac{2L_h}{\pi x^2}\int_0^\infty Z dx.$$

- $Z = ia\gamma\xi_a' \simeq V_\theta$ is approximately the poloidal flow.
- $\Gamma = \gamma\tau_A$ and $\Omega_{*i} = \omega_{*i}\tau_A$ are the normalized growth rate and ion diamagnetic frequency.
- $\lambda_s^2 = \frac{\rho_s^2\Gamma(\Gamma - i\Omega_{*i})}{\left(\Gamma - i\frac{5}{3}\Omega^n_{*e}\right)\left(\Gamma - i\frac{5}{3}\Omega^n_{*i}\right)}$ is a modified ion sound gyroradius length scale squared.
- $\lambda_e^2 = \frac{\Gamma}{\Gamma - i\Omega_{*e}}\left(\frac{1}{S\Gamma} + d_e^2\right)$ is the effective resistive skin depth squared.
- L_h is the inertial ideal MHD length scale.

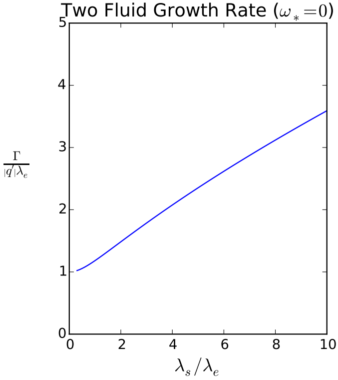
The eigenvalues of the inner layer equation are calculated numerically using a shooting method.

- The layer equation is solved by treating it as an initial value problem.
- The equation is integrated twice:
  - The first integration starts at $x = 0$ and integrates outwards towards an intermediate value of x_m.
  - The second integration starts at $x_r > \max(\lambda_e, \lambda_s)$ and integrates inwards towards x_m.
  - To seed the integration we initially guess the $\int Z dx$.
- We converge on Γ by minimizing the error in Z and Z' at x_m and the the guess of $\int Z dx$.

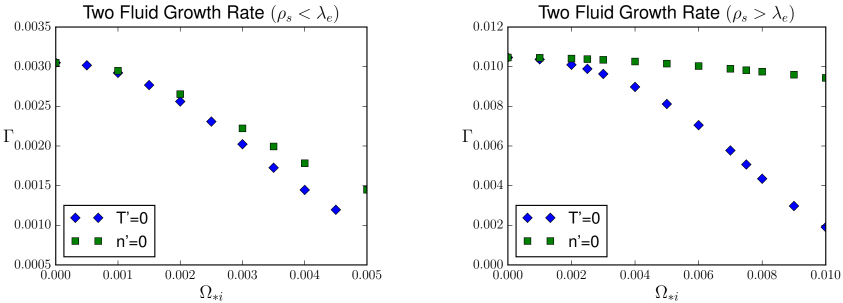


Decoupling of the electrons and ions leads to faster growth rates at large ρ_s.

- The linear growth rate is enhanced when the ion sound length scale is larger than the resistive skin depth ($\lambda_s > \lambda_e$).
  - Here the two-fluid treatment is accurate provided that the ion-sound Larmor radius is larger than the ion Larmor radius ($\rho_s > \rho_i$).
  - The current sheet width is characterized by the resistive skin depth λ_e.
  - The flow "sheet" is characterized by the ion sound length scale λ_s.



Diamagnetic drifts decrease the linear growth rate at fixed ρ_s.



- In the collisional limit $(\lambda_e > \rho_s)$ drifts that arise due to temperature gradients have a similar impact of the growth rate as drifts that arise due to density gradients.
- In the semi-collisional limit $(\lambda_e < \rho_s)$ drifts that arise due density gradients have a larger impact of the growth rate than drifts due to temperature gradients.

A verification effort is underway as part of a larger effort to model the nonlinear evolution of the giant sawtooth.

