Cantori, chaotic coordinates and temperature gradients in chaotic fields

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Motivation

→ Error fields, 3D effects, . . create chaotic fields.

Method

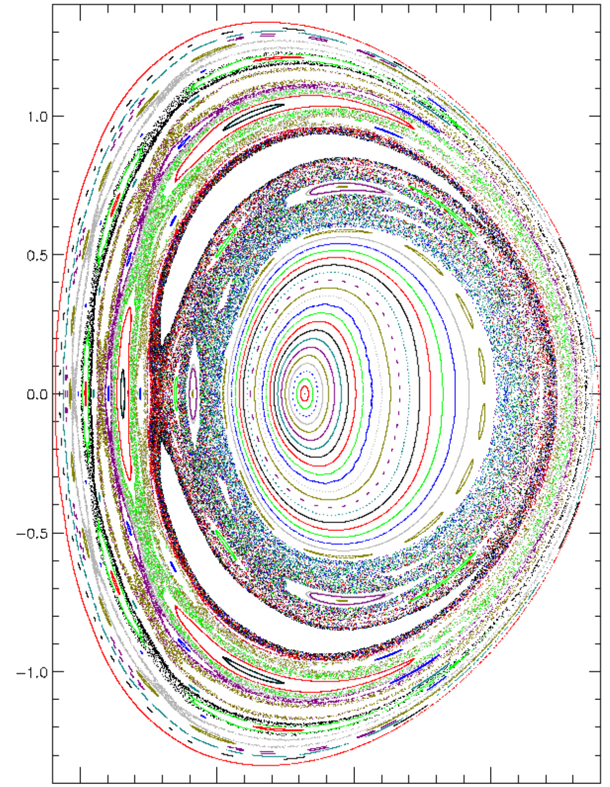
→ Heat transport is solved numerically:

$\nabla \cdot \left(\kappa_{\parallel} \ \mathbf{b} \ \mathbf{b} \cdot \nabla T + \kappa_{\perp} \nabla_{\perp} T \right) = 0$ with $\kappa_{\perp} / \kappa_{\parallel} = 10^{-10}$.

We found

→ Isotherms coincide with cantori,

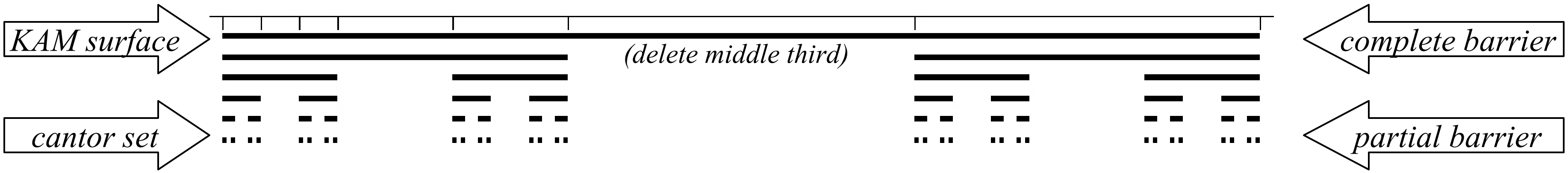
→ Chaotic coordinates, based on *ghost - surfaces*,
solves the temperature profile in a chaotic field.



