The 3D M3D-C^1 Equations[1]

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[1] Has R^2 inserted in first two momentum equations and new gauge condition.

0.0 Preliminaries and Definitions

We use a cylindrical coordinate system (R, φ, Z) and define the 2D gradient operator as:

$$\nabla_\perp a \equiv a_R \hat{R} + a_Z \hat{Z} \tag{0.1}$$

Here, and elsewhere subscripts denote partial differentiation. The velocity field and magnetic vector potential are represented in terms of the constant R_0 and the 5 scalar variables (U, χ, v, f, ψ) as:

$$\mathbf{V} = \nabla U \times \nabla \varphi + \nabla_\perp \chi + v \nabla \varphi \tag{0.2}$$

$$\mathbf{A} = R^2 \nabla \varphi \times \nabla f + \psi \nabla \varphi - R_0 \ln R \hat{Z} \tag{0.3}$$

(This is different than what was used in the original M3D). The constant R_0 is proportional to the total current in the toroidal field magnets, I_{TF}, for an axisymmetric system:

$$R_0 = \frac{\mu_0 I_{TF}}{2\pi} \tag{0.4}$$

Also, note that the new gauge condition implied by (0.3) is:

$$R^2 \nabla_\perp \bullet \frac{1}{R^2} \mathbf{A} \equiv R \frac{\partial}{\partial R} \frac{1}{R} \mathbf{A} \bullet \hat{R} + \frac{\partial}{\partial Z} \mathbf{A} \bullet \hat{Z} = 0 \tag{0.5}$$

The magnetic field is the curl of the vector potential, $\mathbf{B} = \nabla \times \mathbf{A}$, thus from (0.3)

$$\begin{aligned} \mathbf{B} &= \nabla \psi \times \nabla \varphi - \nabla_\perp f' + F \nabla \varphi \\ &= \nabla \psi \times \nabla \varphi - \nabla f' + F^* \nabla \varphi \end{aligned} \tag{0.6}$$

Here, for convenience, we have defined the scalars F and F^* in terms of R_0 and f such that

$$\begin{aligned} F &\equiv R_0 + R^2 \nabla \bullet \nabla_\perp f \\ F^* &\equiv R_0 + R^2 \nabla^2 f = F + f'' \end{aligned} \tag{0.7}$$

We define the operators:

$$\Delta^\dagger u \equiv \nabla \cdot \nabla_\perp u = \frac{\partial^2 u}{\partial R^2} + \frac{1}{R} \frac{\partial u}{\partial R} + \frac{\partial^2 u}{\partial Z^2} \tag{0.9}$$

$$\Delta^* \psi \equiv R^2 \nabla \bullet \frac{1}{R^2} \nabla_\perp \psi = \frac{\partial^2 \psi}{\partial R^2} - \frac{1}{R} \frac{\partial \psi}{\partial R} + \frac{\partial^2 \psi}{\partial Z^2} \tag{0.10}$$

We also define the brackets:

$$[a,b]=[\nabla a\times\nabla b\bullet\nabla\varphi]=\frac{1}{R}\left(a_Z b_R-a_R b_Z\right)$$
$$(a,b)=\nabla a\bullet\nabla b=a_R b_R+a_Z b_Z$$

Note the useful identities:

$$R^2[a,b][c,d]=(a,c)(b,d)-(a,d)(b,c)$$
$$[a,b][c,d]=[c,b][a,d]-[c,a][b,d]$$
$$[a,b](c,d)-[a,d](c,b)=[d,b](a,c)$$

Also, letting the perpendicular volume element be $dV_\perp = RdRdZ$, we have, for any scalar quantities *(a,b,c)*:

$$\iint dV_\perp c[a,b]=-\iint dV_\perp a[c,b]=-\iint dV_\perp b[a,c]$$
$$\iint dV_\perp c(a,b)=-\iint dV_\perp a(c,b)-\iint dV_\perp ac\Delta^\dagger b$$
$$=-\iint dV_\perp \frac{a}{R}(cR,b)-\iint dV_\perp ac\nabla_\perp^2 b$$
$$=-\iint dV_\perp \frac{a}{R^2}\left(cR^2,b\right)-\iint dV_\perp ac\Delta^* b$$

Also, for any vector $\mathbf{A}$ and scalar field $\nu_i(R,Z)$:

$$\iint dV_\perp \nu_i\nabla\varphi\bullet\nabla\times\mathbf{A}=-\iint dV_\perp\nabla\varphi\bullet\nabla\nu_i\times\mathbf{A}$$
$$\iint dRdZR\nu_i\nabla_\perp\bullet\mathbf{A}=-\iint dRdZR\nabla_\perp\nu_i\bullet\mathbf{A}$$

1.0 The Momentum Equation.

The implicit form of the momentum equation time advance is as follows:

$$\begin{aligned}\mathbf{M} &\equiv \left\{\rho-(\theta\delta t)^2 L\right\}\mathbf{V}^{n+1}+\delta t\theta\left\{\rho\mathbf{V}^{n+1}\cdot\nabla\mathbf{V}^{n}+\rho\mathbf{V}^{n}\cdot\nabla\mathbf{V}^{n+1}\right\}+\delta t\theta\nabla\cdot\overleftrightarrow{\Pi}_i^{\,n+1}\\ &=\left\{\rho-(\theta\delta t)^2 L\right\}\mathbf{V}^{n}+\delta t\left\{-\nabla p+(\nabla\times\mathbf{B})\times\mathbf{B}+\rho\mathbf{g}\right\}-\delta t(1-\theta)\nabla\cdot\overleftrightarrow{\Pi}_i^{\,n}\\ &\quad-(1-2\theta)\delta t\left\{\rho\mathbf{V}^{n}\cdot\nabla\mathbf{V}^{n}\right\}\end{aligned} \tag{1.1}$$

Here we have defined the ideal MHD (plus gravity) operator:

$$\begin{aligned}L\{\mathbf{V}\} &= \left\{\nabla\times\left[\nabla\times\left(\mathbf{V}\times\mathbf{B}\right)\right]\right\}\times\mathbf{B}+\left(\nabla\times\mathbf{B}\right)\times\left[\nabla\times\left(\mathbf{V}\times\mathbf{B}\right)\right]\\ &\quad+\nabla\left(\mathbf{V}\bullet\nabla p+\gamma p\nabla\bullet\mathbf{V}\right)-\mathbf{g}\nabla\bullet(\rho\mathbf{V})\end{aligned} \tag{1.2}$$

The ion stress tensor Π_i will be taken to have the form

$$\Pi_i=-\mu\left(\nabla\mathbf{v}+\nabla\mathbf{v}^{\dagger}\right)+h\mu\nabla_{\perp}^{2}\left(\nabla\mathbf{v}+\nabla\mathbf{v}^{\dagger}\right)+\Pi_i^{GV}$$

where μ is a constant viscosity, h is a hyper-viscosity multiplier, and the last term is the gyroviscous part of the tensor, which will be ignored for now. Excluding this piece, the divergence of the physical (non-hyper) part of the stress tensor is

$$\nabla\cdot\overleftrightarrow{\Pi}_i=-\mu\nabla^2\mathbf{v}-\left(2\mu_C-\mu\right)\nabla\left(\nabla\cdot\mathbf{v}\right).$$

1.1 First Projection of the Momentum Equation (U):

$$-\iint dRdZR\nu_i R^2 \nabla\varphi \bullet \nabla\times\mathbf{M} = \iint dRdZR\nabla\varphi \bullet \nabla(R^2\nu_i)\times\mathbf{M} = \iint dRdZRI_1$$

$$I_1 \equiv \left\{ -\frac{\rho}{R^2}\left(R^2\nu_i, U^{n+1}\right) + \rho\left[R^2\nu_i, \chi^{n+1}\right] - (\theta\delta t)^2\left[L_{11}\left\{U^{n+1}\right\} + L_{12}\left\{v^{n+1}\right\} + L_{13}\left\{\chi^{n+1}\right\}\right]\right\}$$

$$+\theta\delta t\left\{\begin{array}{l} -\rho\dfrac{\Delta^* U^{n+1}}{R^2}\left[R^2\nu_i, U^n\right] - \dfrac{\left(U^{n+1}, U^n\right)}{2R^2}\left[R^2\nu_i, \rho\right] - \dfrac{\rho\nu^{n+1}}{R^4}\left(R^2\nu_i, U'^n\right) - \rho\dfrac{\Delta^* U^{n+1}}{R^2}\left(R^2\nu_i, \chi^n\right) \\ +\left[U^{n+1}, \chi^n\right]\left[R^2\nu_i, \rho\right] + \dfrac{\rho\nu^{n+1}}{R^2}\left[R^2\nu_i, \chi'^n\right] - \dfrac{\rho\nu^{n+1}\nu^n}{R^2}\dfrac{\partial\nu_i}{\partial z} - \dfrac{1}{2}\left(\chi^{n+1}, \chi^n\right)\left[R^2\nu_i, \rho\right] \\ -\rho\dfrac{\Delta^* U^n}{R^2}\left[R^2\nu_i, U^{n+1}\right] - \dfrac{\left(U^n, U^{n+1}\right)}{2R^2}\left[R^2\nu_i, \rho\right] - \dfrac{\rho\nu^n}{R^4}\left(R^2\nu_i, U'^{n+1}\right) - \rho\dfrac{\Delta^* U^n}{R^2}\left(R^2\nu_i, \chi^{n+1}\right) \\ +\left[U^n, \chi^{n+1}\right]\left[R^2\nu_i, \rho\right] + \dfrac{\rho\nu^n}{R^2}\left[R^2\nu_i, \chi'^{n+1}\right] - \dfrac{\rho\nu^n\nu^{n+1}}{R^2}\dfrac{\partial\nu_i}{\partial z} - \dfrac{1}{2}\left(\chi^n, \chi^{n+1}\right)\left[R^2\nu_i, \rho\right] \end{array}\right\}$$

$$-\mu\theta\delta t\left[\frac{1}{R^2}\left(\Delta^* R^2\nu_i\right)\left(\Delta^* U\right) + \nu_i\frac{\Delta^* U''}{R^2} - 2\frac{\nu_i}{R^2}\left(\frac{1}{R}\frac{\partial U''}{\partial R} + \frac{\partial\chi''}{\partial z} - \frac{\partial\nu'}{\partial z}\right)\right]^{n+1}$$

$$= \left\{ -\frac{\rho}{R^2}\left(R^2\nu_i, U^n\right) + \rho\left[R^2\nu_i, \chi^n\right] - (\theta\delta t)^2\left[L_{11}\left\{U^n\right\} + L_{12}\left\{v^n\right\} + L_{13}\left\{\chi^n\right\}\right]\right\}$$

$$-(1-2\theta)\delta t\left\{\begin{array}{l} -\rho\dfrac{\Delta^* U^n}{R^2}\left[R^2\nu_i, U^n\right] - \dfrac{\left(U^n, U^n\right)}{2R^2}\left[R^2\nu_i, \rho\right] - \dfrac{\rho\nu^n}{R^4}\left(R^2\nu_i, U'^n\right) - \rho\dfrac{\Delta^* U^n}{R^2}\left(R^2\nu_i, \chi^n\right) \\ +\left[U^n, \chi^n\right]\left[R^2\nu_i, \rho\right] + \dfrac{\rho\nu^n}{R^2}\left[R^2\nu_i, \chi'^n\right] - \dfrac{\rho\nu^n\nu^n}{R^2}\dfrac{\partial\nu_i}{\partial z} - \dfrac{1}{2}\left(\chi^n, \chi^n\right)\left[R^2\nu_i, \rho\right] \end{array}\right\}$$

$$+\delta t\left\{\begin{array}{l} -\left(\dfrac{\Delta^*\psi}{R^2}\right)\left(\left[R^2\nu_i, \psi\right] - \left(R^2\nu_i, f'\right)\right) \\ -\dfrac{F}{R^4}\left(R^2\nu_i, \psi'\right) - \dfrac{F}{R^2}\left[R^2\nu_i, f''\right] - \dfrac{F}{R^2}\left[R^2\nu_i, F\right] + R^2\nu_i\nabla\rho\times\vec{g}\bullet\nabla\varphi \\ +\mu(1-\theta)\left[\dfrac{1}{R^2}\left(\Delta^* R^2\nu_i\right)\left(\Delta^* U\right) + \nu_i\dfrac{\Delta^* U''}{R^2} - 2\dfrac{\nu_i}{R^2}\left(\dfrac{1}{R}\dfrac{\partial U''}{\partial R} + \dfrac{\partial\chi''}{\partial z} - \dfrac{\partial\nu'}{\partial z}\right)\right] \end{array}\right\}^n$$

$$
\begin{aligned}
L_{11}\{U\} = &+\frac{1}{R^2}\left([U,\psi],[\hat{v}_i,\psi]\right)-\frac{1}{R^2}\Delta^*\psi\left[\hat{v}_i,[U,\psi]\right]+\frac{F'}{R^4}\left(U,[\hat{v}_i,\psi]\right)+\frac{F}{R^4}\left(U',[\hat{v}_i,\psi]\right)\\
&-\frac{F}{R^4}\left([U,\psi],\hat{v}_i\right)'-\Delta^*\psi\left(\frac{F}{R^4}[\hat{v}_i,U]\right)'+\frac{1}{R^2}\left[\frac{F}{R^2},U\right](\hat{v}_i,\psi')-\frac{1}{R^2}\left([U,\psi],(\hat{v}_i,f')\right)\\
&-\frac{1}{R^2}\left((U,f'),[\hat{v}_i,\psi]\right)+\frac{1}{R^2}\Delta^*\psi\left[\hat{v}_i,(U,f')\right]-\frac{2F}{R^2}\left[U,\frac{F}{R^2}\right]\hat{v}_{iz}-\frac{F}{R^4}\left(\frac{F}{R^2}(U',\hat{v}_i)\right)'\\
&-F\left(\frac{F'}{R^6}(U,\hat{v}_i)\right)'-\frac{F'}{R^4}\left(U,(\hat{v}_i,f')\right)-\frac{F}{R^4}\left(U',(\hat{v}_i,f')\right)+\frac{F}{R^4}\left((U,f'),\hat{v}_i\right)'\\
&+[\hat{v}_i,f'']\left[\frac{F}{R^2},U\right]+\frac{1}{R^2}\left((U,f'),(\hat{v}_i,f')\right)
\end{aligned}
$$

$$
\begin{aligned}
L_{12}\{v\} = &-\frac{v'}{R^4}\left(\psi,[\hat{v}_i,\psi]\right)+\frac{v'}{R^4}\Delta^*\psi\left[\hat{v}_i,\psi\right]-\frac{v}{R^4}\left(\psi',[\hat{v}_i,\psi]\right)+\frac{v}{R^4}\Delta^*\psi\left[\hat{v}_i,\psi'\right]-\frac{1}{R^2}\left[\frac{v}{R^2},\psi\right](\hat{v}_i,\psi')\\
&-\frac{2F}{R^2}\left[\frac{v}{R^2},\psi\right]\hat{v}_{iz}+F\left(\frac{v}{R^6}(\psi',\hat{v}_i)\right)'+F\left(\frac{v'}{R^6}(\psi,\hat{v}_i)\right)'-\frac{v'}{R^2}\left[[\hat{v}_i,\psi],f'\right]-\frac{v}{R^2}\left[[\hat{v}_i,\psi],f''\right]\\
&-\left[\frac{v}{R^2},\psi\right][\hat{v}_i,f'']+\frac{1}{R^2}\nabla\bullet\frac{v}{R^2}\nabla_\perp f'(\hat{v}_i,\psi')+\frac{v'}{R^4}\left(\psi,(\hat{v}_i,f')\right)+\frac{v}{R^4}\left(\psi',(\hat{v}_i,f')\right)\\
&-\Delta^*\psi\left(\frac{v}{R^4}(f',\hat{v}_i)\right)'+\frac{2F}{R^2}\hat{v}_{iZ}\nabla\bullet\frac{v}{R^2}\nabla_\perp f'-F\left(\frac{v}{R^4}[f',\hat{v}_i]\right)''\\
&+\frac{v'}{R^2}\left[(\hat{v}_i,f'),f'\right]+\frac{v}{R^2}\left[(\hat{v}_i,f'),f''\right]+\nabla\bullet\frac{v}{R^2}\nabla_\perp f'[\hat{v}_i,f'']
\end{aligned}
$$

