Princeton Plasma Physics Laboratory, Princeton, NJ 08543

Resistive MHD in toroidal geometry

Xianzhu Tang

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Abstract

A set of scalar representations of resistive MHD in a toroidal geometry

unknown

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E}; \tag{1}$$

$$\frac{\partial \mathbf{A}}{\partial t} = -\mathbf{E} + \nabla \Phi; \tag{2}$$

$$\mathbf{E} = -\mathbf{v} \times \mathbf{B} + \eta \mathbf{J} \quad \text{with} \quad \eta \equiv \eta^{emu} \frac{1}{v_0 a}; \tag{3}$$

$$\frac{d\mathbf{v}}{dt} = \frac{R^2}{d} \mathbf{J} \times \mathbf{B} - \frac{\epsilon R^2}{d} \nabla p + \mu \frac{R^2}{d} \nabla^2 \mathbf{v} \quad \text{with} \quad \mu \equiv \mu^{emu} \frac{v_0}{a B_0^2} \tag{4}$$

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho \mathbf{v}); \tag{5}$$

$$\frac{dp}{dt} = -\gamma p \nabla \cdot \mathbf{v} + \rho \nabla \cdot \kappa \cdot \nabla \frac{p}{\rho} \quad \text{with} \quad \kappa \equiv \kappa^{emu} \frac{1}{a v_0}. \tag{6}$$

$$\mathbf{v} = R^2 \epsilon \nabla U \times \nabla \varphi + \nabla_\perp \chi + v_\varphi \hat{\varphi}$$

$$\mathbf{B} = \nabla \psi \times \nabla \varphi + \frac{1}{R} \nabla_\perp F + \frac{R_0}{R} I \hat{\varphi}$$

$$F \equiv \frac{\partial f}{\partial \varphi},$$

$$\nabla_\perp^2 F = -\frac{a}{R} \tilde{I}'.$$

$$I = \frac{R}{R_0} B_\varphi = 1 + \epsilon \tilde{I}.$$

$$\mathbf{J} = J_\varphi \hat{\varphi} + \nabla \tilde{I} \times \nabla \varphi - \frac{1}{R} \nabla_\perp F' \times \nabla \varphi + \frac{1}{R^2} \nabla_\perp \psi'$$

$$RJ_\varphi \equiv -C = -\{\Delta^*\psi + \frac{1}{R}\frac{\partial F}{\partial z}\}.$$