Linear calculations use the NIMROD code to evolve the primitive fields.

$$\rho\left(\partial_t\vec{V}+\vec{V}\cdot\nabla\vec{V}\right)=\vec{J}\times\vec{B}-\nabla P-\nabla\cdot\pi_i$$

$$\pi_i=-\rho\nu_{iso}W+\frac{P_i}{4\Omega_{ci}}\left[\hat{b}\times W\cdot\left(I+3\hat{b}\hat{b}\right)-\left(I+3\hat{b}\hat{b}\right)\cdot W\times\hat{b}\right]$$

$$W=\nabla\vec{V}+\nabla\vec{V}^T-2/3I\nabla\cdot\vec{V}$$

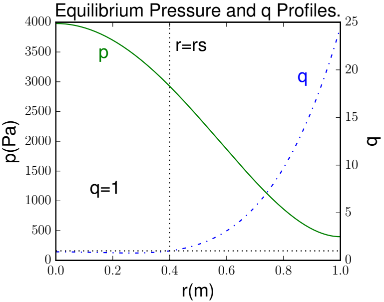
$$\partial_t n+\nabla\cdot(n\vec{V})=\nabla\cdot(D\nabla n-D_h\nabla\nabla^2 n)$$

$$n\left(\partial_t T_s+\vec{V}_s\cdot\nabla T_s\right)=-\left(\gamma-1\right)P_s\nabla\cdot\vec{V}_s-\left(\gamma-1\right)\nabla\cdot\vec{q}_s$$

$$\partial_t\vec{B}=-\nabla\times\left[\eta\vec{J}-\vec{V}\times\vec{B}+\frac{1}{ne}\left(\vec{J}\times\vec{B}-T_e\nabla n\right)+\mu_0 d_e^2\partial_t\vec{J}\right]+k_{divb}\nabla\nabla\cdot\vec{B}$$

- Artificial particle diffusivity and magnetic divergence diffusions are used to provide numerical stability.
- Gyro-viscosity and two-fluid corrections to Ohm's law are included in two-fluid calculations.

Calculations are performed in a screw pinch that is $n = 1$ ideal kink unstable.

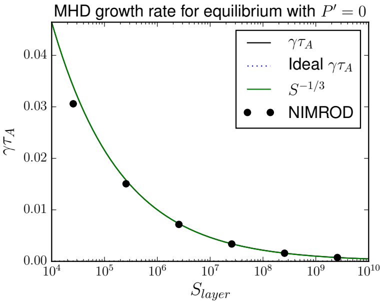


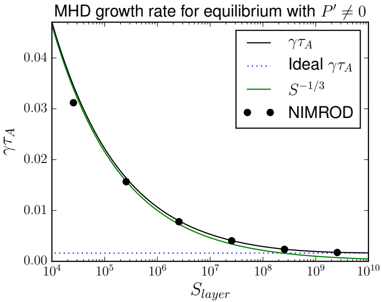
Equilibrium Parameters:

q_0	0.9
$q(a)$	24.5
B_0	1T
R/a	30
β_0	1%

- Equilibria are generated by specifying the pressure and safety factor.
- Two equilibria are studied: one with a uniform pressure and the other with a spatially varying pressure profile.
  - This allows for the study of two fluid modifications to the kink with and without diamagnetic drift stabilization.
- A strongly sheared q profile is needed to produce thin layers.

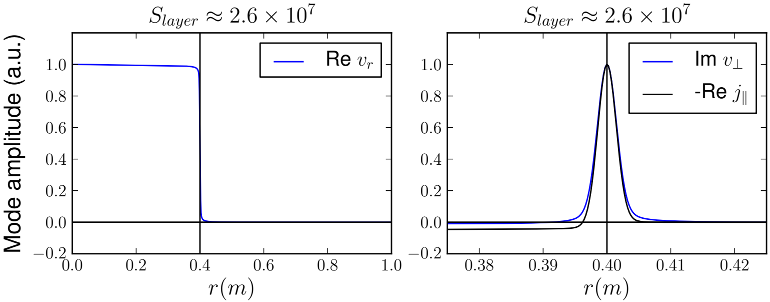
Resistive MHD calculations capture the transition from resistive to ideal kink.