$$
\begin{aligned}
L_{13}\{\chi\} = &-\frac{1}{R^2}\left((\chi,\psi),[\hat{v}_i,\psi]\right)+\frac{1}{R^2}\Delta^*\psi\left[\hat{v}_i,(\chi,\psi)\right]-\frac{F'}{R^2}\left[[\hat{v}_i,\psi],\chi\right]-\frac{F}{R^2}\left[[\hat{v}_i,\psi],\chi'\right]\\
&+\frac{F}{R^4}\left((\chi,\psi),\hat{v}_i\right)'+\frac{1}{R^2}(\hat{v}_i,\psi')\nabla\bullet\frac{F}{R^2}\nabla_\perp\chi-\frac{1}{R^2}\Delta^*\psi\left(\frac{F}{R^2}(\chi,\hat{v}_i)\right)'+\frac{1}{R^2}\left((\chi,\psi),(\hat{v}_i,f')\right)\\
&+\frac{1}{R^2}\left(R^2[f',\chi],[\hat{v}_i,\psi]\right)-\frac{1}{R^2}\Delta^*\psi\left[\hat{v}_i,R^2[f',\chi]\right]+\frac{2F}{R^2}\hat{v}_{iZ}\nabla\bullet\frac{F}{R^2}\nabla_\perp\chi\\
&-\frac{F}{R^4}\left(F[\chi,\hat{v}_i]\right)''+\frac{F'}{R^2}\left[(\hat{v}_i,f'),\chi\right]+\frac{F}{R^2}\left[(\hat{v}_i,f'),\chi'\right]-\frac{F}{R^4}\left(R^2[f',\chi],\hat{v}_i\right)'\\
&+\nabla\bullet\frac{F}{R^2}\nabla_\perp\chi[\hat{v}_i,f'']-\frac{1}{R^2}\left(R^2[f',\chi],(\hat{v}_i,f')\right)
\end{aligned}
$$

.NOTE: used identity: $R^2\left[\hat{v}_i,\frac{F}{R^2}\right]-[\hat{v}_i,F]=-\frac{2F}{R^2}\hat{v}_{iz}$

1.2 Second Projection of the Momentum Equation (ν):

$$\iint dRdZR\nu_i R^2 \nabla\varphi \bullet \mathbf{M} = \iint dRdZR\left\{I_2\right\}$$

$$\begin{aligned}
I_2 &= \left\{\rho\nu_i\nu^{n+1} - (\theta\delta t)^2\left[L_{21}\left\{U^{n+1}\right\} + L_{22}\left\{\nu^{n+1}\right\} + L_{23}\left\{\chi^{n+1}\right\}\right]\right\} \\
&\quad + \theta\delta t\rho\nu_i\left\{\left[\nu^{n+1}, U^n\right] + \left(\nu^{n+1}, \chi^n\right) + \frac{\nu^{n+1}\nu'^n}{R^2}\left[\nu^n, U^{n+1}\right] + \left(\nu^n, \chi^{n+1}\right) + \frac{\nu^n\nu'^{n+1}}{R^2}\right\} \\
&\quad - \delta t\theta\left[\nu_i\mu\Delta^*\nu + \frac{2\mu_C\nu_i}{R^2}\nu'' - \left(2\mu_C - \mu\right)\left(\nu_i, \chi'\right) + \frac{2\mu\nu_i}{R}\left(\frac{\partial\chi'}{\partial R} - \frac{1}{R}\frac{\partial U'}{\partial z}\right)\right]^{n+1} \\
&= \left\{\rho\nu_i\nu^n - (\theta\delta t)^2\left[L_{21}\left\{U^n\right\} + L_{22}\left\{\nu^n\right\} + L_{23}\left\{\chi^n\right\}\right]\right\} \\
&\quad + \delta t\left\{\begin{array}{l}
-(1-2\theta)\rho\nu_i\left\{\left[\nu, U\right] + \left(\nu, \chi\right) + \frac{\nu\nu'}{R^2}\right\} - \frac{\nu_i}{R^2}\left(\psi, \psi'\right) + \nu_i\left[f'', \psi\right] + \nu_i\left[F, \psi\right] \\
+\nu_i\left[f', \psi'\right] - \nu_i\left(f', f''\right) - \nu_i\left(f', F\right) - \nu_i p' \\
+(1-\theta)\left[\nu_i\mu\Delta^*\nu + \frac{2\mu_C\nu_i}{R^2}\nu'' - \left(2\mu_C - \mu\right)\left(\nu_i, \chi'\right) + \frac{2\mu\nu_i}{R}\left(\frac{\partial\chi'}{\partial R} - \frac{1}{R}\frac{\partial U'}{\partial z}\right)\right]
\end{array}\right\}^n
\end{aligned}$$

$$L_{21}\{U\} = -\frac{\nu_i}{R^2}\left([U,\psi],\psi\right)' - \left(\nu_i \frac{F'}{R^4}(U,\psi)\right)' - \left(\nu_i \frac{F}{R^4}(U',\psi)\right)' - [U,\psi][F,\nu_i]$$

$$+R^2[\nu_i,\psi]\left[\frac{F}{R^2},U\right] + \nu_i\left[f',[U,\psi]\right]' + \frac{\nu_i}{R^2}\left((U,f'),\psi\right)' + \nu_i \frac{F'}{R^2}[F,U] + \nu_i \frac{F}{R^2}[F,U']$$

$$+\nu_i\left(\frac{F'}{R^2}[f',U] + \frac{F}{R^2}[f',U']\right)' - R^2\nabla\bullet\left(\nu_i\nabla_\perp f'\right)\left[\frac{F}{R^2},U\right] - \nu_i\left[F,(U,f')\right]$$

$$-\nu_i\left[f',(U,f')\right]' - \nu_i[U,p]'$$

$$L_{22}\{v\} = -R^2[\nu_i,\psi]\left[\frac{v}{R^2},\psi\right] + \left(\nu_i \frac{v'}{R^4}(\psi,\psi)\right)' + \left(\nu_i \frac{v}{R^4}(\psi',\psi)\right)' - \nu_i \frac{v'}{R^2}[F,\psi]$$

$$-\nu_i \frac{v}{R^2}[F,\psi'] + R^2\nabla\bullet\left(\nu_i\nabla_\perp f'\right)\left[\frac{v}{R^2},\psi\right] + R^2[\nu_i,\psi]\nabla\bullet\frac{v}{R^2}\nabla_\perp f' - \frac{\nu_i}{R^2}\left(v'[f',\psi] + v[f'',\psi]\right)'$$

$$-\frac{\nu_i}{R^2}\left(v'[f',\psi] + v[f',\psi']\right)' + \nu_i \frac{v'}{R^2}(F,f') + \nu_i \frac{v}{R^2}(F,f'') + \frac{\nu_i}{R^2}\left(v'(f',f') + v(f',f'')\right)'$$

$$-\nabla\bullet\frac{v}{R^2}\nabla_\perp f'\left[R^2\nabla\bullet\left(\nu_i\nabla_\perp f'\right)\right] + \frac{\nu_i}{R^2}(vp')' + \gamma\frac{\nu_i}{R^2}(pv')'$$

$$L_{23}\{\chi\} = +\frac{\nu_i}{R^2}\left((\chi,\psi),\psi\right)' - \frac{F}{R^2}\left(R^2[\nu_i,\psi],\chi\right) + (\chi,\psi)[F,\nu_i] - \frac{\nu_i}{R^2}\left(F'[\chi,\psi]\right)'$$

$$-\frac{\nu_i}{R^2}\left(F[\chi',\psi]\right)' - \frac{\nu_i}{R^2}\left(R^2[f',\chi],\psi\right)' - \nu_i\left[f',(\chi,\psi)\right]' + \nu_i \frac{F'}{R^2}(F,\chi) + \nu_i \frac{F}{R^2}(F,\chi')$$

$$+\left(\nu_i \frac{F'}{R^2}(f',\chi) + \nu_i \frac{F}{R^2}(f',\chi')\right)' - R^2\nabla\bullet\left(\nu_i\nabla_\perp f'\right)\nabla\bullet\frac{F}{R^2}\nabla_\perp\chi - R^2[f',\chi][F,\nu_i]$$

$$-R^2[f',\chi][f',\nu_i]' + \left(\nu_i(\chi,p) + \nu_i\gamma p\nabla_\perp^2\chi\right)'$$

.

1.3 Third Projection of the Momentum Equation (χ):

$$\iint dRdZR\nu_i \nabla_\perp \bullet \mathbf{M} = -\iint dRdZR\nabla_\perp \nu_i \bullet \mathbf{M} = \int RdRdZI_3$$

$$\begin{aligned}
I_3 &= -\rho\left(\nu_i, \chi^{n+1}\right) + \nu_i\left[\rho, U^{n+1}\right] - (\theta\delta t)^2\left[L_{31}\left\{U^{n+1}\right\} + L_{32}\left\{v^{n+1}\right\} + L_{33}\left\{\chi^{n+1}\right\}\right] \\
&+\theta\delta t\rho\left[\begin{array}{l}
\frac{1}{R^2}\Delta^* U^{n+1}\left(U^n, \nu_i\right) - \frac{1}{2}\left(\nu_i, \frac{1}{R^2}\left(U^{n+1}, U^n\right)\right) - \frac{v^{n+1}}{R^2}\left[\nu_i, U'^n\right] - \Delta^* U^{n+1}\left[\nu_i, \chi^n\right] \\
-\frac{v^{n+1}}{R^2}\left(\chi'^n, \nu_i\right) + \frac{v^{n+1}v^n}{R^3}\frac{\partial \nu_i}{\partial R} - \left(\nu_i, \left[\chi^{n+1}, U^n\right]\right) - \frac{1}{2}\left(\nu_i, \left(\chi^{n+1}, \chi^n\right)\right) \\
\frac{1}{R^2}\Delta^* U^n\left(U^{n+1}, \nu_i\right) - \frac{1}{2}\left(\nu_i, \frac{1}{R^2}\left(U^n, U^{n+1}\right)\right) - \frac{v^n}{R^2}\left[\nu_i, U'^{n+1}\right] - \Delta^* U^n\left[\nu_i, \chi^{n+1}\right] \\
-\frac{v^n}{R^2}\left(\chi'^{n+1}, \nu_i\right) + \frac{v^n v^{n+1}}{R^3}\frac{\partial \nu_i}{\partial R} - \frac{1}{2}\left(\nu_i, \left(\chi^n, \chi^{n+1}\right)\right) - \left(\nu_i, \left[\chi^n, U^{n+1}\right]\right)
\end{array}\right] \\
&-\delta t\theta 2\mu_C \Delta^\dagger \nu_i \Delta^\dagger \chi^{n+1} \\
&= -\rho\left(\nu_i, \chi^{n+1}\right) + \nu_i\left[\rho, U^n\right] - (\theta\delta t)^2\left[L_{31}\left\{U^n\right\} + L_{32}\left\{v^n\right\} + L_{33}\left\{\chi^n\right\}\right] \\
&+\delta t\left\{\begin{array}{l}
-(1-2\theta)\rho\left[\begin{array}{l}
\frac{1}{R^2}\Delta^* U\left(U, \nu_i\right) - \frac{1}{2}\left(\nu_i, \frac{1}{R^2}\left(U, U\right)\right) - \frac{v}{R^2}\left[\nu_i, U'\right] - \Delta^* U\left[\nu_i, \chi\right] \\
-\frac{v}{R^2}\left(\chi', \nu_i\right) + \frac{v^2}{R^3}\frac{\partial \nu_i}{\partial R} - \frac{1}{2}\left(\nu_i, \left(\chi, \chi\right)\right) - \left(\nu_i, \left[\chi, U\right]\right)
\end{array}\right] \\
\Delta^*\psi\left[\frac{1}{R^2}\left(\psi, \nu_i\right) + \left[\nu_i, f'\right]\right] + \frac{F}{R^2}\left[\left(\nu_i, F\right) + \left(\nu_i, f''\right) - \left[\nu_i, \psi'\right]\right] \\
+\left(\nu_i, p\right) + \nu_i \vec{g} \bullet \nabla\rho + (1-\theta)2\mu_C \Delta^\dagger \nu_i \Delta^\dagger \chi
\end{array}\right\}^n
\end{aligned}$$

$$L_{31} = -\frac{1}{R^2}\left([U,\psi],(v_i,\psi)\right) + \frac{1}{R^2}\Delta^*\psi\left([U,\psi],v_i\right) + \frac{1}{R^2}\left((U,f'),(v_i,\psi)\right)$$

$$-\frac{1}{R^2}\left([U,\psi],R^2[v_i,f']\right) - \frac{1}{R^2}\Delta^*\psi\left((U,f'),v_i\right) - \frac{F}{R}\left[v_i,[U,\psi]\right]'$$

$$+[v_i,\psi']\left[\frac{F}{R^2},U\right] - \frac{F'}{R^4}\left(U,(v_i,\psi)\right) - \frac{F}{R^4}\left(U',(v_i,\psi)\right) + \Delta^*\psi\left(\frac{F}{R^4}(U,v_i)\right)' + \frac{F}{R}\left[v_i,(U,f')\right]'$$

$$-\left[\frac{F}{R^2},U\right](v_i,f'') - \frac{F'}{R^4}\left(U,R^2[v_i,f']\right) - \frac{F}{R^4}\left(U',R^2[v_i,f']\right) - F\left(\frac{F}{R^3}[v_i,U]\right)''$$

$$+F\Delta^* v_i\left[\frac{F}{R^2},U\right] + \frac{1}{R^2}\left((U,f'),R^2[v_i,f']\right) + \nabla_\perp^2 v_i[p,U]$$

$$L_{32} = +\frac{v'}{R^4}\left(\psi,(v_i,\psi)\right) + \frac{v}{R^4}\left(\psi',(v_i,\psi)\right) - \Delta^*\psi\left(\frac{v}{R^4}(\psi,v_i)\right)' - [v_i,\psi']\left[\frac{v}{R^2},\psi\right] + (v_i,f'')\left[\frac{v}{R^2},\psi\right]$$

$$+\frac{v'}{R^4}\left(\psi,R^2[v_i,f']\right) + \frac{v}{R^4}\left(\psi',R^2[v_i,f']\right) + \Delta^*\psi\left(\frac{v}{R^2}[f',v_i]\right)' - \frac{v'}{R^2}\left[f',(v_i,\psi)\right]$$

$$-\frac{v}{R^2}\left[f'',(v_i,\psi)\right] + [v_i,\psi']\nabla\bullet\frac{v}{R^2}\nabla_\perp f' + F\left(\frac{v}{R^3}[v_i,\psi]\right)'' - F\Delta^* v_i\left[\frac{v}{R^2},\psi\right] - F\left(\frac{v}{R^3}(f',v_i)\right)''$$

$$+\left[F\Delta^* v_i - (v_i,f'')\right]\nabla\bullet\frac{v}{R^2}\nabla_\perp f' - \frac{v'}{R^2}\left[f',R^2[v_i,f']\right] - \frac{v}{R^2}\left[f'',R^2[v_i,f']\right]$$

$$+\nabla_\perp^2 v_i\left(\frac{v}{R^2}p'\right) + \nabla_\perp^2 v_i\left(\gamma p\left(\frac{v'}{R^2}\right)\right)$$

$$L_{33} = +\frac{1}{R^2}\left((\chi,\psi),(v_i,\psi)\right) - \frac{1}{R^2}\Delta^*\psi\left((\chi,\psi),v_i\right)$$

$$-\frac{1}{R^2}\left(R^2[f',\chi],(v_i,\psi)\right) + \frac{1}{R^2}\left((\chi,\psi),R^2[v_i,f']\right) + \frac{1}{R^2}\Delta^*\psi\left(R^2[f',\chi],v_i\right)$$

$$+\frac{F}{R}\left[v_i,(\chi,\psi)\right]' - \frac{F'}{R^2}\left[\chi,(v_i,\psi)\right] - \frac{F}{R^2}\left[\chi',(v_i,\psi)\right] + \frac{1}{R^2}\Delta^*\psi\left(F[\chi,v_i]\right)' - \frac{F'}{R^2}\left[\chi,R^2[v_i,f']\right]$$

$$-\frac{F}{R^2}\left[\chi',R^2[v_i,f']\right] - \frac{F}{R}\left[v_i,R^2[f',\chi]\right]' + \left[F\Delta^* v_i + [v_i,\psi'] - (v_i,f'')\right]\nabla\bullet\frac{F}{R^2}\nabla_\perp\chi$$

$$-F\left(\frac{F}{R^3}(\chi,v_i)\right)'' - \frac{1}{R^2}\left(R^2[f',\chi],R^2[v_i,f']\right) + \nabla_\perp^2 v_i\left((\chi,p) + \gamma p\left(\nabla_\perp^2\chi\right)\right)$$