eg. M3D simulation of CDX-U

Field-line transport is restricted by irrational field-lines

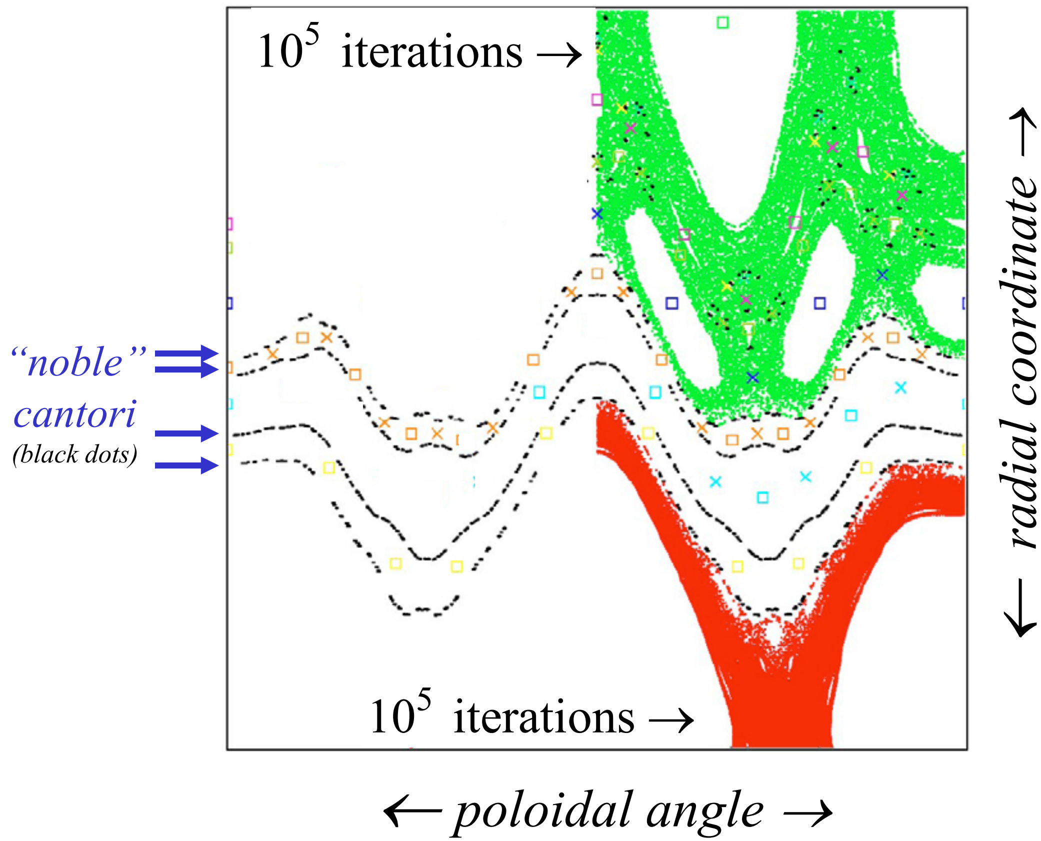
→ *the irrational KAM surfaces disintegrate into invariant irrational sets ≡ cantori, which continue to restrict field-line transport even after the onset of chaos.*



→ *KAM surfaces* ***stop*** *radial field-line transport*

→ *broken KAM surfaces ≡cantori* ***do not stop, but do slow down*** *radial field-line transport*

Poincaré plot (model field → next slide)



Cantori are approximated by high-order periodic orbits;

$\rightarrow$ *high-order (minimizing) periodic orbits are located using variational methods;*

- Magnetic field-lines, $\mathbf{B} = \nabla \times \mathbf{A}$, are stationary curves C of the action integral $S = \int_C \mathbf{A} \cdot \mathbf{dl}$,

  where $\mathbf{A} = \psi \,\nabla\theta - \chi \,\nabla\phi$ and $\chi(\psi,\theta,\phi) = \psi^2/2 + \sum k_{mn}(\psi) \cos(m\theta - n\phi)$.

- Setting $\delta S = 0$ gives $\dot{\theta} = B^\theta / B^\phi = \dot{\theta}(\psi,\theta,\phi)$ and $\dot{\psi} = B^\psi / B^\phi$.

- A piecewise linear, $\theta(\phi) = \theta_i + (\theta_{i+1} - \theta_i) / \Delta\phi$, trial curve

  allows analytic evaluation of the action integral, $S = S(\theta_0, \theta_1, \theta_2 \;...) \rightarrow$ *fast*!

- To find (p,q) periodic curves, use Newton's method to find $\partial S / \partial \theta_i = 0 \rightarrow$ *robust*!

  with constraint $\phi_N = 2\pi q$, $\theta_N = \theta_0 + 2\pi p$.

where robust means is not sensitive to Lyapunov error

- Two types of periodic orbit: O : stable, action-minimax

  X : unstable, action-minimizing $\rightarrow$ *cantori as* $p/q \rightarrow$ *irrational*

Character of solution determined by 2nd derivative=Hessian

Ghost-surfaces constructed via action-gradient flow between the stable & unstable periodic orbits.