- The resistive interchange scaling $\gamma\tau_A = S^{-1/3}$ is an excellent approximation for the uniform pressure equilibrium with $S < 10^{10}$.
  - Here the ideal drive is weak: $\gamma_{ideal}\tau_A = 4.2 \times 10^{-5}$
- The pressure gradient is the dominant source of free energy for the nonuniform pressure equilibrium.
  - Ideal behavior is recovered for $S \gtrsim 10^9$.
  - Here the ideal growth rate is $\gamma_{ideal}\tau_A = 1.6 \times 10^{-3}$.
- The validity of the analytic theory breaks down due to a finite layer width at small S.

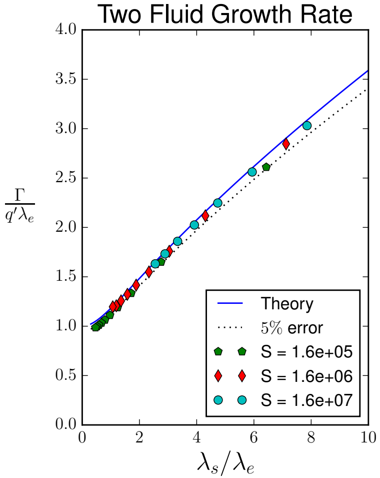
The radial velocity resembles the “top hot” trial function at large S.



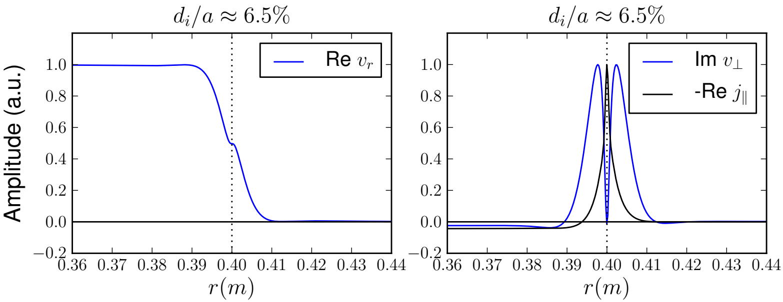
- Figures show the mode structure for the nonuniform pressure equilibrium at $S = 2.6 \times 10^7$.
- The radial flow is proportional to the displacement.
- As predicted from MHD theory, the momentum and current layers have the same width.
  - Note that the horizontal axes use different scales in the two plots.

NIMROD accurately calculates the growth rate in the transition from the collisional to the semi-collisional regime in the absences of drifts.

- The linear growth rates agree with theory to within 5% error for a wide range or parameters.
  - This agreement has been verified for $\rho_s/\lambda_e \lesssim 60$ and $S \lesssim 10^9$.
- These calculations use the uniform pressure equilibrium.
- The theoretical growth rate is calculated assuming $\lambda_h = 0$.

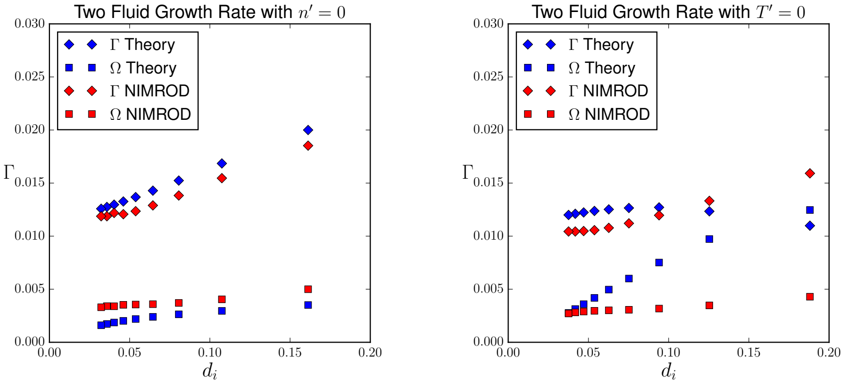


The separation of layer widths is observed at large d_i.