2.0 Magnetic Field Evolution Equations

Next, consider the evolution of the magnetic field:

$$\mathbf{N} \equiv \frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E}, \tag{2.1}$$

The electric field is given by Ohm's law (including two-fluid terms),

$$\mathbf{E} + \mathbf{v} \times \mathbf{B} = \eta \mathbf{J} + \frac{\mathbf{J} \times \mathbf{B} - \nabla p_e - \nabla \cdot \overleftrightarrow{\Pi}_e}{\rho} \tag{2.2}$$

The electron stress tensor will take the form of a hyperresistivity with coefficient λ:

$$\overleftrightarrow{\Pi}_e = \lambda \eta \rho \nabla \mathbf{J} \tag{2.3}$$

Note that Equation (2.1) can be written in terms of the vector potential $\mathbf{A}$:

$$\nabla \times \frac{\partial \mathbf{A}}{\partial t} = -\nabla \times \mathbf{E}, \tag{2.4}$$

or

$$\mathbf{O} \equiv R^2 \nabla \varphi \times \nabla_\perp \dot{f} + \dot{\psi} \nabla \varphi = -\mathbf{E} - \nabla \Phi \tag{2.5}$$

Here, we have introduced the scalar potential Φ.

2.1 First Projection of the Magnetic Field Evolution Equation (ψ)

Operating on (2.5) with $\nu_i R^2 \nabla \varphi \cdot$ and integrating gives

$$\iint dRdZR\nu_i \nabla \varphi \bullet \mathbf{O} = \iint dRdZRJ_2 \tag{2.6}$$

$$\begin{aligned} J_2 = \nu_i \dot{\psi} &= -\nu_i [\psi, U] - \nu_i (\psi, \chi) - \nu_i (f', U) + R^2 \nu_i [f', \chi] - \frac{1}{R^2}\left(\eta \nu_i R^2, \psi\right) \\ &-\nu_i \Phi' + \frac{\nu_i}{\rho R^2}(\psi, \psi') + \frac{\nu_i}{\rho}(f', f'') + \frac{\nu_i}{\rho}[\psi, f']' + \frac{\nu_i}{\rho} p_e' \\ &-R^2 \Delta^\dagger f \left[\psi, \frac{\nu_i}{\rho}\right] - R^2 \Delta^\dagger f \left(\frac{\nu_i}{\rho}, f'\right) - \frac{\nu_i}{\rho} R^2 \Delta^\dagger f \Delta^\dagger f' + h\text{-r terms} \end{aligned} \tag{2.7}$$

NOTE: we used the identity: $\frac{\nu_i}{\rho}(F, f') = -R^2 \Delta^\dagger f \left(\frac{\nu_i}{\rho}, f'\right) - \frac{\nu_i}{\rho} R^2 \Delta^\dagger f \Delta^\dagger f'$

2.2 Second Projection of the Magnetic Field Evolution Equation (f)

$$\iint dRdZR\nu_i \nabla\varphi \bullet \mathbf{N} = \iint dRdZRJ_1 \tag{2.8}$$

$$\begin{aligned}
J_1 \equiv -\left(\nu_i, \dot{f}\right) &= \frac{1}{R^2}(R_0 + R^2\Delta^\dagger f)[\nu_i, U] + \frac{1}{R^2}(R_0 + R^2\Delta^\dagger f)(\nu_i, \chi) \\
&+ \frac{\nu}{R^2}(\nu_i, f') - \frac{\nu}{R^2}[\nu_i, \psi] + \frac{\eta}{R^2}[\nu_i, \psi'] - \frac{\eta}{R^2}(\nu_i, f'') + R^2\Delta^\dagger f \nabla \bullet \frac{\eta}{R^2}\nabla \nu_i \\
&+ \frac{\Delta^* \psi}{R^2\rho}\left[(\nu_i, f') - [\nu_i, \psi]\right] - \left(\frac{R_0 + R^2\Delta^\dagger f}{R^2\rho}\right)\left\{[\nu_i, f''] + \frac{1}{R^2}(\nu_i, \psi')\right\} \\
&- \frac{F}{R^2\rho}[\nu_i, F] - \frac{[\nu_i, p_e]}{\rho} + \text{h-r terms}
\end{aligned} \tag{2.9}$$

Note: The last term (in red) can be evaluated as in the following. It does not require more than two derivatives on the unknown function at the new time level f^{n+1}:

$$\begin{aligned}
-\left(\frac{F}{R^2\rho}\right)[\nu_i, F] &= -\left(\frac{F^n}{R^2\rho}\right)[\nu_i, F^n] - \theta\delta t\left\{\left(\frac{\dot{F}}{R^2\rho}\right)[\nu_i, F^n] + \left(\frac{F^n}{R^2\rho}\right)[\nu_i, \dot{F}]\right\} \\
&= -(1-2\theta)\left(\frac{F^n}{R^2\rho}\right)[\nu_i, F^n] - \theta\left\{\left(\frac{F^{n+1}}{R^2\rho}\right)[\nu_i, F^n] + \left(\frac{F^n}{R^2\rho}\right)[\nu_i, F^{n+1}]\right\} \\
&= -(1-2\theta)\left(\frac{F^n}{R^2\rho}\right)[\nu_i, F^n] - \theta\left\{\left(\frac{F^{n+1}}{R^2\rho}\right)[\nu_i, F^n] - \left(F^{n+1}\right)\left[\nu_i, \frac{F^n}{R^2\rho}\right]\right\} \\
&= -(1-2\theta)\left(\frac{F^n}{R^2\rho}\right)[\nu_i, F^n] + \theta\left(R_0 + R^2\Delta^\dagger f^{n+1}\right)F^n\left[\nu_i, \frac{1}{R^2\rho}\right]
\end{aligned}$$

2.3 Third Magnetic Field Equation for the Scalar Potential (Φ)

It follows from (0.5) and (2.5) that

$$\nabla_\perp \bullet \frac{1}{R^2}\nabla\Phi = -\nabla_\perp \cdot \frac{1}{R^2}\mathbf{E} \tag{2.10}$$

Multiply by the basis function and integrate over the area.

Thus, the integrand is

$$
\begin{aligned}
-\frac{1}{R^2}(\nu_i,\Phi) = -\nu\left(\frac{1}{R^4}(\nu_i,\psi)+\frac{1}{R^2}[\nu_i,f']\right)+\left(\frac{R_0+R^2\Delta^\dagger f}{R^4}\right)(\nu_i,U)-\Delta^\dagger f[\nu_i,\chi] \\
+\eta\left[\frac{1}{R^4}(\nu_i,\psi')+\frac{1}{R^2}[\nu_i,f'']\right]-\Delta^\dagger f[\nu_i,\eta] \\
+\frac{1}{\rho R^4}(R_0+R^2\Delta^\dagger f)[\nu_i,\psi']-\frac{(R_0+R^2\Delta^\dagger f)}{\rho R^4}(\nu_i,f'')-\frac{F}{\rho R^4}(\nu_i,F) \\
-\frac{1}{\rho R^4}(\Delta^*\psi)(\nu_i,\psi)-\frac{1}{\rho R^2}(\Delta^*\psi)[\nu_i,f']-\frac{1}{\rho R^2}(\nu_i,p_e).
\end{aligned}
\tag{2.12}
$$

Similarly:

$$
\begin{aligned}
-\frac{F}{\rho R^2}(\nu_i,F) &= -(1-2\theta)\left(\frac{F^n}{R^2\rho}\right)(\nu_i,F^n)-\theta\left\{\left(\frac{F^{n+1}}{R^2\rho}\right)(\nu_i,F^n)+\left(\frac{F^n}{R^2\rho}\right)(\nu_i,F^{n+1})\right\} \\
&= -(1-2\theta)\left(\frac{F^n}{R^2\rho}\right)(\nu_i,F^n)-\theta\left\{\left(\frac{F^{n+1}}{R^2\rho}\right)(\nu_i,F^n)-F^{n+1}\nabla\bullet\left(\frac{F^n}{R^2\rho}\nabla\nu_i\right)\right\} \\
&= -(1-2\theta)\left(\frac{F^n}{R^2\rho}\right)(\nu_i,F^n)+\theta(R_0+R^2\Delta^\dagger f^{n+1})F^n\nabla\bullet\left(\frac{1}{R^2\rho}\nabla\nu_i\right)
\end{aligned}
$$

2.4 Alternate form for the ψ equation:

We could combine (2.7) and (2.12) to get the single equation:

$$
\begin{aligned}
&-\frac{1}{R^2}\left(\nu_i,\dot{\psi}\right)=\frac{1}{R^2}\left(\nu_i,[\psi,U]\right)+\frac{1}{R^2}\left(\nu_i,\frac{1}{R}\left(f',U\right)\right)-\left(\frac{F'}{R^4}\right)\left(\nu_i,U\right)-\left(\frac{F}{R^4}\right)\left(\nu_i,U'\right)\\
&+\frac{\nu'}{R^2}\left(\frac{1}{R^2}\left(\nu_i,\psi\right)+\left[\nu_i,f'\right]\right)+\frac{\nu}{R^2}\left(\frac{1}{R^2}\left(\nu_i,\psi'\right)+\left[\nu_i,f''\right]\right)\\
&+\frac{1}{R^2}\left(\nu_i,\left(\psi,\chi\right)\right)-\frac{1}{R^2}\left(\nu_i,R^2\left[f',\chi\right]\right)+\frac{1}{R^2}F'\left[\nu_i,\chi\right]+\frac{1}{R^2}F\left[\nu_i,\chi'\right]\\
&+\eta\left[\frac{1}{R^2}\Delta^*\nu_i\Delta^*\psi-\frac{1}{R^4}\left(\nu_i,\psi''\right)-\frac{1}{R^2}\left[\nu_i,f'''\right]\right]-\frac{\eta}{R^2}\left[\nu_i,F'\right]\\
&-\frac{1}{R^2}\left(\nu_i,\frac{1}{\rho R^2}\left(\psi,\psi'\right)\right)+\frac{1}{\rho R^4}\left(\Delta^*\psi'\right)\left(\nu_i,\psi\right)+\frac{1}{\rho R^4}\left(\Delta^*\psi\right)\left(\nu_i,\psi'\right)\\
&-\frac{1}{R^2}\left(\nu_i,\frac{1}{\rho}\left[\psi',f'\right]\right)-\frac{1}{R^2}\left(\nu_i,\frac{1}{\rho}\left[\psi,f''\right]\right)+\frac{1}{\rho R^2}\left(\Delta^*\psi'\right)\left[\nu_i,f'\right]\\
&+\frac{1}{\rho R^2}\left(\Delta^*\psi\right)\left[\nu_i,f''\right]-\frac{1}{\rho}\frac{F'}{R^4}\left[\nu_i,\psi'\right]-\frac{1}{R^2}\left(\nu_i,\frac{1}{\rho}\left[\psi,F\right]\right)-\frac{1}{\rho}\frac{F}{R^4}\left[\nu_i,\psi''\right]\\
&+\frac{1}{\rho}\frac{F'}{R^4}\left(\nu_i,F\right)+\frac{1}{\rho}\frac{F}{R^4}\left(\nu_i,F'\right)+\frac{1}{\rho}\frac{F}{R^4}\left(\nu_i,f'''\right)-\frac{1}{R^2}\left(\nu_i,\frac{1}{\rho}\left(f',F\right)\right)+\frac{1}{\rho}\frac{F'}{R^4}\left(\nu_i,f''\right)\\
&-\frac{1}{R^2}\left(\nu_i,\frac{1}{\rho}\left(f',f''\right)\right)+\frac{1}{\rho R^2}\left(\nu_i,p_e'\right)-\frac{1}{R^2}\left(\nu_i,\frac{1}{\rho}p_e'\right)
\end{aligned}
$$

Because the above expression contains derivatives of *F*, we would have to make F an auxiliary variable. It would have to be solved simultaneously from the definition:

$$
-\left(R^2\nu_i,f\right)=\nu_i\left(F-R_0\right)
$$

3.0 The Density Equation (ρ)

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \left(\rho \mathbf{v} \right) = D_\rho \nabla^2 \rho.$$

$$\nu_i \dot{\rho} = -\nu_i \left[\rho, U \right] + \rho \left(\nu_i, \chi \right) - \frac{1}{R^2} \nu_i \left(\rho v \right)' + D_\rho \left(-\left(\nu_i, \rho \right) + \frac{1}{R^2} \nu_i \rho'' \right)$$

4.0 The Pressure Equation (*p*)

$$\frac{1}{(\gamma - 1)} \left[\frac{\partial p}{\partial t} + \nabla \cdot \left(p \mathbf{v} \right) \right] = -p \nabla \cdot \mathbf{v} - \overleftrightarrow{\Pi} : \nabla \mathbf{v} - \nabla \cdot \left(\mathbf{q}_i + \mathbf{q}_e \right) + \frac{\mathbf{J}}{\rho} \cdot \left[\eta \rho \mathbf{J} + \frac{\rho^\gamma}{(\gamma - 1)} \nabla \left(\frac{p_e}{\rho^\gamma} \right) \right]$$

$$\nu_i \dot{p} = -\nu_i [p, U] - \nu_i \left(p, \chi \right) - \frac{1}{R^2} p' \nu_i v + \gamma \left(\nu_i p, \chi \right) - \gamma \frac{1}{R^2} p \nu_i v' + \nu_i S_\kappa + \nu_i S_P$$

Here, we have defined the effective source terms:

$$\nu_i S_\kappa \equiv (\gamma - 1) \nu_i \nabla \bullet \left[\kappa \nabla p \right] = (\gamma - 1) \kappa \left[-\left(\nu_i, p \right) + \frac{\nu_i}{R^2} p'' \right]$$

$$\nu_i S_p \equiv \frac{\nu_i}{\rho} \mathbf{J} \bullet \nabla p_e - \frac{\gamma \nu_i p_e}{\rho^2} \mathbf{J} \bullet \nabla \rho + (\gamma - 1) \left[\eta J^2 + \mu \left| \nabla V \right|^2 \right]$$