C. Golé, J. Differ. Equations **97**, 140 1992., R. S. MacKay and M. R. Muldoon, Phys. Lett. A **178**, 245, 1993.

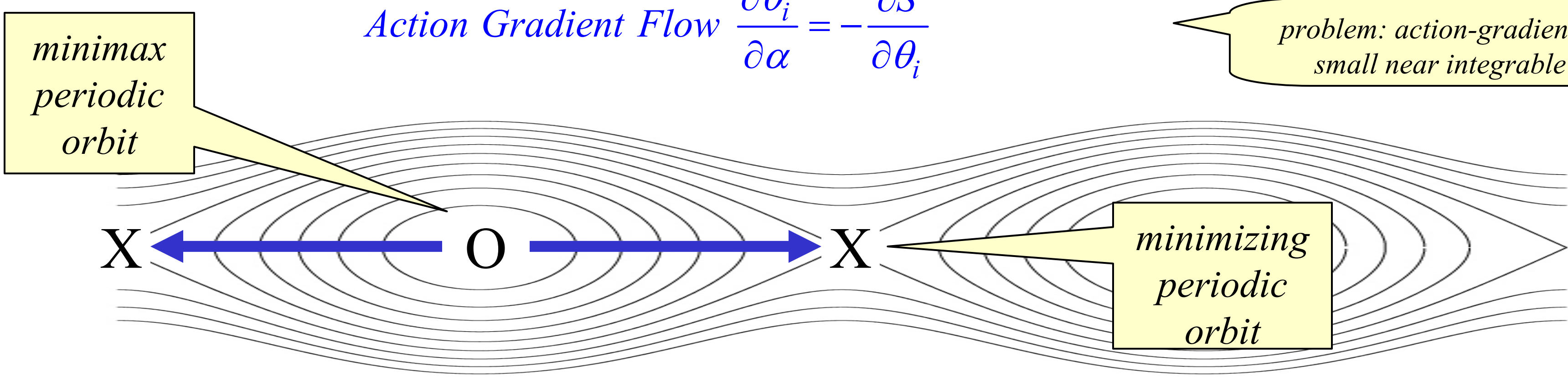
- At the minimax (stable) periodic orbit, the eigenvector of the Hessian, $\partial^2 S / \partial^2 \theta_{ij}$, with negative eigenvalue indicates the direction in which the action integral decreases.

- Pushing trial curve from minimax (stable) p/q orbit down action-gradient flow to minimizing (unstable) p/q orbit defines *ghost - surfaces*,

- Ghost-surfaces may be thought of as rational coordinate surfaces that pass through island chains.

Action Gradient Flow $\dfrac{\partial \theta_i}{\partial \alpha} = -\dfrac{\partial S}{\partial \theta_i}$

problem: action-gradient flow is small near integrable limit



Ghost-surfaces are almost identical to quadratic-flux-minimizing surfaces.

Dewar, Hudson & Price, Phys. Lett. A, 1994; Hudson & Dewar, Phys. Lett. A, 2009.

