MHD Simulation Perturbations

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1 The Magnetic Field and Currents

We begin with the magnetic vector potential $\mathbf{A}$, where

$$\mathbf{B} = \nabla \times \mathbf{A}. \tag{1}$$

Since Faraday's law tells us that

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E}, \tag{2}$$

it follows that

$$\frac{\partial \mathbf{A}}{\partial t} = -\mathbf{E} + \nabla \Phi \tag{3}$$

for any scalar field Φ. To specify $\mathbf{A}$, it is convenient to choose the gauge

$$\nabla_\perp \cdot \mathbf{A} \ = \ 0 \tag{4}$$

When we apply the $\nabla_\perp \cdot$ operator to (3), we then find that

$$\nabla_\perp^2 \Phi \ = \ \nabla_\perp \cdot \mathbf{E}. \tag{5}$$

It is evident that the gauge (4) places no restrictions on the toroidal component of $\mathbf{A}$, A_φ; the most general form of $\mathbf{A}$ satisfying (4) is therefore

$$\mathbf{A} \ = \ \nabla_\perp f \times \hat{\varphi} \ + \ \psi \nabla \varphi \tag{6}$$

where f and ψ are general scalar fields.

2 The Velocity Components

We expect the strong toroidal field to keep poloidal flows nearly incompressible. Perfect conservation of toroidal flux across a poloidal plane could be expressed as

$$\frac{\partial B_\varphi}{\partial t} + \nabla_\perp \cdot (B_\varphi \mathbf{v}_\perp) = 0. \tag{7}$$

For $\epsilon \ll 1$, the toroidal flux is dominated by the unchanging $R_0 B_{\varphi_0}/R$ component; thus to zeroth order in ϵ,

$$\nabla_\perp \cdot \frac{\mathbf{v}_\perp}{R} = 0.$$

In analogy with the derivation of the expression for the vector potential (6) arising from the gauge condition (4), we could then write

$$\frac{\mathbf{v}}{R} = \epsilon \nabla_\perp U \times \hat{\varphi} \; + \; v_\varphi \nabla \varphi$$

for some scalar functions U and v_φ; however to allow, more generally for *some* compressibility, we add an extra, compressible term to get

$$\mathbf{v} = R^2 \epsilon \nabla_\perp U \times \nabla \varphi \; + \; \nabla_\perp \chi \; + \; v_\varphi \hat{\varphi} \tag{8}$$

The first scalar velocity component to investigate is the stream function U. We can extract it by operating on $\mathbf{v}$ with $-\nabla\varphi \cdot \nabla\times$:

$$-\frac{1}{\epsilon} \nabla \varphi \cdot (\nabla \times \mathbf{v}) = \Delta^\dagger U.$$

3 Perturbation

The perturbation is prescribed through the poloidal flux function, ψ and velocity stream function U through

$$\delta W = pm \star \psi \star cos(n(\zeta - \theta))$$

where
$W \equiv \Delta^\dagger U$ the poloidal vorticity field,
pm the perturbation magnitude ($\sim 10^{-3}$),
n is the toroidal mode number,
θ the poloidal angle,
and ζ the toroidal angle.
There is no need to add the poloidal mode number m because the toroidicity of the equilibrium couples all the poloidal modes together; the n eigenmode will most likely consist of a spectrum of poloidal mode numbers m. However, if you know you're looking for a mode with primary m significantly different from n, you might be able to achieve faster convergence if you change the expression for perturbed vorticity to

$$\delta W = pm \star \psi \star cos(n * \zeta - m * \theta)$$

Then we solve the elliptic equation

$$\Delta^\dagger U = W$$

for the poloidal stream function U, with Dirichlet boundary condition $U_{bdy} = 0$. Some people may use random numbers as their perturbations.