- The width of the flow layer scales with ρ_s.
- The current layer width depends on both the resistive layer width and the electron skin depth.
  - In the collisionless limit the current layer width scales linearly with electron skin depth.
- Figures show the mode structure for $S = 2.6 \times 10^7$.

There is considerable disagreement in the calculated growth rate when drift effects are included.



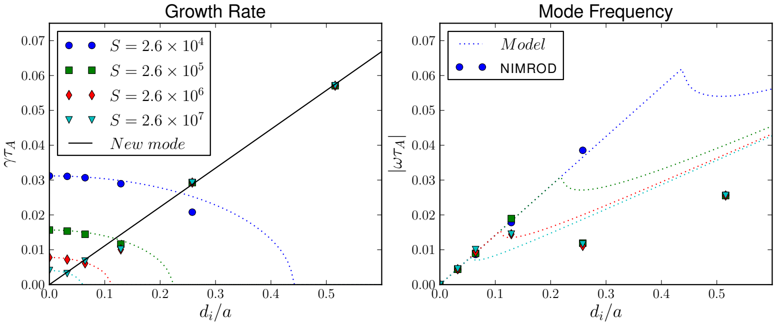
- Reasonable agreement is observed between NIMROD and the theory in the uniform density calculation, where pressure gradients are due to temperature gradients.
  - Here the drifts have a small effect on the growth rate.
- There is considerable disagreement in the uniform temperature calculation.
- Increasing d_i increases both ρ_s and ω_* in these calculations.

NIMROD reproduces the two-fluid linear kink behavior in several regimes.

- Resistive MHD calculations correctly calculate the linear growth rate in both the inertial and resistive regimes.
  - Agreement between theory and calculations is limited by the validity of small layer approximation.
- The two-fluid calculations agree with the theory of Zakharov and Rogers to within 5% error in the absence of drifts.
  - Agreement is obtained for a wide range of parameters ρ_s and S.
- The agreement breaks down when in calculations where the diamagnetic drifts have a significant effect on the growth rate.
  - Further work is needed to understand the source of the disagreement.

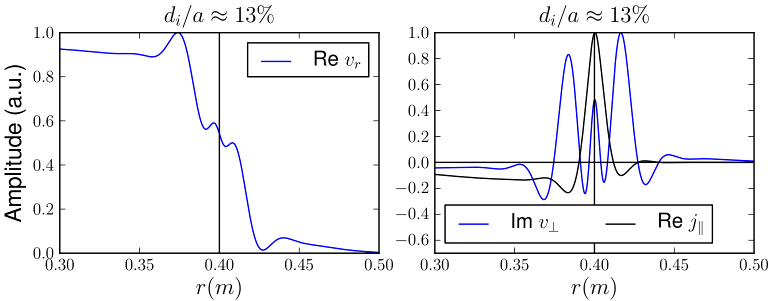
Extra Slides

Drifts reduce the linear growth rate at small d_i for all cases with a finite pressure gradient.



- The calculated growth rates are approximated by a simple model: $\omega = \omega_{*i} + i\sqrt{\gamma_{MHD}^2 - \omega_{*i}^2/4}$.
- Here the nonuniform pressure equilibrium is used.
- The 1/1 kink in not the dominant mode at large d_i.
  - The new mode is characterized by a large $v_{\|}$.
  - This modes scales linearly with d_i and is insensitive to S.

Large oscillations in the mode structure are observed when the drifts have a significant impact on the growth rate.



- These oscillations are characteristic of drift stabilization.
- Figures show the mode structure for $S = 2.6 \times 10^5$.
- Here the growth rate is 25% smaller than the MHD growth rate.