- Quadratic-flux-minimizing surfaces, S,

minimize $\varphi_2 \equiv \frac{1}{2}\int_S (B \cdot N)^2 \, d\theta d\phi$

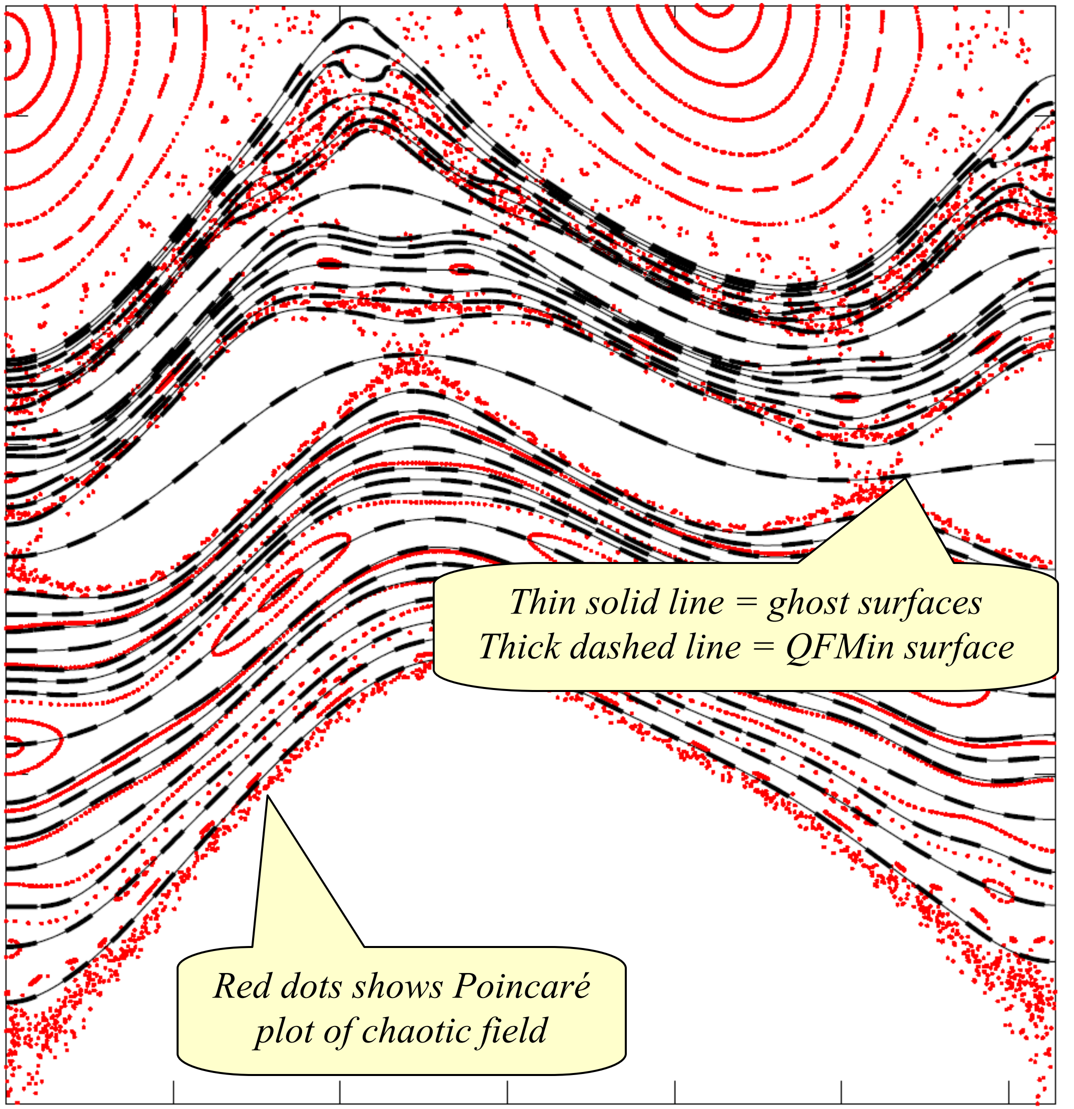
where N is normal to the surface.

- A constrained variational principle for rational pseudo-orbits was found

$$S = \int_C \mathbf{A} \cdot \mathbf{dl} - \nu \left(\int \theta d\zeta - a \right)$$

**constraint of fixed "area" a;*
**ν is Lagrange multiplier ;*
**numerically is much faster ;*

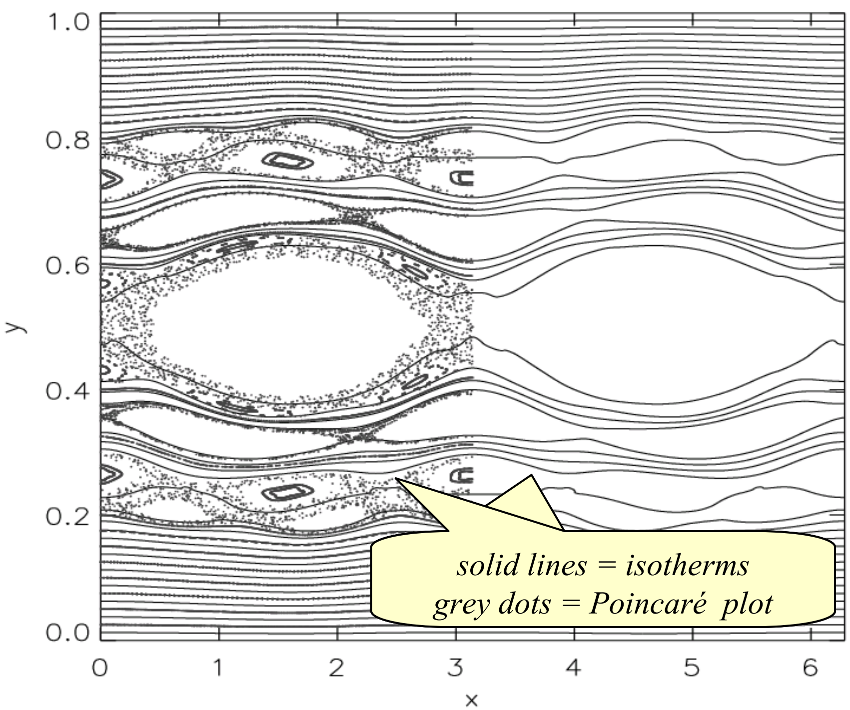
- Numerical evidence suggests ghost-surfaces and QFMin surfaces are the same; confirmed to1st-order;