(

5.0 Electron Pressure (*p*$_e$)

$$\frac{1}{(\gamma - 1)} \left[\frac{\partial p_e}{\partial t} + \nabla \bullet \left(p_e \mathbf{V} \right) \right] = -p_e \nabla \bullet \mathbf{V} + \frac{\mathbf{J}}{\rho} \bullet \left[\frac{\rho^\gamma}{(\gamma - 1)} \nabla \left(p_e / \rho^\gamma \right) + \eta \rho \mathbf{J} \right] + \nabla \left(\frac{\mathbf{J}}{\rho} \right) : \overleftrightarrow{\Pi}_e - \nabla \bullet \mathbf{q}_e + Q_\Delta$$

$$\nu_i \dot{p}_e = -\nu_i [p_e, U] - \nu_i \left(p_e, \chi \right) - \frac{1}{R^2} p_e' \nu_i v + \gamma \left(\nu_i p_e, \chi \right) - \gamma \frac{1}{R^2} p_e \nu_i v' + \nu_i S_{\kappa e} + \nu_i S_{Pe}$$

Here, we have defined the effective source terms:

$$\nu_i S_{\kappa e} \equiv (\gamma - 1) \nu_i \nabla \bullet \left[\kappa \nabla p_e \right] = (\gamma - 1) \kappa \left[-\left(\nu_i, p_e \right) + \frac{\nu_i}{R^2} p_e'' \right]$$

$$\nu_i S_{pe} \equiv \frac{\nu_i}{\rho} \mathbf{J} \bullet \nabla p_e - \frac{\gamma \nu_i p_e}{\rho^2} \mathbf{J} \bullet \nabla \rho + (\gamma - 1) \eta J^2$$

6.0 Linearized Forms in the Absence of Flow

Assume all quantities consist of a stationary real axisymmetric equilibrium component together with a small, complex time-varying linear perturbation with toroidal mode number n:

$$U(R,\varphi,Z,t) = U_0(R,Z) + \mathrm{Re}\{\tilde{U}(R,Z,t)e^{in\varphi}\}$$

Here Re{a} means "take the real part of a". In the following, for notational convenience we let $\tilde{U}(R,Z,t) \to U(R,Z,t)$. In this section, we assume there is no equilibrium flow.

6.1 The Velocity Matrix Equations

The velocity equations can be written in matrix form:

$$\begin{bmatrix} S_{11}^{v} & S_{12}^{v} & S_{13}^{v} \\ S_{21}^{v} & S_{22}^{v} & S_{23}^{v} \\ S_{31}^{v} & S_{32}^{v} & S_{33}^{v} \end{bmatrix} \bullet \begin{bmatrix} U \\ v \\ \mathrm{X} \end{bmatrix}^{n+1} = \begin{bmatrix} D_{11}^{v} & D_{12}^{v} & D_{13}^{v} \\ D_{21}^{v} & D_{22}^{v} & D_{23}^{v} \\ D_{31}^{v} & D_{32}^{v} & D_{33}^{v} \end{bmatrix} \bullet \begin{bmatrix} U \\ v \\ \mathrm{X} \end{bmatrix}^{n} + \begin{bmatrix} R_{11}^{v} & R_{12}^{v} & R_{13}^{v} \\ R_{21}^{v} & R_{22}^{v} & R_{23}^{v} \\ R_{31}^{v} & R_{32}^{v} & R_{33}^{v} \end{bmatrix} \begin{bmatrix} \psi \\ f \\ P \end{bmatrix}^{n} + \begin{bmatrix} O_1^{v} \\ O_2^{v} \\ O_3^{v} \end{bmatrix}$$

$$S_{11}^{v} U^{n+1} = \left\{ \begin{array}{l} -\dfrac{\rho}{R^2}\left(R^2 v_i, U^{n+1}\right) - \theta\delta t\mu\left[\dfrac{1}{R^2}\Delta^*(R^2 v_i)\Delta^* U^{n+1} - v_i\dfrac{n^2}{R^2}\Delta^* U^{n+1} + \dfrac{2n^2}{R^3} v_i U_R^{n+1}\right] \\ -(\theta\delta t)^2 \left[\begin{array}{l} \dfrac{1}{R^2}\left(\left[U^{n+1},\psi\right],\left[R^2 v_i,\psi\right]\right) - \dfrac{1}{R^2}\Delta^*\psi\left[R^2 v_i,\left[U^{n+1},\psi\right]\right] \\ +\dfrac{inF}{R^4}\left(U^{n+1},\left[R^2 v_i,\psi\right]\right) - in\dfrac{F}{R^4}\left(\left[U^{n+1},\psi\right],R^2 v_i\right) - \dfrac{inF}{R^4}\Delta^*\psi\left[R^2 v_i, U^{n+1}\right] \\ -2F\left[U^{n+1},\dfrac{F}{R^2}\right]v_{iZ} + n^2\dfrac{F^2}{R^6}\left(U^{n+1},R^2 v_i\right) + v_i R^2\nabla\varphi\bullet\nabla\left[\rho, U^{n+1}\right]\times\vec{g} \end{array} \right] \end{array} \right\}$$

$$S_{12}^{v} v^{n+1} = -\mu\theta\delta t\left[\frac{2inv_i}{R^2} v_Z^{n+1}\right] - \left(\theta\delta t\right)^2 \left[\begin{array}{l} -\dfrac{inv^{n+1}}{R^4}\left(\psi,\left[R^2 v_i,\psi\right]\right) + \Delta^*\psi\dfrac{inv^{n+1}}{R^4}\left[R^2 v_i,\psi\right] - 2F\left[\dfrac{v^{n+1}}{R^2},\psi\right]v_{iZ} \\ -n^2\dfrac{Fv^{n+1}}{R^6}\left(\psi,R^2 v_i\right) + v_i inR^2\nabla\varphi\bullet\nabla\left[\dfrac{1}{R^2}\left(\rho v^{n+1}\right)\right]\times\vec{g} \end{array} \right]$$

$$S_{13}^{v}\chi^{n+1} = \left\{ \begin{array}{l} \rho\left[v_i R^2,\chi^{n+1}\right] - \mu\theta\delta t\left[\dfrac{2n^2 v_i}{R^2}\chi_Z^{n+1}\right] \\ -(\theta\delta t)^2 \left[\begin{array}{l} -\dfrac{1}{R^2}\left(\left(\chi^{n+1},\psi\right),\left[R^2 v_i,\psi\right]\right) + \dfrac{1}{R^2}\Delta^*\psi\left[R^2 v_i,\left(\chi^{n+1},\psi\right)\right] - in\dfrac{F}{R^2}\left[\left[R^2 v_i,\psi\right],\chi^{n+1}\right] \\ +in\dfrac{F}{R^4}\left(\left(\chi^{n+1},\psi\right),R^2 v_i\right) - in\Delta^*\psi\dfrac{F}{R^4}\left(\chi^{n+1},R^2 v_i\right) + \left(2F\nabla\bullet\dfrac{F}{R^2}\nabla_\perp\chi^{n+1}\right)v_{iZ} \\ +n^2\dfrac{F^2}{R^4}\left[\chi^{n+1},R^2 v_i\right] + v_i R^2\nabla\varphi\bullet\nabla\left[\left(\rho,\chi^{n+1}\right) + \rho\Delta^\dagger\chi^{n+1}\right]\times\vec{g} \end{array} \right] \end{array} \right\}$$

$$D_{11}^{v}U^{n}=\left\{\begin{array}{l}-\frac{\rho}{R^{2}}\left(R^{2}v_{i},U^{n}\right)+(1-\theta)\delta t\mu\left[\frac{1}{R^{2}}\Delta^{*}R^{2}v_{i}\Delta^{*}U^{n}-v_{i}\frac{n^{2}}{R^{2}}\Delta^{*}U^{n}+\frac{2n^{2}}{R^{3}}v_{i}U_{R}^{n}\right]\\ -(\theta\delta t)^{2}\left[\begin{array}{l}\frac{1}{R^{2}}\left(\left[U^{n},\psi\right],\left[R^{2}v_{i},\psi\right]\right)-\frac{1}{R^{2}}\Delta^{*}\psi\left[R^{2}v_{i},\left[U^{n},\psi\right]\right]\\ +\frac{inF}{R^{4}}\left(U^{n},\left[R^{2}v_{i},\psi\right]\right)-in\frac{F}{R^{4}}\left(\left[U^{n},\psi\right],R^{2}v_{i}\right)-\frac{in}{R^{2}}\Delta^{*}\psi\left(\frac{F}{R^{2}}\left[R^{2}v_{i},U^{n}\right]\right)\\ -2F\left[U^{n},\frac{F}{R^{2}}\right]v_{iZ}+n^{2}\frac{F^{2}}{R^{6}}\left(U^{n},R^{2}v_{i}\right)+v_{i}R^{2}\nabla\varphi\bullet\nabla\left[\rho,U^{n}\right]\times\vec{g}\end{array}\right]\end{array}\right\}$$

$$D_{12}^{v}v^{n}=\mu(1-\theta)\delta t\left[\frac{2inv_{i}}{R^{2}}v_{Z}^{n}\right]-\left(\theta\delta t\right)^{2}\left[\begin{array}{l}-\frac{inv^{n}}{R^{4}}\left(\psi,\left[R^{2}v_{i},\psi\right]\right)+\Delta^{*}\psi\frac{inv^{n}}{R^{4}}\left[R^{2}v_{i},\psi\right]-2F\left[\frac{v^{n}}{R^{2}},\psi\right]v_{iZ}\\ -n^{2}\frac{Fv^{n}}{R^{6}}\left(\psi,R^{2}v_{i}\right)+v_{i}inR^{2}\nabla\varphi\bullet\nabla\left[\frac{1}{R^{2}}\left(\rho v^{n}\right)\right]\times\vec{g}\end{array}\right]$$

$$D_{13}^{v}\chi^{n}=\left\{\begin{array}{l}\rho\left[v_{i}R^{2},\chi^{n}\right]+(1-\theta)\mu\delta t\left[\frac{2n^{2}v_{i}}{R^{2}}\chi_{Z}^{n}\right]\\ -(\theta\delta t)^{2}\left[\begin{array}{l}-\frac{1}{R^{2}}\left(\left(\chi^{n},\psi\right),\left[R^{2}v_{i},\psi\right]\right)+\frac{1}{R^{2}}\Delta^{*}\psi\left[R^{2}v_{i},\left(\chi^{n},\psi\right)\right]-in\frac{F}{R^{2}}\left[\left[R^{2}v_{i},\psi\right],\chi^{n}\right]\\ +in\frac{F}{R^{4}}\left(\left(\chi^{n},\psi\right),R^{2}v_{i}\right)-in\Delta^{*}\psi\frac{F}{R^{4}}\left(\chi^{n},R^{2}v_{i}\right)+\left(2F\nabla\bullet\frac{F}{R^{2}}\nabla_{\perp}\chi^{n}\right)v_{iZ}\\ +n^{2}\frac{F^{2}}{R^{4}}\left[\chi^{n},R^{2}v_{i}\right]+v_{i}R^{2}\nabla\varphi\bullet\nabla\left[\left(\rho,\chi^{n}\right)+\rho\Delta^{\dagger}\chi^{n}\right]\times\vec{g}\end{array}\right]\end{array}\right\}$$

$$R_{11}^{v}\Psi^{n}=\delta t\left\{-\left(\frac{\Delta^{*}\psi_{0}}{R^{2}}\right)\left[R^{2}v_{i},\psi^{n}\right]-\left(\frac{\Delta^{*}\psi^{n}}{R^{2}}\right)\left[R^{2}v_{i},\psi_{0}\right]-in\frac{F_{0}}{R^{4}}\left(R^{2}v_{i},\psi^{n}\right)\right\}$$

$$R_{12}^{v}f^{n}=\delta t\left\{\frac{in}{R^{2}}\Delta^{*}\psi_{0}\left(R^{2}v_{i},f^{n}\right)+n^{2}\frac{F_{0}}{R^{2}}\left[R^{2}v_{i},f\right]-2v_{iZ}F_{0}\Delta^{\dagger}f^{n}\right\}$$

$$R_{13}^{v}P^{n}=0$$

$$O_{1}^{v}=\delta tv_{i}R^{2}\nabla\rho\times\vec{g}\bullet\nabla\varphi$$

NOTE: we made use of the identity:

$$-\frac{F_{0}}{R^{2}}\left[R^{2}v_{i},R^{2}\Delta^{\dagger}f^{n}\right]-\Delta^{\dagger}f^{n}\left[R^{2}v_{i},F_{0}\right]=R^{2}\Delta^{\dagger}f^{n}\left[R^{2}v_{i},\frac{F_{0}}{R^{2}}\right]-\Delta^{\dagger}f^{n}\left[R^{2}v_{i},F_{0}\right]=-2F_{0}v_{iZ}\Delta^{\dagger}f^{n}$$

$$S_{21}^{v}U^{n+1} = \theta\delta t\frac{2\nu_i\mu}{R^2}inU_Z^{n+1}$$

$$-(\theta\delta t)^2\left\{\begin{array}{l}-in\frac{\nu_i}{R^2}\left(\left[U^{n+1},\psi\right],\psi\right)+n^2\nu_i\frac{F}{R^4}\left(U^{n+1},\psi\right)-\left[U^{n+1},\psi\right]\left[F,\nu_i\right]\\+R^2\left[\frac{F}{R^2},U^{n+1}\right]\left[\nu_i,\psi\right]+in\nu_i\frac{F}{R^2}\left[F,U^{n+1}\right]-in\nu_i\left[U^{n+1},p\right]\end{array}\right\}$$

$$S_{22}^{v}v^{n+1} = \left\{\begin{array}{l}\rho\nu_i v^{n+1}-\theta\delta t\left[\mu\nu_i\Delta^{*}v^{n+1}-\frac{2n^2\mu_C\nu_i}{R^2}v^{n+1}\right]\\-(\theta\delta t)^2\left[-R^2\left[\nu_i,\psi\right]\left[\frac{v^{n+1}}{R^2},\psi\right]-n^2\frac{\nu_i}{R^4}v^{n+1}\left(\psi,\psi\right)-in\frac{\nu_i}{R^2}v^{n+1}\left[F,\psi\right]-n^2\nu_i\gamma p\frac{v^{n+1}}{R^2}\right]\end{array}\right\}$$

$$S_{23}^{v}\chi^{n+1} = \left\{\begin{array}{l}-\theta\delta tin\left[-(2\mu_C-\mu)\left(\nu_i,\chi^{n+1}\right)+2\frac{\mu\nu_i}{R}\chi_R^{n+1}\right]\\-(\theta\delta t)^2\left[\begin{array}{l}in\frac{\nu_i}{R^2}\left(\left(\chi^{n+1},\psi\right),\psi\right)-\frac{F}{R^2}\left(R^2\left[\nu_i,\psi\right],\chi^{n+1}\right)+\left(\chi^{n+1},\psi\right)\left[F,\nu_i\right]\\+n^2\frac{\nu_i}{R^2}F\left[\chi^{n+1},\psi\right]+in\nu_i\frac{F}{R^2}\left(F,\chi^{n+1}\right)+in\nu_i\left(\chi^{n+1},p\right)+in\nu_i\gamma p\nabla_{\perp}^2\chi^{n+1}\end{array}\right]\end{array}\right\}$$