$$\begin{aligned}
\frac{\partial \tilde{I}}{\partial t} &= -\epsilon R\hat{\varphi}\cdot\nabla_\perp\tilde{I}\times\nabla_\perp U - \tilde{I}\Delta^*\chi - \nabla_\perp\tilde{I}\cdot\nabla_\perp\chi \\
&\quad +R\hat{\varphi}\cdot[\nabla(\frac{v_\varphi}{R})\times\nabla_\perp\psi] - v_\varphi\frac{a}{R}\frac{\partial\tilde{I}}{\partial\varphi} + R\nabla_\perp F\cdot\nabla_\perp(\frac{v_\varphi}{R}) \\
&\quad +\nabla_\perp\eta\cdot\nabla_\perp\tilde{I} - \frac{1}{R}\nabla_\perp\eta\cdot\nabla_\perp F' - \frac{1}{R}(\nabla_\perp\eta\times\nabla_\perp\psi')\cdot\hat{\varphi} \\
&\quad +\eta[\Delta^*\tilde{I} - \frac{1}{R}\nabla_\perp^2 F' + \frac{2}{R^2}(\frac{\partial F'}{\partial R} + \frac{\partial\psi'}{\partial Z})] \qquad (7) \\
\nabla_\perp^2 f &= -\tilde{I}/R \qquad (8) \\
\frac{\partial\psi}{\partial t} &= -RE_\varphi + \frac{\partial\Phi}{\partial\varphi} \\
&= \epsilon R(\nabla U\times\nabla\psi)_\varphi + \epsilon R(\nabla_\perp F\cdot\nabla U) - \nabla_\perp\chi\cdot\nabla\psi \\
&\quad +(\nabla_\perp\chi\times\nabla_\perp F)_\varphi - \eta RJ_\varphi + \frac{\partial\Phi}{\partial\varphi} \qquad (9) \\
\nabla_\perp^2\Phi &= \epsilon\nabla_\perp\tilde{I}\cdot\nabla_\perp U + \epsilon\tilde{I}\nabla_\perp^2 U + \nabla_\perp\chi\times\nabla_\perp(\frac{\tilde{I}}{R})\cdot\hat{\varphi} \\
&\quad -\nabla_\perp(\frac{v_\varphi}{R})\cdot\nabla_\perp\psi - \frac{v_\varphi}{R}\nabla_\perp^2\psi - \nabla_\perp F\times\nabla_\perp(\frac{v_\varphi}{R})\cdot\hat{\varphi} \\
&\quad -\eta\frac{1}{R^2}[\frac{\partial\tilde{I}}{\partial z} - \frac{1}{R}(\frac{\partial F'}{\partial Z} - \frac{\partial\psi'}{\partial R}) + \frac{\partial}{\partial\varphi}(RJ_\varphi)] \\
&\quad +\frac{1}{R}(\nabla_\perp\eta\times\nabla_\perp\tilde{I})\cdot\hat{\varphi} - \frac{1}{R^2}(\nabla_\perp\eta\times\nabla_\perp F')\cdot\hat{\varphi} + \frac{1}{R^2}\nabla_\perp\eta\cdot\nabla_\perp\psi' \qquad (10) \\
\frac{\partial}{\partial t}\Delta^\dagger U &= \epsilon R[\nabla_\perp U\times\nabla_\perp(\Delta^\dagger U)]\cdot\hat{\varphi} - \nabla_\perp\chi\cdot\nabla_\perp(\Delta^\dagger U) - \frac{v_\varphi}{R}\frac{\partial}{\partial\varphi}\Delta^\dagger U \\
&\quad -2\epsilon\Delta^\dagger U\frac{\partial U}{\partial Z} - \Delta^\dagger\chi\Delta^\dagger U \\
&\quad -\nabla_\perp(\frac{v_\varphi}{R})\cdot\nabla_\perp(\frac{\partial U}{\partial\varphi}) + \frac{2R_0}{R^2}v_\varphi\frac{\partial v_\varphi}{\partial Z} \\
&\quad +R_0[\nabla_\perp(\frac{v_\varphi}{R})\times\nabla_\perp(\frac{\partial\chi}{\partial\varphi})]\cdot\nabla\varphi \\
&\quad -R_0\mathbf{B}\cdot\nabla(\frac{RJ_\varphi}{d}) + R_0\mathbf{J}\cdot\nabla(\frac{R_0 I}{d}) \\
&\quad +\frac{2}{d}\frac{\partial p}{\partial Z} - \frac{R}{d^2}(\nabla_\perp d\times\nabla_\perp p)\cdot\hat{\varphi}. \\
&\quad -R_0\nabla\varphi\cdot\nabla\times[\mu\frac{R^2}{d}\nabla^2\mathbf{v}] \qquad (11) \\
\frac{\partial v_\varphi}{\partial t} &= -R\epsilon(\nabla_\perp v_\varphi\times\nabla_\perp U)\cdot\hat{\varphi} - \nabla_\perp\chi\cdot\nabla_\perp v_\varphi - \frac{v_\varphi}{R}\frac{\partial v_\varphi}{\partial\varphi} \\
&\quad +\frac{1}{d}(\nabla_\perp\tilde{I}\times\nabla_\perp\psi)\cdot\hat{\varphi} + \frac{1}{d}(\nabla_\perp\tilde{I}\cdot\nabla_\perp F) - \frac{1}{Rd}(\nabla_\perp F'\times\nabla_\perp\psi)\cdot\hat{\varphi}
\end{aligned}$$