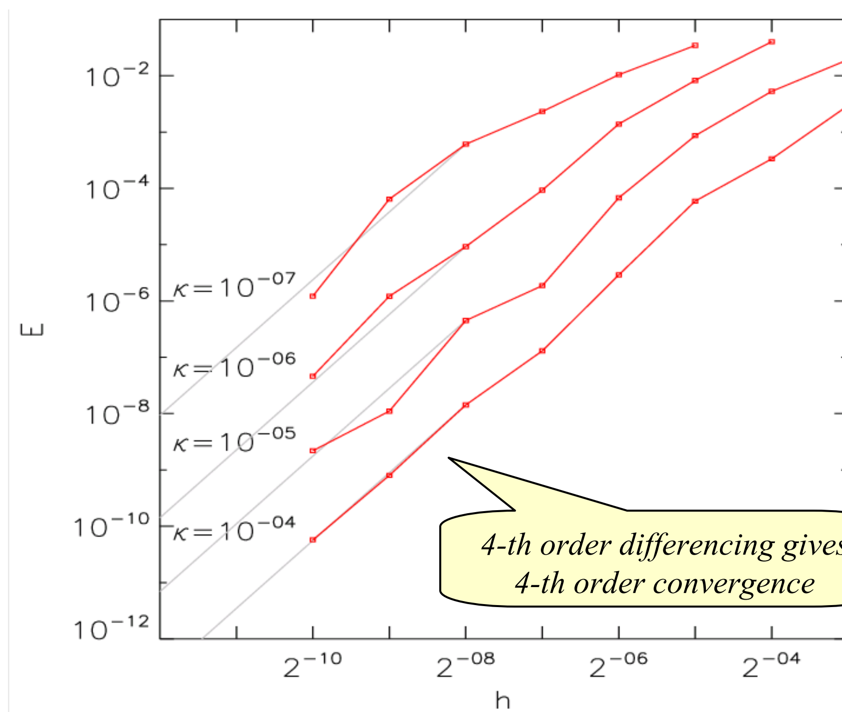
Numerical method for solving anisotropic heat transport exploits field-line coordinates

- heat flux $\nabla \cdot \mathbf{q} = 0$, where $\mathbf{q} = \kappa_{\|}\mathbf{b}\ \mathbf{b} \cdot \nabla T + \kappa_{\perp} \nabla T$; strongly anisotropic ;
- parallel relaxation: use field-alligned coordinates $\mathbf{B} = \nabla \alpha \times \nabla \beta$, so $\nabla_{\|}^2 T = B^{\phi} \dfrac{\partial}{\partial \phi}\left(\dfrac{B^{\phi}}{B^2} \dfrac{\partial T}{\partial \phi} \right)$
- perpendicular relaxation: approximated by $\nabla_{\perp}{}^2 T = \partial_{xx}^2 T + \partial_{yy}^2 T$
- solve sparse linear system iteratively on numerical grid, resolution $= 2^{12} \times 2^{12}$