$$D_{21}^{v}U^{n} = -(1-\theta)\delta t\frac{2\nu_i\mu}{R^2}inU_Z^{n}$$

$$-(\theta\delta t)^2\left\{\begin{array}{l}-in\frac{\nu_i}{R^2}\left(\left[U^{n},\psi\right],\psi\right)+n^2\nu_i\frac{F}{R^4}\left(U^{n},\psi\right)-\left[U^{n},\psi\right]\left[F,\nu_i\right]\\+R^2\left[\frac{F}{R^2},U^{n}\right]\left[\nu_i,\psi\right]+in\nu_i\frac{F}{R^2}\left[F,U^{n}\right]-in\nu_i\left[U^{n},p\right]\end{array}\right\}$$

$$D_{22}^{v}v^{n} = \left\{\begin{array}{l}\rho\nu_i v^{n}+(1-\theta)\delta t\left[\mu\nu_i\Delta^{*}v^{n}-\frac{2n^2\mu_C\nu_i}{R^2}v^{n}\right]\\-(\theta\delta t)^2\left[-R^2\left[\nu_i,\psi\right]\left[\frac{v^{n}}{R^2},\psi\right]-n^2\frac{\nu_i}{R^4}v^{n}\left(\psi,\psi\right)-in\frac{\nu_i}{R^2}v^{n}\left[F,\psi\right]-n^2\nu_i\gamma p\frac{v^{n}}{R^2}\right]\end{array}\right\}$$

$$D_{23}^{v}\chi^{n} = \left\{\begin{array}{l}(1-\theta)\delta tin\left[-(2\mu_C-\mu)\left(\nu_i,\chi^{n}\right)+2\frac{\mu\nu_i}{R}\chi_R^{n}\right]\\-(\theta\delta t)^2\left[\begin{array}{l}in\frac{\nu_i}{R^2}\left(\left(\chi^{n},\psi\right),\psi\right)-\frac{F}{R^2}\left(R^2\left[\nu_i,\psi\right],\chi^{n+1}\right)+\left(\chi^{n},\psi\right)\left[F,\nu_i\right]\\+n^2\frac{\nu_i}{R^2}F\left[\chi^{n},\psi\right]+in\nu_i\frac{F}{R^2}\left(F,\chi^{n}\right)+in\nu_i\left(\chi^{n},p\right)+in\nu_i\gamma p\nabla_{\perp}^2\chi^{n}\end{array}\right]\end{array}\right\}$$

$$R_{21}^{v}\psi^{n} = \delta t\left\{-in\frac{\nu_i}{R^2}\left(\psi_0,\psi^n\right)+\nu_i\left[F,\psi^n\right]\right\}$$

$$R_{22}^{v} f^{n} = \delta t\left\{-n^2\nu_i\left[f^n,\psi\right]-R^2\Delta^{\dagger} f^n\left[\nu_i,\psi\right]-in\nu_i\left(f^n,F\right)\right\}$$

$$R_{23}^{v}P^{n} = -in\delta t\nu_i p^n$$

$$O_2^{v} = 0$$

$$S_{31}^{v}U^{n+1} = \left\{\begin{array}{l} +\nu_i\left[\rho,U^{n+1}\right] \\ -(\theta\delta t)^2\left[\begin{array}{l} -\frac{1}{R^2}\left(\left[U^{n+1},\psi\right],\left(\nu_i,\psi\right)\right)+\frac{1}{R^2}\Delta^{*}\psi\left(\left[U^{n+1},\psi\right],\nu_i\right) \\ -in\frac{F}{R}\left[\nu_i,\left[U^{n+1},\psi\right]\right]-in\frac{F}{R^4}\left(U^{n+1},\left(\nu_i,\psi\right)\right) \\ +in\Delta^{*}\psi\frac{F}{R^4}\left(U^{n+1},\nu_i\right)+n^2\frac{F^2}{R^3}\left[\nu_i,U^{n+1}\right] \\ +\left[\frac{F}{R^2},U^{n+1}\right]F\Delta^{*}\nu_i-\nabla_{\perp}^2\nu_i\left[U^{n+1},p\right]-\nu_i\vec{g}\bullet\nabla\left[\rho,U^{n+1}\right] \end{array}\right] \end{array}\right\}$$

$$S_{32}^{v}v^{n+1} = -(\theta\delta t)^2\left[\begin{array}{l} +in\frac{v^{n+1}}{R^4}\left(\psi,\left(\nu_i,\psi\right)\right)-in\Delta^{*}\psi\frac{v^{n+1}}{R^4}\left(\psi,\nu_i\right) \\ -n^2F\frac{v^{n+1}}{R^3}\left[\nu_i,\psi\right]-F\Delta^{*}\nu_i\left[\frac{v^{n+1}}{R^2},\psi\right]+in\nabla_{\perp}^2\nu_i\gamma p\frac{v^{n+1}}{R^2}-\nu_i\vec{g}\bullet\nabla\left[\frac{1}{R^2}\left(\rho v^{n+1}\right)\right] \end{array}\right]$$

$$S_{33}^{v}\mathrm{X}^{n+1} = \left\{\begin{array}{l} -\rho\left(\nu_i,\chi^{n+1}\right)-\delta t\theta 2\mu_C\Delta^{\dagger}\nu_i\Delta^{\dagger}\chi^{n+1} \\ -(\theta\delta t)^2\left[\begin{array}{l} \frac{1}{R^2}\left(\left(\chi^{n+1},\psi\right),\left(\nu_i,\psi\right)\right)-\frac{1}{R^2}\Delta^{*}\psi\left(\left(\chi^{n+1},\psi\right),\nu_i\right) \\ +in\frac{F}{R}\left[\nu_i,\left(\chi^{n+1},\psi\right)\right]-in\frac{F}{R^2}\left[\chi^{n+1},\left(\nu_i,\psi\right)\right]+\Delta^{*}\psi\frac{in}{R^2}F\left[\chi^{n+1},\nu_i\right] \\ +n^2\frac{F^2}{R^3}\left(\chi^{n+1},\nu_i\right)+F\Delta^{*}\nu_i\nabla\bullet\frac{F}{R^2}\nabla_{\perp}\chi^{n+1} \\ +\nabla_{\perp}^2\nu_i\left(\left(\chi^{n+1},p\right)+\gamma p\nabla_{\perp}^2\chi^{n+1}\right)-\nu_i\vec{g}\bullet\nabla\left[\left(\rho,\chi^{n+1}\right)+\rho\Delta^{\dagger}\chi^{n+1}\right] \end{array}\right] \end{array}\right\}$$

$$D_{31}^{v}U^{n+1}=\left\{\begin{array}{l}+\nu_i\left[\rho,U^n\right]\\ -(\theta\delta t)^2\left[\begin{array}{l}-\frac{1}{R^2}\left(\left[U^n,\psi\right],\left(\nu_i,\psi\right)\right)+\frac{1}{R^2}\Delta^*\psi\left(\left[U^n,\psi\right],\nu_i\right)\\ -in\frac{F}{R}\left[\nu_i,\left[U^n,\psi\right]\right]-in\frac{F}{R^4}\left(U^n,\left(\nu_i,\psi\right)\right)\\ +in\Delta^*\psi\frac{F}{R^4}\left(U^n,\nu_i\right)+n^2\frac{F^2}{R^3}\left[\nu_i,U^n\right]\\ +\left[\frac{F}{R^2},U^n\right]F\Delta^*\nu_i-\nabla_\perp^2\nu_i\left[U^n,p\right]-\nu_i\vec{g}\bullet\nabla\left[\rho,U^n\right]\end{array}\right]\end{array}\right\}$$

$$D_{32}^{v}\nu^{n+1}=-(\theta\delta t)^2\left[\begin{array}{l}+in\frac{v^n}{R^4}\left(\psi,\left(\nu_i,\psi\right)\right)-in\Delta^*\psi\frac{v^n}{R^4}\left(\psi,\nu_i\right)\\ -n^2F\frac{v^n}{R^3}\left[\nu_i,\psi\right]-F\Delta^*\nu_i\left[\frac{v^n}{R^2},\psi\right]+in\nabla_\perp^2\nu_i\gamma p\frac{v^n}{R^2}-\nu_i\vec{g}\bullet\nabla\left[\frac{1}{R^2}\left(\rho v^n\right)\right]\end{array}\right]$$

$$D_{33}^{v}\mathrm{X}^{n+1}=\left\{\begin{array}{l}-\rho\left(\nu_i,\chi^n\right)+(1-\theta)\delta t2\mu_C\Delta^\dagger\nu_i\Delta^\dagger\chi^n\\ -(\theta\delta t)^2\left[\begin{array}{l}\frac{1}{R^2}\left(\left(\chi^n,\psi\right),\left(\nu_i,\psi\right)\right)-\frac{1}{R^2}\Delta^*\psi\left(\left(\chi^n,\psi\right),\nu_i\right)\\ +in\frac{F}{R}\left[\nu_i,\left(\chi^n,\psi\right)\right]-in\frac{F}{R^2}\left[\chi^n,\left(\nu_i,\psi\right)\right]+\Delta^*\psi\frac{in}{R^2}F\left[\chi^n,\nu_i\right]\\ +n^2\frac{F^2}{R^3}\left(\chi^n,\nu_i\right)+F\Delta^*\nu_i\nabla\bullet\frac{F}{R^2}\nabla_\perp\chi^n\\ +\nabla_\perp^2\nu_i\left(\left(\chi^n,p\right)+\gamma p\nabla_\perp^2\chi^n\right)-\nu_i\vec{g}\bullet\nabla\left[\left(\rho,\chi^n\right)+\rho\Delta^\dagger\chi^n\right]\end{array}\right]\end{array}\right\}$$

$$R_{31}^{v}\Psi^n=\delta t\left\{\frac{\Delta^*\psi^n}{R^2}\left(\psi_0,\nu_i\right)+\frac{\Delta^*\psi_0}{R^2}\left(\psi^n,\nu_i\right)-in\frac{F}{R^2}\left[\nu_i,\psi^n\right]\right\}$$

$$R_{32}^{v}f^n=\delta t\left\{in\Delta^*\psi\left[\nu_i,f^n\right]-n^2\frac{F_0}{R^2}\left(\nu_i,f^n\right)-\Delta^\dagger f^nF_0\Delta^*\nu_i\right\}$$

$$R_{33}^{v}P^n=\delta t\left(\nu_i,p^n\right)$$

$$O_v^3=\delta t\nu_i\vec{g}\bullet\nabla\rho$$

NOTE: We used the identity: $+\frac{F_0}{R^2}\left(\nu_i,F\right)+\frac{F}{R^2}\left(\nu_i,F_0\right)=-\Delta^\dagger fF_0\Delta^*\nu_i$

6.2 The Magnetic Field Matrix Equations

The Linearized Magnetic Field Evolution Equations can be written:

$$\begin{bmatrix} S_{11}^b & S_{12}^b & S_{13}^b \\ S_{21}^b & S_{22}^b & S_{23}^b \\ S_{31}^b & S_{32}^b & S_{33}^b \end{bmatrix} \bullet \begin{bmatrix} \psi \\ f \\ \Phi \end{bmatrix}^{n+1} = \begin{bmatrix} D_{11}^b & D_{12}^b & D_{13}^b \\ D_{21}^b & D_{22}^b & D_{23}^b \\ D_{31}^b & D_{32}^b & D_{33}^b \end{bmatrix} \bullet \begin{bmatrix} \psi \\ f \\ \Phi \end{bmatrix}^{n} + \begin{bmatrix} R_{11}^b & R_{12}^b & R_{13}^b \\ R_{21}^b & R_{22}^b & R_{23}^b \\ R_{31}^b & R_{32}^b & R_{33}^b \end{bmatrix} \bullet \begin{bmatrix} U \\ \nu \\ \chi \end{bmatrix}^{n+1}$$

$$+ \begin{bmatrix} Q_{11}^b & Q_{12}^b & Q_{13}^b \\ Q_{21}^b & Q_{22}^b & Q_{23}^b \\ Q_{31}^b & Q_{32}^b & Q_{33}^b \end{bmatrix} \bullet \begin{bmatrix} U \\ \nu \\ \chi \end{bmatrix}^{n} + \begin{bmatrix} O_1 \\ O_2 \\ O_3 \end{bmatrix}$$

$$S_{11}^b \psi^{n+1} = \nu_i \psi^{n+1} - \delta t \theta \left[-\frac{1}{R^2}\left(\eta \nu_i R^2, \psi^{n+1}\right) + \frac{\nu_i}{\rho}\left[\psi^{n+1}, F_0\right] + \frac{in}{\rho R^2}\nu_i\left(\psi_0, \psi^{n+1}\right)\right]$$

$$S_{12}^b f^{n+1} = -\delta t \theta \left[-R^2 \Delta^\dagger f^{n+1}\left[\psi_0, \frac{\nu_i}{\rho}\right] - \frac{n^2}{\rho}\nu_i\left[\psi_0, f^{n+1}\right] + \frac{in}{\rho}\nu_i\left(f^{n+1}, F_0\right)\right]$$

$$S_{13}^b \Phi^{n+1} = in\delta t \nu_i \theta \Phi^{n+1}$$

NOTE: $$\frac{\nu_i}{\rho}\left(f', R^2\Delta^\dagger f\right) = -R^2\Delta^\dagger f\left(f', \frac{\nu_i}{\rho}\right) - \frac{\nu_i}{\rho}R^2(\Delta^\dagger f)(\Delta^\dagger f')$$

$$D_{11}^b \psi^n = \nu_i \psi^n + \delta t(1-\theta)\left[-\frac{1}{R^2}\left(\eta \nu_i R^2, \psi^n\right) + \frac{\nu_i}{\rho}\left[\psi^n, F_0\right] + \frac{in}{\rho R^2}\nu_i\left(\psi_0, \psi^n\right)\right]$$

$$D_{12}^b f^n = \delta t(1-\theta)\left[-R^2 \Delta^\dagger f^n\left[\psi_0, \frac{\nu_i}{\rho}\right] - \frac{n^2}{\rho}\nu_i\left[\psi_0, f^n\right] + \frac{in}{\rho}\nu_i\left(f^n, F_0\right)\right]$$

$$D_{13}^b \Phi^n = -in\delta t \nu_i (1-\theta)\Phi^n$$

$$R_{11}^b U^{n+1} = -\delta t \nu_i \theta\left[\psi_0, U^{n+1}\right]$$

$$R_{12}^b \nu^{n+1} = 0$$

$$R_{13}^b \chi^{n+1} = -\delta t \nu_i \theta\left(\psi_0, \chi^{n+1}\right)$$

$$O_1 = +\delta t\left[in\theta\frac{\nu_i}{\rho}p_e^{n+1} + (1-\theta)in\frac{\nu_i}{\rho}p_e^n\right]$$

$$Q_{11}^{b}U^{n} = -\delta t \nu_i (1-\theta)\left[\psi_0, U^{n}\right]$$

$$Q_{12}^{b}\nu^{n} = 0$$

$$Q_{13}^{b}\chi^{n} = -\delta t \nu_i (1-\theta)\left(\psi_0, \chi^{n}\right)$$

$$S_{21}^{b}\psi^{n+1} = -\theta\delta t\left[in\frac{\eta}{R^2}\left[\nu_i, \psi^{n+1}\right] - \frac{1}{R^2\rho}\left[\nu_i, \psi_0\right]\Delta^{*}\psi^{n+1} - \frac{1}{R^2\rho}\left[\nu_i, \psi^{n+1}\right]\Delta^{*}\psi_0 - in\frac{F_0}{R^4\rho}\left(\nu_i, \psi^{n+1}\right)\right]$$