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\begin{aligned}
& -\frac{1}{Rd}\nabla_\perp F' \cdot \nabla F - \frac{1}{Rd}\nabla_\perp \psi' \cdot \nabla_\perp \psi + \frac{1}{Rd}\hat{\varphi} \cdot (\nabla_\perp \psi' \times \nabla_\perp F) \\
& -\epsilon v_\varphi \frac{\partial U}{\partial Z} - \frac{v_\varphi}{R}\frac{\partial \chi}{\partial R} \\
& -\frac{\epsilon R}{d}\frac{\partial p}{\partial \varphi} \\
& +\hat{\varphi} \cdot [\mu \frac{R^2}{d}\nabla^2 \mathbf{v}] \qquad (12)
\end{aligned}
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\begin{aligned}
\frac{\partial}{\partial t}(\frac{\partial \chi}{\partial R}) &= -\epsilon R \nabla_\perp R \times \nabla_\perp \frac{\partial U}{\partial t} \cdot \hat{\varphi} \\
& +\frac{v_\varphi^2}{R}\nabla_\perp R \cdot \nabla_\perp R - \frac{\partial}{\partial R}(\frac{v_\perp^2}{2}) - \frac{v_\varphi}{R}\nabla_\perp R \cdot \nabla_\perp (\frac{\partial \chi}{\partial \varphi}) \\
& +\epsilon R \Delta^\dagger U[\epsilon R \nabla_\perp R \cdot \nabla_\perp U - (\nabla_\perp R \times \nabla_\perp \chi) \cdot \hat{\varphi}] \\
& -\epsilon v_\varphi [\nabla_\perp R \times \nabla_\perp (\frac{\partial U}{\partial \varphi})] \cdot \hat{\varphi} \\
& +\nabla R \cdot (\frac{R^2}{d}\mathbf{J} \times \mathbf{B}) \\
& -\frac{\epsilon R^2}{d}\frac{\partial p}{\partial R} \\
& +\nabla R \cdot (\mu \frac{R^2}{d}\nabla^2 \mathbf{v}) \qquad (13)
\end{aligned}
$$

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\begin{aligned}
\frac{\partial}{\partial t}(\frac{\partial \chi}{\partial Z}) &= -\epsilon R \nabla_\perp Z \times \nabla_\perp \frac{\partial U}{\partial t} \cdot \hat{\varphi} \\
& +\frac{v_\varphi^2}{R}\nabla_\perp Z \cdot \nabla_\perp R - \frac{\partial}{\partial Z}(\frac{v_\perp^2}{2}) - \frac{v_\varphi}{R}\nabla_\perp Z \cdot \nabla_\perp (\frac{\partial \chi}{\partial \varphi}) \\
& +\epsilon R \Delta^\dagger U[\epsilon R \nabla_\perp Z \cdot \nabla_\perp U - (\nabla_\perp Z \times \nabla_\perp \chi) \cdot \hat{\varphi}] \\
& -\epsilon v_\varphi [\nabla_\perp Z \times \nabla_\perp (\frac{\partial U}{\partial \varphi})] \cdot \hat{\varphi} \\
& +\nabla Z \cdot (\frac{R^2}{d}\mathbf{J} \times \mathbf{B}) \\
& -\frac{\epsilon R^2}{d}\frac{\partial p}{\partial Z} \\
& +\nabla Z \cdot (\mu \frac{R^2}{d}\nabla^2 \mathbf{v}) \qquad (14)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial \rho}{\partial t} &= -\rho[\Delta^\dagger \chi + 2\epsilon \frac{\partial U}{\partial Z} + \frac{1}{R}\frac{\partial v_\varphi}{\partial \varphi}] \\
& +R\epsilon \nabla_\perp U \times \nabla_\perp \rho \cdot \hat{\varphi} - \nabla_\perp \chi \cdot \nabla_\perp \rho - \frac{v_\varphi}{R}\frac{\partial \rho}{\partial \varphi} \qquad (15) \\
\frac{\partial p}{\partial t} &= R\epsilon \nabla_\perp U \times \nabla_\perp p \cdot \hat{\varphi} - \nabla_\perp \chi \cdot \nabla_\perp p - \frac{v_\varphi}{R}\frac{\partial \rho}{\partial \varphi} \\
& -\gamma p[\Delta^\dagger \chi + 2\epsilon \frac{\partial U}{\partial Z} + \frac{1}{R}\frac{\partial v_\varphi}{\partial \varphi}]
\end{aligned}
$$

$$
\begin{aligned}
& +\rho\nabla\cdot\kappa\cdot\nabla\left(\frac{p}{\rho}\right) && (16)\\
c_a &= \nabla^2 a - \frac{a_x}{R} && (17)\\
\frac{dc_a}{dt} &= \frac{R}{R_0}\left\{[\mathbf{u},c_a]+[w,a]+2[\mathbf{u}_x,a_x]+2[\mathbf{u}_y,a_y]\right\}+2\epsilon[\mathbf{u},a_x]+2\frac{\epsilon}{R}\mathbf{u}_y a_x \\
& +\frac{R}{R_0}\left\{\nabla\mathbf{u}\cdot\nabla(\nabla^2 F)+\nabla w\times\nabla F+2\nabla\mathbf{u}_x\cdot\nabla F_x\right\} && (18)
\end{aligned}
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