Poincaré plot

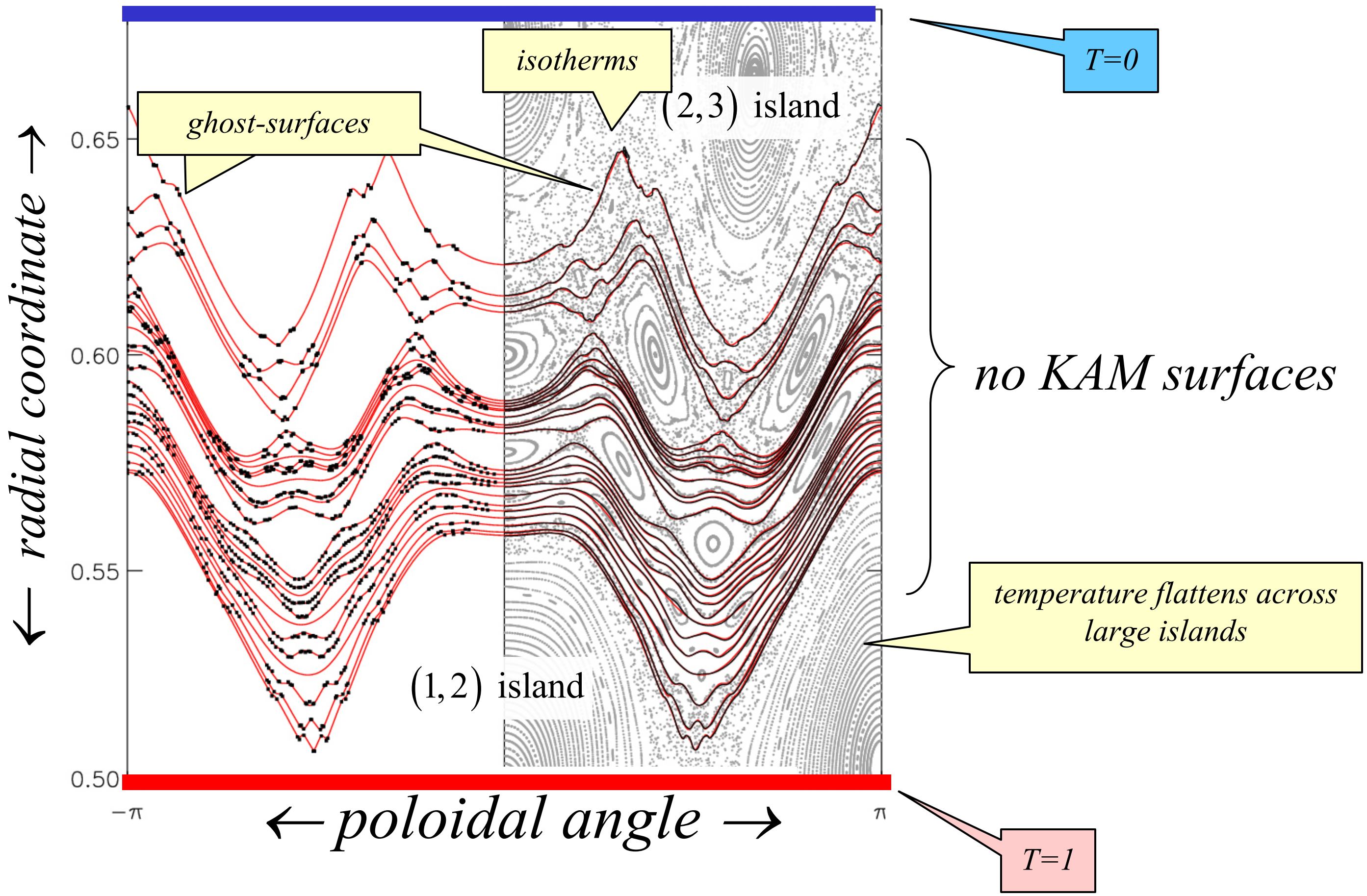


Error vs grid resolution



Steady state temperature is solved numerically; isotherms coincide with ghost-surfaces.