$$S_{22}^{b}f^{n+1} = -\left(\nu_i, f^{n+1}\right) - \theta\delta t\left[\begin{array}{l} R^2\Delta^{\dagger}f^{n+1}\nabla_{\perp}\bullet\frac{\eta}{R^2}\nabla_{\perp}\nu_i + n^2\frac{\eta}{R^2}\left(\nu_i, f^{n+1}\right) \\ +\frac{in}{R^2\rho}\Delta^{*}\psi_0\left(\nu_i, f^{n+1}\right) + \frac{F_0}{R^2\rho}n^2\left[\nu_i, f^{n+1}\right] + R^2\Delta^{\dagger}f^{n+1}F_0\left[\nu_i, \frac{1}{R^2\rho}\right] \end{array}\right]$$

$$S_{23}^{b}\Phi^{n+1} = 0$$

$$D_{21}^{b}\psi^{n} = (1-\theta)\delta t\left[in\frac{\eta}{R^2}\left[\nu_i, \psi^{n}\right] - \frac{1}{R^2\rho}\left[\nu_i, \psi_0\right]\Delta^{*}\psi^{n} - \frac{1}{R^2\rho}\left[\nu_i, \psi^{n}\right]\left(\Delta^{*}\psi_0\right) - in\frac{F_0}{R^4\rho}\left(\nu_i, \psi^{n}\right)\right]$$

$$D_{22}^{b}f^{n} = -\left(\nu_i, f^{n}\right) + (1-\theta)\delta t\left[\begin{array}{l} R^2\Delta^{\dagger}f^{n}\nabla_{\perp}\bullet\frac{\eta}{R^2}\nabla_{\perp}\nu_i + n^2\frac{\eta}{R^2}\left(\nu_i, f^{n}\right) \\ +\frac{in}{R^2\rho}\Delta^{*}\psi_0\left(\nu_i, f^{n}\right) + \frac{F_0}{R^2\rho}n^2\left[\nu_i, f^{n}\right] + R^2\Delta^{\dagger}f^{n}F_0\left[\nu_i, \frac{1}{R^2\rho}\right] \end{array}\right]$$

$$D_{23}^{b}\Phi^{n} = 0$$

$$R_{21}^{b}U^{n+1} = \theta\delta t\left(\frac{F_0}{R^2}\right)\left[\nu_i, U^{n+1}\right]$$

$$R_{22}^{b}\nu^{n+1} = -\theta\delta t\frac{\nu^{n+1}}{R^2}\left[\nu_i, \psi_0\right]$$

$$R_{23}^{b}\chi^{n+1} = \theta\delta t\left(\frac{F_0}{R^2}\right)\left(\nu_i, \chi^{n+1}\right)$$

$$O_2 = -\delta t\left[\theta\frac{\left[\nu_i, p_e^{\,n+1}\right]}{\rho} + (1-\theta)\frac{\left[\nu_i, p_e^{n}\right]}{\rho}\right]$$

$$Q_{21}^{b}U^{n} = (1-\theta)\delta t\left(\frac{F_0}{R^2}\right)\left[\nu_i, U^n\right]$$

$$Q_{22}^{b}\nu^{n} = -(1-\theta)\delta t\,\frac{\nu^n}{R^2}\left[\nu_i, \psi_0\right]$$

$$Q_{23}^{b}\chi^{n} = (1-\theta)\delta t\left(\frac{F_0}{R^2}\right)\left(\nu_i, \chi^n\right)$$

$$S_{31}^{b}\psi^{n+1} = -\eta\frac{in}{R^2}\left(\nu_i, \psi^{n+1}\right) - in\frac{F_0}{\rho R^2}\left[\nu_i, \psi^{n+1}\right] + \frac{1}{\rho R^2}\Delta^*\psi_0\left(\nu_i, \psi^{n+1}\right) + \frac{1}{\rho R^2}\Delta^*\psi^{n+1}\left(\nu_i, \psi_0\right)$$

$$S_{32}^{b}f^{n+1} = +\eta n^2\left[\nu_i, f^{n+1}\right] + R^2\Delta^{\dagger}f^{n+1}\left[\nu_i, \eta\right] + \frac{in}{\rho}\Delta^*\psi_0\left[\nu_i, f^{n+1}\right]$$

$$-\frac{n^2F_0}{\rho R^2}\left(\nu_i, f^{n+1}\right) - R^2\Delta^{\dagger}f^{n+1}F_0\nabla_{\perp}\bullet\frac{1}{\rho R^2}\nabla_{\perp}\nu_i$$

$$S_{33}^{b}\Phi^{n+1} = -\left(\nu_i, \Phi^{n+1}\right)$$

Note: $+\frac{F_0}{\rho R^2}\left(\nu_i, F\right) + \frac{F}{\rho R^2}\left(\nu_i, F_0\right) = -R^2(\Delta^{\dagger}f)F_0\nabla_{\perp}\bullet\frac{1}{\rho R^2}\nabla_{\perp}\nu_i$

$$D_{31}^{b}\psi^{n} = 0$$

$$D_{32}^{b}f^{n} = 0$$

$$D_{33}^{b}\Phi^{n} = 0$$

$$R_{31}^{b}U^{n+1} = +\left(\frac{F_0}{R^2}\right)\left(\nu_i, U^{n+1}\right)$$

$$R_{32}^{b}\nu^{n+1} = -\frac{\nu^{n+1}}{R}\left(\frac{1}{R}\left(\nu_i, \psi_0\right)\right)$$

$$R_{33}^{b}\chi^{n+1} = -F_0\left[\nu_i, \chi^{n+1}\right]$$

$$O_2 = -\frac{1}{\rho}\left(\nu_i, p_e^{n+1}\right)$$

$$Q_{31}^{b}U^{n} = 0$$

$$Q_{32}^{b}\nu^{n} = 0$$

$$Q_{33}^{b}\chi^{n} = 0$$

6.3 The Density Matrix Equations

The linearized form without flow is:

$$S_{11}^{n} \bullet N^{n+1} = D_{11}^{n} \bullet N^{n} + \begin{bmatrix} R_{11}^{n} & R_{12}^{n} & R_{13}^{n} \end{bmatrix} \bullet \begin{bmatrix} U \\ \nu \\ \mathrm{X} \end{bmatrix}^{n+1} + \begin{bmatrix} Q_{11}^{n} & Q_{12}^{n} & Q_{13}^{n} \end{bmatrix} \bullet \begin{bmatrix} U \\ \nu \\ \mathrm{X} \end{bmatrix}^{n}$$

The matrix elements are given by:

$$S_{11}^{n} N_{j}^{n+1} = \nu_i \rho^{n+1} + D_\rho \theta \delta t \left(\left(\nu_i, \rho^{n+1} \right) + \frac{n^2}{R^2} \nu_i \rho^{n+1} \right)$$

$$D_{11}^{n} N_{j}^{n} = \nu_i \rho^{n} - D_\rho (1-\theta) \delta t \left(\left(\nu_i, \rho^{n} \right) + \frac{n^2}{R^2} \nu_i \rho^{n} \right)$$

$$R_{11}^{n} U^{n+1} = -\theta \delta t \nu_i \left[\rho, U^{n+1} \right]$$

$$R_{12}^{n} \nu^{n+1} = -in\theta \delta t \frac{1}{R^2} \nu_i \rho \nu^{n+1}$$

$$R_{13}^{n} \mathrm{X}^{n+1} = +\theta \delta t \rho \left(\nu_i, \chi^{n+1} \right)$$

$$Q_{11}^{n} U^{n} = -(1-\theta) \delta t \nu_i \left[\rho, U^{n} \right]$$

$$Q_{12}^{n} \nu^{n} = -in(1-\theta) \delta t \frac{1}{R^2} \nu_i \rho \nu^{n}$$

$$Q_{13}^{n} \mathrm{X}^{n} = (1-\theta) \delta t \rho \left(\nu_i, \chi^{n} \right)$$

6.4 The Pressure Matrix Equations

The Linearized form for the pressure advance equation in the absence of flow is:

$$S_{11}^{p} \bullet P^{n+1} = D_{11}^{p} \bullet P^{n} + \begin{bmatrix} R_{11}^{p} & R_{12}^{p} & R_{13}^{p} \end{bmatrix} \bullet \begin{bmatrix} U \\ V_z \\ \mathrm{X} \end{bmatrix}^{n+1} + \begin{bmatrix} Q_{11}^{p} & Q_{12}^{p} & Q_{13}^{p} \end{bmatrix} \bullet \begin{bmatrix} U \\ V_z \\ \mathrm{X} \end{bmatrix}^{n} + O_{1}^{p}$$

$$S_{11}^{p} P_{j}^{n+1} = \nu_i p^{n+1} + \theta \delta t (\gamma - 1) \kappa \left[\left(\nu_i, p^{n+1} \right) + \frac{\nu_i n^2}{R^2} p^{n+1} \right]$$

$$D_{11}^{p} P_{j}^{n} = \nu_i p^{n} - (1-\theta) \delta t (\gamma - 1) \kappa \left[\left(\nu_i, p^{n+1} \right) + \frac{\nu_i n^2}{R^2} p^{n+1} \right]$$

$$R_{11}^{p} U_{j}^{n+1} = -\delta t \theta \nu_i [p, U^{n+1}]$$

$$R_{12}^{p} \nu^{n+1} = -in\gamma \delta t \theta \frac{1}{R^2} p \nu_i \nu^{n+1}$$

$$R_{13}^{p} \mathrm{X}_{j}^{n+1} = -\delta t \theta \nu_i \left(p, \chi^{n+1} \right) + \delta t \theta \gamma \left(\nu_i p, \chi^{n+1} \right)$$

$$Q_{11}^{p}U^{n} = -\delta t(1-\theta)\nu_i[p, U^{n}]$$

$$Q_{12}^{p}v^{n} = -in\gamma\delta t(1-\theta)\frac{1}{R^{2}}\, p\nu_i v^{n}$$

$$Q_{13}^{p}\mathrm{X}^{n} = -\delta t(1-\theta)\nu_i\left(p, \chi^{n}\right) + \delta t(1-\theta)\gamma\left(\nu_i p, \chi^{n}\right)$$

$$O_{1}^{p} = 0$$

6.5 The Electron Pressure Matrix Equations

These are not needed initially. They can be the same as for the pressure equations.

6.6 Summary of linear 2-field reduced system:

The integrands for the weak form of the nonlinear equations are:

$$-\frac{1}{R^2}\left(\nu_i R^2,\dot{U}\right)=\frac{\Delta^*\dot{U}}{R^2}\left[\nu_i R^2,U\right]-\frac{\Delta^*\psi}{R^2}\left[\nu_i R^2,\psi\right]-\frac{F}{R^4}\left(\nu_i R^2,\psi'\right)$$
$$+\mu\left[\frac{\Delta^* U}{R^2}\Delta^*(R^2\nu_i)+\frac{\nu_i}{R^2}\Delta^* U''-\frac{2\nu_i}{R^3}U''_R\right] \tag{6.6.1}$$

$$-\frac{1}{R^2}\left(\nu_i,\dot{\psi}\right)=\frac{1}{R^2}\left(\nu_i,[\psi,U]\right)-\left(\frac{F}{R^4}\right)\left(\nu_i,U'\right)+\eta\left[\frac{1}{R^2}\Delta^*\nu_i\Delta^*\psi-\frac{1}{R^4}\left(\nu_i,\psi''\right)\right] \tag{6.6.2}$$

For the linear solution, these are time differenced as follows:

$$-\frac{1}{R^2}\left(\nu_i R^2,U^{n+1}\right)-(\theta\delta t)^2 L_{11}\left\{U^{n+1}\right\}-\mu\theta\delta t\left[\frac{\Delta^* U}{R^2}\Delta^*\left(R^2\nu_i\right)-n^2\nu_i\frac{\Delta^* U}{R^2}+2n^2\frac{\nu_i}{R^3}U_R\right]^{n+1}$$
$$=-\frac{1}{R^2}\left(\nu_i R^2,U^n\right)-(\theta\delta t)^2 L_{11}\left\{U^n\right\}$$
$$+\delta t\left\{\begin{array}{l}-\left(\frac{\Delta^*\psi_0}{R^2}\right)\left[\nu_i R^2,\psi\right]-\left(\frac{\Delta^*\psi}{R^2}\right)\left[\nu_i R^2,\psi_0\right]-in\frac{F_0}{R^4}\left(\nu_i R^2,\psi\right)\\ +\mu(1-\theta)\left[\frac{\Delta^* U}{R^2}\Delta^*\left(\nu_i R^2\right)-n^2\nu_i\frac{\Delta^* U}{R^2}+2n^2\frac{\nu_i}{R^3}U_R\right]\end{array}\right\} \tag{6.6.3}$$

$$-\frac{1}{R^2}\left(\nu_i,\psi^{n+1}\right)-\theta\delta t\left[\frac{1}{R^2}\left(\nu_i,\left[\psi_0,U^{n+1}\right]\right)-in\frac{F_0}{R^4}\left(\nu_i,U\right)^{n+1}+\eta\frac{1}{R^2}\Delta^*\nu_i\Delta^*\psi^{n+1}+\eta\frac{n^2}{R^4}\left(\nu_i,\psi^{n+1}\right)\right]$$
$$=-\frac{1}{R^2}\left(\nu_i,\psi^n\right)+(1-\theta)\delta t\left[\frac{1}{R^2}\left(\nu_i,\left[\psi_0,U^n\right]\right)-in\frac{F_0}{R^4}\left(\nu_i,U\right)^n+\eta\frac{1}{R^2}\Delta^*\nu_i\Delta^*\psi^n+\eta\frac{n^2}{R^4}\left(\nu_i,\psi^n\right)\right] \tag{6.6.4}$$

Here we have defined the operator:

$$L_{11}\{U\}=\frac{1}{R^2}\left([U,\psi_0],\left[\nu_i R^2,\psi_0\right]\right)-\frac{1}{R^2}\Delta^*\psi_0\left[\nu_i R^2,[U,\psi_0]\right]-2F_0\left[U,\frac{F_0}{R^2}\right]\nu_{iZ}$$
$$+in\frac{F_0}{R^4}\left(U,\left[\nu_i R^2,\psi_0\right]\right)-in\frac{F_0}{R^4}\left([U,\psi_0],R^2\nu_i\right)-in\frac{F_0}{R^4}\Delta^*\psi_0\left[R^2\nu_i,U\right] \tag{6.6.5}$$
$$+n^2\frac{F_0^2}{R^6}\left(U,R^2\nu_i\right)$$

We can put these equations in the form:

$$
\begin{aligned}
S_{11}^{v} U^{n+1} &= D_{11}^{v} U^{n} + R_{11}^{v} \psi^{n} \\
S_{11}^{b} \psi^{n+1} &= D_{11}^{b} \psi^{n} + R_{11}^{b} U^{n+1}
\end{aligned}
\tag{6.6.6}
$$