→ *ghost-surface for high-order periodic orbits "fill in the gaps" in the irrational cantori;*
→ *ghost-surfaces and isotherms are almost indistinguishable; suggests T=T(s);*



Chaotic-coordinates simplifies temperature profile

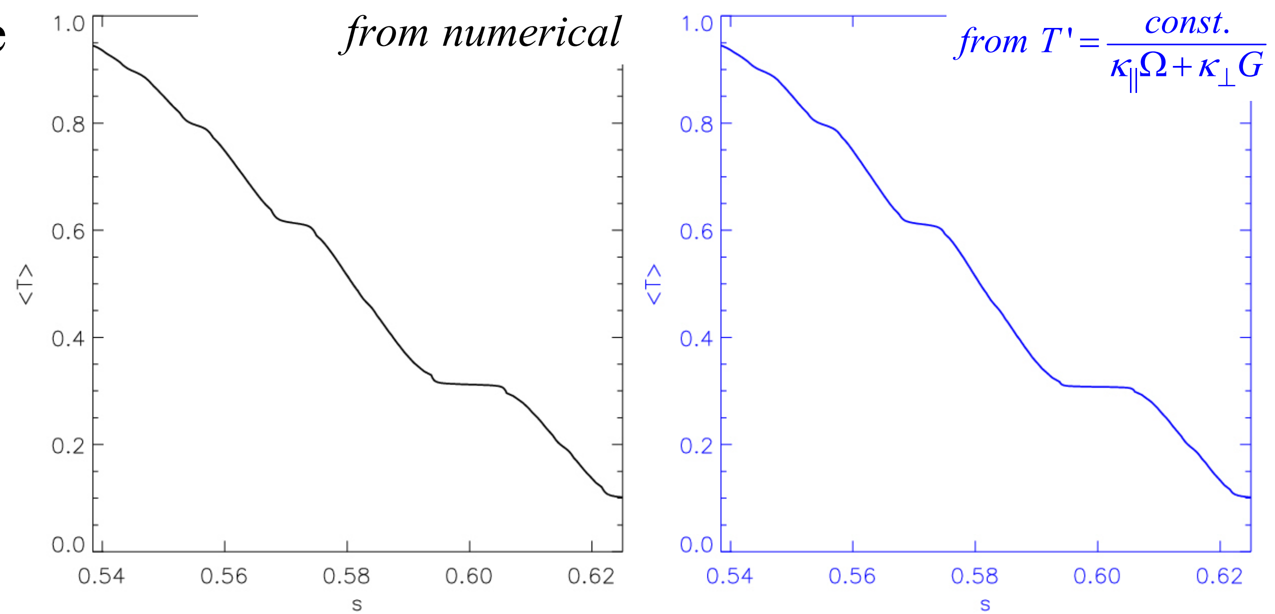
→ ghost-surfaces can be used as radial coordinate surfaces → chaotic-coordinates (s,θ,ϕ)

- From $0 = \dfrac{\partial}{\partial s}\int_V \nabla\cdot\mathbf{q}\, dV = \dfrac{\partial}{\partial s}\int_{\partial V}\mathbf{q}\cdot\mathbf{n}\, d\sigma$ assume $T = T(s)$ to derive $T' = \dfrac{const.}{\kappa_{\parallel}\Omega + \kappa_{\perp} G}$

  for quadratic-flux $\Omega = \int d\sigma g^{ss} (B_n / B)^2$, and metric $G = \int d\sigma g^{ss}$, where $g^{ss} = \nabla s\cdot\nabla s$, $B_n = \mathbf{B}\cdot\nabla s / |\nabla s|$
- in the "ideal limit" $\kappa_{\perp} \to 0$, $T' \to \infty$ on irrational KAM surfaces where $\Omega = 0$;
- non-zero $\kappa_{\perp}$ ensures $T(s)$ is smooth, T' peaks on minimal-Ω surfaces (noble cantori).

Temperature Profile

$(\kappa_{\perp} / \kappa_{\parallel} = 10^{-10})$



Summary

→ in chaotic fields, anisotropic heat transport is restricted by irrational field-lines ≡ cantori

→ ghost-surfaces are closely related to quadratic-flux minimizing surfaces, and a simple numerical construction has been introduced;

→ interpolating a suitable selection of ghost-surfaces allows chaotic-magnetic-coordinates to be constructed

→ the temperature takes the form $T=T(s)$, where s labels the chaotic coordinate surfaces, and an expression for the temperature gradient is derived.

Future Work

→ For a practical implementation of this theory, eg. in MHD codes, the following points must be addressed:

→ *what is the best selection of rational p/q ghost-surfaces for a given chaotic field, and*

→ *how does the best selection of ghost-surfaces depend on* $\kappa_\perp$ *?*

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