Here,

$$
\begin{aligned}
S_{11}^{v} = &-\frac{1}{R^2}\left(\nu_i R^2, U\right) - (\theta\delta t)^2 \left\{ \begin{array}{l} \frac{1}{R^2}\left(\left[U,\psi_0\right],\left[\nu_i R^2,\psi_0\right]\right) - \frac{1}{R^2}\Delta^* \psi_0 \left[\nu_i R^2,\left[U,\psi_0\right]\right] \\ -2F_0\left[U,\frac{F_0}{R^2}\right]\frac{\partial \nu_i}{\partial Z} + n^2 \frac{F_0^2}{R^6}\left(U, R^2 \nu_i\right) \\ +in\frac{F_0}{R^4}\left(U,\left[R^2\nu_i,\psi_0\right]\right) - in\frac{F_0}{R^4}\left(\left[U,\psi_0\right], R^2\nu_i\right) - in\frac{F_0}{R^4}\Delta^*\psi_0\left[R^2\nu_i, U\right] \end{array} \right\} \\
&-\mu\theta\delta t\left[\frac{1}{R^2}\Delta^*\left(R^2\nu_i\right)\left(\Delta^* U\right) - n^2\nu_i\frac{\Delta^* U}{R^2} + 2n^2\frac{\nu_i}{R^2}\left(\frac{1}{R}\frac{\partial U}{\partial R}\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
D_{11}^{v} = &-\frac{1}{R^2}\left(\nu_i R^2, U^n\right) - (\theta\delta t)^2 \left\{ \begin{array}{l} \frac{1}{R^2}\left(\left[U,\psi_0\right],\left[\nu_i R^2,\psi_0\right]\right) - \frac{1}{R^2}\Delta^* \psi_0 \left[\nu_i R^2,\left[U,\psi_0\right]\right] \\ -2F_0\left[U,\frac{F_0}{R^2}\right]\frac{\partial \nu_i}{\partial Z} + n^2 \frac{F_0^2}{R^6}\left(U, R^2 \nu_i\right) \\ +in\frac{F_0}{R^4}\left(U,\left[R^2\nu_i,\psi_0\right]\right) - in\frac{F_0}{R^4}\left(\left[U,\psi_0\right], R^2\nu_i\right) - in\frac{F_0}{R^4}\Delta^*\psi_0\left[R^2\nu_i, U\right] \end{array} \right\} \\
&+\mu(1-\theta)\left[\frac{1}{R^2}\Delta^*\left(R^2\nu_i\right)\left(\Delta^* U\right) - n^2\nu_i\frac{\Delta^* U}{R^2} + 2n^2\frac{\nu_i}{R^2}\left(\frac{1}{R}\frac{\partial U}{\partial R}\right)\right]
\end{aligned}
$$

$$
R_{11}^{v} = +\delta t\left\{-\left(\frac{\Delta^*\psi_0}{R^2}\right)\left[\nu_i R^2,\psi\right] - \left(\frac{\Delta^*\psi}{R^2}\right)\left[\nu_i R^2,\psi_0\right] - in\frac{F_0}{R^4}\left(\nu_i R^2,\psi\right)\right\}
$$

$$
S_{11}^{b} = -\frac{1}{R^2}\left(\nu_i,\psi^{n+1}\right) - \eta\delta t\theta\left[\frac{1}{R^2}\Delta^*\nu_i\Delta^*\psi + \frac{n^2}{R^4}\left(\nu_i,\psi\right)\right]
$$

$$
D_{11}^{b} = -\frac{1}{R^2}\left(\nu_i,\psi^{n}\right) + (1-\theta)\eta\delta t\left[\frac{1}{R^2}\Delta^*\nu_i\Delta^*\psi + \frac{n^2}{R^4}\left(\nu_i,\psi\right)\right]
$$

$$
R_{11}^{b} = \delta t\frac{1}{R^2}\left(\nu_i,\left[\psi_0, U^{n+1}\right]\right) - in\delta t\frac{F_0}{R^4}\left(\nu_i, U\right)^{n+1}
$$

7.0 Linearized Form with Flow

New terms arising from equilibrium flow. For now, neglect terms with perturbed density

7.1 Momentum Equation

$$S_{11}^{v}U^{n+1} = +\theta\delta t\left\{\begin{array}{l} -\rho\dfrac{\Delta^{*}U^{n+1}}{R^{2}}\left[R^{2}v_{i},U_{0}\right]-\dfrac{\left(U^{n+1},U_{0}\right)}{R^{2}}\left[R^{2}v_{i},\rho\right]-\rho\dfrac{\Delta^{*}U^{n+1}}{R^{2}}\left(R^{2}v_{i},\chi_{0}\right) \\ +\left[U^{n+1},\chi_{0}\right]\left[R^{2}v_{i},\rho\right]-\rho\dfrac{\Delta^{*}U_{0}}{R^{2}}\left[R^{2}v_{i},U^{n+1}\right]-in\dfrac{\rho v_{0}}{R^{4}}\left(R^{2}v_{i},U^{n+1}\right) \end{array}\right\}$$

$$S_{12}^{v}v^{n+1} = +\theta\delta t\left\{-2\frac{\rho v^{n+1}v_{0}}{R^{2}}v_{iZ}\right\}$$

$$S_{13}^{v}\chi^{n+1} = +\theta\delta t\left\{-\left(\chi^{n+1},\chi_{0}\right)\left[R^{2}v_{i},\rho\right]-\rho\frac{\Delta^{*}U_{0}}{R^{2}}\left(R^{2}v_{i},\chi^{n+1}\right)+\left[U_{0},\chi^{n+1}\right]\left[R^{2}v_{i},\rho\right]+in\frac{\rho v_{0}}{R^{2}}\left[R^{2}v_{i},\chi^{n+1}\right]\right\}$$

$$D_{11}^{v}U^{n} = -(1-\theta)\delta t\left\{\begin{array}{l} -\rho\dfrac{\Delta^{*}U^{n}}{R^{2}}\left[R^{2}v_{i},U_{0}\right]-\dfrac{\left(U^{n},U_{0}\right)}{R^{2}}\left[R^{2}v_{i},\rho\right]-\rho\dfrac{\Delta^{*}U^{n}}{R^{2}}\left(R^{2}v_{i},\chi_{0}\right) \\ +\left[U^{n},\chi_{0}\right]\left[R^{2}v_{i},\rho\right]-\rho\dfrac{\Delta^{*}U_{0}}{R^{2}}\left[R^{2}v_{i},U^{n}\right]-in\dfrac{\rho v_{0}}{R^{2}}\left(v_{i},U^{n}\right) \end{array}\right\}$$

$$D_{12}^{v}v^{n} = -(1-\theta)\delta t\left\{-2\frac{\rho v^{n}v_{0}}{R^{2}}v_{iZ}\right\}$$

$$D_{13}^{v}\chi^{n} = -(1-\theta)\delta t\left\{-\left(\chi^{n},\chi_{0}\right)\left[R^{2}v_{i},\rho\right]-\rho\frac{\Delta^{*}U_{0}}{R^{2}}\left(R^{2}v_{i},\chi^{n}\right)+\left[U_{0}\ ,\chi^{n}\right]\left[R^{2}v_{i},\rho\right]+in\frac{\rho v_{0}}{R^{2}}\left[R^{2}v_{i},\chi^{n}\right]\right\}$$

$$S_{21}^{v}U^{n+1} = +\theta\delta t\rho v_{i}\left[v_{0},U^{n+1}\right]$$

$$S_{22}^{v}v^{n+1} = +\theta\delta t\rho v_{i}\left\{\left[v^{n+1},U_{0}\right]+\left(v^{n+1},\chi_{0}\right)+in\frac{v_{0}v^{n+1}}{R^{2}}\right\}$$

$$S_{23}^{v}\chi^{n+1} = +\theta\delta t\rho v_{i}\left(v_{0},\chi^{n+1}\right)$$

$$D_{21}^{v}U^{n} = -(1-\theta)\delta t\rho v_{i}\left[v_{0},U^{n}\right]$$

$$D_{22}^{v}v^{n} = -(1-\theta)\delta t\rho v_{i}\left\{\left[v^{n},U_{0}\right]+\left(v^{n},\chi_{0}\right)+in\frac{v_{0}v^{n}}{R^{2}}\right\}$$

$$D_{23}^{v}\chi^{n} = -(1-\theta)\delta t\rho v_{i}\left(v_{0},\chi^{n}\right)$$

$$S_{31}^{v}U^{n+1} = \theta\delta t\rho\begin{bmatrix} \frac{1}{R^2}\Delta^{*}U^{n+1}\left(U_0,\nu_i\right)+\frac{1}{R^2}\Delta^{*}U_0\left(U^{n+1},\nu_i\right)-\left(\nu_i,\frac{1}{R^2}\left(U^{n+1},U_0\right)\right)-\Delta^{*}U^{n+1}\left[\nu_i,\chi_0\right] \\ -in\frac{\nu_0}{R^2}\left[\nu_i,U^{n+1}\right]-\left(\nu_i,\left[\chi_0,U^{n+1}\right]\right) \end{bmatrix}$$

$$S_{32}^{v}\nu^{n+1} = 2\theta\delta t\rho\frac{\nu^{n+1}\nu_0}{R^3}\frac{\partial\nu_i}{\partial R}$$

$$S_{33}^{v}\mathrm{X}^{n+1} = -\theta\delta t\rho\left[\Delta^{*}U_0\left[\nu_i,\chi^{n+1}\right]+\left(\nu_i,\left[\chi^{n+1},U_0\right]\right)+in\frac{\nu_0}{R^2}\left(\chi^{n+1},\nu_i\right)+\left(\nu_i,\left(\chi^{n+1},\chi_0\right)\right)\right]$$

$$D_{31}^{v}U^{n} = -(1-\theta)\delta t\rho\begin{bmatrix} \frac{1}{R^2}\Delta^{*}U^{n}\left(U_0,\nu_i\right)+\frac{1}{R^2}\Delta^{*}U_0\left(U^{n},\nu_i\right)-\left(\nu_i,\frac{1}{R^2}\left(U^{n},U_0\right)\right)-\Delta^{*}U^{n}\left[\nu_i,\chi_0\right] \\ -in\frac{\nu_0}{R^2}\left[\nu_i,U^{n}\right]-\left(\nu_i,\left[\chi_0,U^{n}\right]\right) \end{bmatrix}$$

$$D_{32}^{v}\nu^{n} = -(1-\theta)\delta t\rho\left[2\frac{\nu^{n}\nu_0}{R^3}\frac{\partial\nu_i}{\partial R}\right]$$

$$D_{33}^{v}\mathrm{X}^{n} = (1-\theta)\delta t\rho\left[\Delta^{*}U_0\left[\nu_i,\chi^{n}\right]+\left(\nu_i,\left[\chi^{n},U_0\right]\right)+in\frac{\nu_0}{R^2}\left(\chi^{n},\nu_i\right)+\left(\nu_i,\left(\chi^{n},\chi_0\right)\right)\right]$$

7.2 Magnetic Field Equation

$$S_{11}^{b}\psi^{n+1} = \theta\delta t\nu_i\left[\psi^{n+1},U_0\right]+\theta\delta t\nu_i\left(\psi^{n+1},\chi_0\right)$$

$$S_{12}^{b}f^{n+1} = +in\theta\delta t\nu_i\left(f^{n+1},U_0\right)-in\theta\delta tR^2\nu_i\left[f^{n+1},\chi_0\right]$$

$$D_{11}^{b}\psi^{n} = -(1-\theta)\delta t\nu_i\left[\psi^{n},U_0\right]-(1-\theta)\delta t\nu_i\left(\psi^{n},\chi_0\right)$$

$$D_{12}^{b}f^{n} = -in(1-\theta)\delta t\nu_i\left(f^{n},U_0\right)+in(1-\theta)\delta tR^2\nu_i\left[f^{n},\chi_0\right]$$

$$S_{21}^{b}\psi^{n+1} = +\theta\delta t\frac{\nu_0}{R^2}\left[\nu_i,\psi^{n+1}\right]$$

$$S_{22}^{b}f^{n+1} = -\theta\delta t\left(\Delta^{\dagger}f^{n+1}\right)\left\{\left[\nu_i,U_0\right]+\left(\nu_i,\chi_0\right)\right\}-\theta in\delta t\frac{\nu_0}{R^2}\left(\nu_i,f^{n+1}\right)$$

$$D_{21}^{b}\psi^{n} = -(1-\theta)\delta t\frac{\nu_0}{R^2}\left[\nu_i,\psi^{n}\right]$$

$$D_{22}^{b}f^{n} = +(1-\theta)\delta t\left(\Delta^{\dagger}f^{n}\right)\left\{\left[\nu_i,U_0\right]+\left(\nu_i,\chi_0\right)\right\}+(1-\theta)in\delta t\frac{\nu_0}{R^2}\left(\nu_i,f^{n}\right)$$

$$S_{31}^{b}\psi^{n+1} = \theta\frac{\nu_0}{R^2}\left(\nu_i,\psi^{n+1}\right)$$
$$S_{32}^{b}f^{n+1} = -\theta\left(\Delta^{\dagger}f^{n+1}\right)\left(\nu_i,U_0\right) - \theta R^2\Delta^{\dagger}f^{n+1}\left[\nu_i,\chi_0\right] + in\theta\nu_0\left[\nu_i,f^{n+1}\right]$$

$$D_{31}^{b}\psi^{n} = -(1-\theta)\frac{\nu_0}{R^2}\left(\nu_i,\psi^{n}\right)$$
$$D_{32}^{b}f^{n} = +(1-\theta)\left(\Delta^{\dagger}f^{n}\right)\left(\nu_i,U_0\right) + (1-\theta)R^2\Delta^{\dagger}f^{n}\left[\nu_i,\chi_0\right] - in(1-\theta)\nu_0\left[\nu_i,f^{n}\right]$$

7.3 The Density Matrix Equations

The additional terms when equilibrium flow is present are:

$$S_{11}^{n}N_j^{n+1} = \theta\delta t\nu_i\left[\rho^{n+1},U_0\right] - \theta\delta t\rho^{n+1}\left(\nu_i,\chi_0\right) + in\theta\delta t\frac{1}{R^2}\nu_i\left(\rho^{n+1}\nu_0\right)$$
$$D_{11}^{n}N_j^{n} = -(1-\theta)\delta t\nu_i\left[\rho^{n},U_0\right] + (1-\theta)\delta t\rho^{n}\left(\nu_i,\chi_0\right) - in(1-\theta)\delta t\frac{1}{R^2}\nu_i\left(\rho^{n}\nu_0\right)$$

7.4 The Pressure Matrix Equations

The new terms in the linearized pressure equation due to flow are:

$$\begin{aligned}
S_{11}^{p}\bullet P^{n+1} &= \delta t\theta\nu_i[p^{n+1},U_0] + \delta t\theta\nu_i\left(p^{n+1},\chi_0\right) + in\delta t\theta\frac{1}{R^2}p^{n+1}\nu_i\nu_0 \\
&\quad -\delta t\theta\gamma\left(\nu_i p^{n+1},\chi_0\right) + in\gamma\delta t\theta\frac{1}{R^2}p^{n+1}\nu_i\nu_0 \\
D_{11}^{p}\bullet P^{n} &= -\delta t(1-\theta)\nu_i[p^{n},U_0] - \delta t(1-\theta)\nu_i\left(p^{n},\chi_0\right) - in\delta t(1-\theta)\frac{1}{R^2}p^{n}\nu_i\nu_0 \\
&\quad +\delta t(1-\theta)\gamma\left(\nu_i p^{n},\chi_0\right) - in\gamma\delta t(1-\theta)\frac{1}{R^2}p^{n}\nu_i\nu_0
\end{aligned}$$

7.5 The Electron Pressure Matrix Equations

Not yet done.

8.0 The reduced (2-variable) 3D nonlinear equations (numvar=1)

Consider the 3D nonlinear equations involving only U and ψ. The PDEs are:

$$\Delta^* \dot{U} = -R^2 \left[\frac{\Delta^* \dot{U}}{R^2}, U \right] + R^2 \left[\frac{\Delta^* \psi}{R^2}, \psi \right] + \frac{F}{R^2} \Delta^* \psi' + \left(\frac{F}{R^2}, \psi' \right) \tag{8.1}$$

$$\dot{\psi} = -\left[\psi, U \right] + \eta \Delta^* \psi - \Phi' \tag{8.2}$$

$$-\nabla \bullet \frac{1}{R^2} \nabla_\perp \Phi' = \nabla \bullet \frac{1}{R^2} \left[\frac{F}{R^2} \nabla_\perp U' + \eta \frac{1}{R^2} \nabla_\perp \psi'' \right] \tag{8.3}$$

Equations (8.2) and (8.3) together imply:

$$\nabla \bullet \frac{1}{R^2} \nabla \dot{\psi} = -\nabla \bullet \frac{1}{R^2} \nabla \left[\psi, U \right] + \nabla \bullet \frac{1}{R^2} \nabla \eta \Delta^* \psi + \nabla \bullet \frac{1}{R^2} \left[\frac{F}{R^2} \nabla_\perp U' + \eta \frac{1}{R^2} \nabla_\perp \psi'' \right] \tag{8.3b}$$

Note that (8.3) differs from the gauge condition presently used in the full system. Operate on (8.1) with $\frac{U}{R^2}$ and (8.3b) by ψ, and integrate over the domain to get the energy equation:

$$\iiint \frac{1}{R^2} \left(\psi, \dot{\psi} \right) + \frac{1}{R^2} \left(U, \dot{U} \right) = -\frac{\eta}{R^2} \left[(\Delta^* \psi)^2 + \frac{1}{R^2} \left(\psi', \psi' \right) \right] \tag{8.4}$$

The integrands for the weak form of these equations are:

$$-\frac{1}{R^2} \left(\nu_i R^2, \dot{U} \right) = \frac{\Delta^* \dot{U}}{R^2} \left[\nu_i R^2, U \right] - \frac{\Delta^* \psi}{R^2} \left[\nu_i R^2, \psi \right] - \frac{F}{R^4} \left(\nu_i R^2, \psi' \right) \tag{8.5}$$

$$-\left(\nu_i, \dot{\psi} \right) = \left[\left(\nu_i, \left[\psi, U \right] \right) - \frac{F}{R^2} \left(\nu_i, U' \right) \right] + \eta \left[\Delta^\dagger \nu_i \Delta^* \psi - \frac{1}{R^2} \left(\nu_i, \psi'' \right) \right] \tag{8.6}$$

These are time differenced as follows

$$
\begin{aligned}
&-\frac{1}{R^2}\left(\nu_i R^2, U^{n+1}\right)-(\theta\delta t)^2 L_{11}\left\{U^{n+1}\right\}+\theta\delta t\left\{-\frac{\Delta^* U^{n+1}}{R^2}\left[\nu_i R^2, U^n\right]-\frac{\Delta^* U^n}{R^2}\left[\nu_i R^2, U^{n+1}\right]\right\} \\
&-\mu\theta\delta t\left[\frac{1}{R^2}\Delta^*\left(R^2\nu_i\right)\left(\Delta^* U\right)+\nu_i\frac{\Delta^* U''}{R^2}-2\frac{\nu_i}{R^2}\left(\frac{1}{R}\frac{\partial U''}{\partial R}\right)\right]^{n+1} \\
&=-\frac{1}{R^2}\left(\nu_i R^2, U^n\right)-(\theta\delta t)^2 L_{11}\left\{U^n\right\}-(1-2\theta)\delta t\left\{-\frac{\Delta^* U^n}{R^2}\left[\nu_i R^2, U^n\right]\right\} \\
&+\delta t\left\{\begin{array}{l}-\left(\frac{\Delta^*\psi}{R^2}\right)\left[\nu_i R^2,\psi\right]-\frac{F}{R^4}\left(\nu_i R^2,\psi'\right) \\ +\mu(1-\theta)\left[\frac{1}{R^2}\Delta^*\left(R^2\nu_i\right)\left(\Delta^* U\right)+\nu_i\frac{\Delta^* U''}{R^2}-2\frac{\nu_i}{R^2}\left(\frac{1}{R}\frac{\partial U''}{\partial R}\right)\right]\end{array}\right\}^n
\end{aligned}
\tag{8.7}
$$

$$
\begin{aligned}
-\frac{1}{R^2}\left(\nu_i,\psi^{n+1}\right)=&-\frac{1}{R^2}\left(\nu_i,\psi^n\right)+\theta\delta t\frac{1}{R^2}\left(\nu_i,\left[\psi^{n+1},U^{n+1}\right]\right) \\
&+(1-\theta)\delta t\frac{1}{R^2}\left(\nu_i,\left[\psi^n,U^{n+1}\right]\right)-\delta t\frac{F}{R^4}\left(\nu_i,U'\right)^{n+1} \\
&+\eta\delta t\theta\left[\frac{1}{R^2}\Delta^*\nu_i\Delta^*\psi-\frac{1}{R^4}\left(\nu_i,\psi''\right)\right] \\
&+(1-\theta)\eta\delta t\theta\left[\frac{1}{R^2}\Delta^*\nu_i\Delta^*\psi-\frac{1}{R^4}\left(\nu_i,\psi''\right)\right]
\end{aligned}
\tag{8.8}
$$

Here we have defined the operator:

$$
\begin{aligned}
L_{11}\{U\}=&\frac{1}{R^2}\left(\left[U,\psi\right],\left[R^2\nu_i,\psi\right]\right)-\frac{1}{R^2}\Delta^*\psi\left[R^2\nu_i,\left[U,\psi\right]\right]-2F\left[U,\frac{F}{R^2}\right]\frac{\partial\nu_i}{\partial Z} \\
&+\frac{F}{R^4}\left(U',\left[\nu_i R^2,\psi\right]\right)-\frac{F}{R^4}\left(\left[U',\psi\right],R^2\nu_i\right)-\frac{F}{R^4}\Delta^*\psi\left[\nu_i R^2,U'\right]-\frac{F^2}{R^6}\left(U'',R^2\nu_i\right) \\
&-\frac{F}{R^4}\left(\left[U,\psi'\right],R^2\nu_i\right)+\left[\frac{F}{R^2},U\right]\frac{1}{R^2}\left(\nu_i R^2,\psi'\right)
\end{aligned}
\tag{8.9}
$$

8.1 The finite difference equations

We do a finite difference approximation in the φ direction. All the variables in the 2D plane j are given by U_j and ψ_j. We apply centered finite differences in the φ direction, thus:

$$U' = \frac{1}{2\delta\phi}\left[U_{j+1} - U_{j-1}\right]$$

$$U'' = \frac{1}{(\delta\phi)^2}\left[U_{j+1} - 2U_j + U_{j-1}\right]$$

$$\psi' = \frac{1}{2\delta\phi}\left[\psi_{j+1} - \psi_{j-1}\right]$$

$$\psi'' = \frac{1}{(\delta\phi)^2}\left[\psi_{j+1} - 2\psi_j + \psi_{j-1}\right]$$

The velocity and field advance equations can then be written as a pair of block-tridiagonal equations which can be solved successively:

$$\begin{aligned} A_j U_{j+1}^{n+1} + B_j U_j^{n+1} + C_j U_{j-1}^{n+1} + D_j &= 0 \\ E_j \Psi_{j+1}^{n+1} + F_j \Psi_j^{n+1} + G_j \Psi_{j-1}^{n+1} + H_j &= 0 \end{aligned} \tag{8.10}$$

Here, the coefficient matrices are defined as follows:

$$A_j U_{j+1}^{n+1} = -(\theta\delta t)^2 \begin{bmatrix} +\dfrac{F}{2\delta\phi R^4}\left(U_{j+1}^{n+1}, \left[\nu_i R^2, \psi\right]\right) - \dfrac{F}{2\delta\phi R^4}\left(\left[U_{j+1}^{n+1}, \psi\right], R^2\nu_i\right) \\ -\dfrac{F}{2\delta\phi R^4}\Delta^*\psi\left[\nu_i R^2, U_{j+1}^{n+1}\right] - \dfrac{F^2}{(\delta\phi)^2 R^6}\left(U_{j+1}^{n+1}, R^2\nu_i\right) \end{bmatrix}$$
$$-\mu\theta\delta t\left[\frac{\nu_i}{(\delta\phi)^2 R^2}\Delta^* U_{j+1}^{n+1} - \frac{2}{(\delta\phi)^2}\frac{\nu_i}{R^3}\frac{\partial U_{j+1}^{n+1}}{\partial R}\right]$$

$$C_j U_{j-1}^{n+1} = -(\theta\delta t)^2 \begin{bmatrix} -\dfrac{F}{2\delta\phi R^4}\left(U_{j-1}^{n+1}, \left[\nu_i R^2, \psi\right]\right) + \dfrac{F}{2\delta\phi R^4}\left(\left[U_{j-1}^{n+1}, \psi\right], R^2\nu_i\right) \\ +\dfrac{F}{2\delta\phi R^4}\Delta^*\psi\left[\nu_i R^2, U_{j-1}^{n+1}\right] - \dfrac{F^2}{(\delta\phi)^2 R^6}\left(U_{j-1}^{n+1}, R^2\nu_i\right) \end{bmatrix}$$
$$-\mu\theta\delta t\left[\frac{\nu_i}{(\delta\phi)^2 R^2}\Delta^* U_{j-1}^{n+1} - \frac{2}{(\delta\phi)^2}\frac{\nu_i}{R^3}\frac{\partial U_{j-1}^{n+1}}{\partial R}\right]$$

$$
\begin{aligned}
B_j U_j^{n+1} = & -\frac{1}{R^2}\left(\nu_i R^2, U_j^{n+1}\right) + \theta\delta t\left\{-\frac{\Delta^* U_j^{n+1}}{R^2}\left[\nu_i R^2, U_j^n\right] - \frac{\Delta^* U_j^n}{R^2}\left[\nu_i R^2, U_j^{n+1}\right]\right\} \\
& -(\theta\delta t)^2\left[\begin{array}{l}
\frac{1}{R^2}\left(\left[U_j^{n+1},\psi\right],\left[\nu_i R^2,\psi\right]\right) - \frac{1}{R^2}\Delta^*\psi\left[\nu_i R^2,\left[U_j^{n+1},\psi\right]\right] \\
-2F\left[U_j^{n+1},\frac{F}{R^2}\right]\frac{\partial \nu_i}{\partial Z} + \frac{2F^2}{(\delta\phi)^2 R^6}\left(U_j^{n+1}, R^2\nu_i\right) \\
-\frac{F}{R^4}\left(\left[U_j^{n+1},\psi'\right], R^2\nu_i\right) + \left[\frac{F}{R^2}, U_j^{n+1}\right]\frac{1}{R^2}\left(R^2\nu_i,\psi'\right)
\end{array}\right] \\
& -\mu\theta\delta t\left[\frac{1}{R^2}\Delta^*\left(R^2\nu_i\right)\left(\Delta^* U_j^{n+1}\right) - \frac{2\nu_i}{(\delta\phi)^2 R^2}\Delta^* U_j^{n+1} + \frac{4}{(\delta\phi)^2}\frac{\nu_i}{R^3}\frac{\partial U_j^{n+1}}{\partial R}\right]
\end{aligned}
$$

$$
\begin{aligned}
D_j = & \frac{1}{R^2}\left(\nu_i R^2, U_j^n\right) - (1-2\theta)\delta t\frac{\Delta^* U_j^n}{R^2}\left[\nu_i R^2, U_j^n\right] \\
& +(\theta\delta t)^2\left[\begin{array}{l}
\frac{1}{R^2}\left(\left[U_j^n,\psi\right],\left[\nu_i R^2,\psi\right]\right) - \frac{1}{R^2}\Delta^*\psi\left[\nu_i R^2,\left[U_j^n,\psi\right]\right] - 2F\left[U_j^n,\frac{F}{R^2}\right]\frac{\partial \nu_i}{\partial Z} \\
+\frac{F}{2\delta\phi R^4}\left(\left[U_{j+1}^n - U_{j-1}^n\right],\left[\nu_i R^2,\psi\right]\right) - \frac{F}{2\delta\phi R^4}\left(\left[\left[U_{j+1}^n - U_{j-1}^n\right],\psi\right], R^2\nu_i\right) \\
-\frac{F}{2\delta\phi R^4}\Delta^*\psi\left[R^2\nu_i,\left[U_{j+1}^n - U_{j-1}^n\right]\right] - \frac{F^2}{(\delta\phi)^2 R^6}\left(\left[U_{j+1}^n - 2U_j^n + U_{j-1}^n\right], R^2\nu_i\right) \\
-\frac{F}{R^4}\left(\left[U_j^n,\psi'\right], R^2\nu_i\right) + \left[\frac{F}{R^2}, U_j^n\right]\frac{1}{R^2}\left(\nu_i R^2,\psi'\right)
\end{array}\right] \\
& -\delta t\left\{\begin{array}{l}
-\left(\frac{\Delta^*\psi_j}{R^2}\right)\left[\nu_i R^2,\psi_j\right] - \frac{F}{2\delta\phi R^4}\left(\nu_i R^2,\left[\psi_{j+1} - \psi_{j-1}\right]\right) \\
+\mu(1-\theta)\left[\begin{array}{l}
\frac{1}{R^2}\left(\Delta^* R^2\nu_i\right)\left(\Delta^* U_j^n\right) + \frac{\nu_i}{(\delta\phi)^2 R^2}\left[\Delta^* U_{j+1}^n - 2\Delta^* U_j^n + \Delta^* U_{j-1}^n\right] \\
-\frac{2}{(\delta\phi)^2}\frac{\nu_i}{R^3}\left[\frac{\partial U_{j+1}^n}{\partial R} - 2\frac{\partial U_j^n}{\partial R} + \frac{\partial U_{j-1}^n}{\partial R}\right]
\end{array}\right]
\end{array}\right\}
\end{aligned}
$$

$$
\begin{aligned}
E_j &= \frac{\eta\delta t\theta}{(\delta\phi)^2 R^4}\left(\nu_i, \psi_{j+1}^{n+1}\right) \\
G_j &= \frac{\eta\delta t\theta}{(\delta\phi)^2 R^4}\left(\nu_i, \psi_{j-1}^{n+1}\right) \\
F_j &= -\frac{1}{R^2}\left(\nu_i, \psi_j^{n+1}\right) - \theta\delta t\frac{1}{R^2}\left(\nu_i,\left[\psi_j^{n+1}, U^{n+1}\right]\right) \\
& -\eta\delta t\theta\left[\frac{1}{R^2}\Delta^*\nu_i\Delta^*\psi_j^{n+1} + \frac{2}{(\delta\phi)^2 R^4}\left(\nu_i,\psi_j^{n+1}\right)\right]
\end{aligned}
$$

$$H_j = \frac{1}{R^2}\left(\nu_i, \psi_j^n\right) - (1-\theta)\delta t \frac{1}{R^2}\left(\nu_i, \left[\psi_j^n, U^{n+1}\right]\right) + \delta t \frac{F}{R^4}\left(\nu_i, U'\right)^{n+1}$$
$$-(1-\theta)\eta\delta t\left[\frac{1}{R^2}\Delta^* \nu_i \Delta^* \psi_j^n - \frac{1}{(\delta\phi)^2 R^4}\left(\nu_i, \left[\psi_{j+1}^n - 2\psi_j^n + \psi_{j-1}^n\right]\right)\right]$$

Appendix A: Some Intermediate Vectors and Scalars

$$\mathbf{J} \equiv \nabla \times \mathbf{B} = \nabla F^* \times \nabla \varphi + \frac{1}{R^2} \nabla_\perp \psi' - \Delta^* \psi \nabla \varphi$$

$$\mathbf{V} \times \mathbf{B} = \left\{ [U, \psi] - (U, f') + \frac{F}{R^2} U' - (\chi, \psi) - R^2 [\chi, f'] - \frac{1}{R^2} \psi' v \right\} \nabla \varphi$$

$$- \frac{F}{R^2} \nabla U + \frac{v}{R^2} \nabla \psi + \{ F \nabla_\perp \chi + v \nabla_\perp f' \} \times \nabla \varphi$$

$$\mathbf{V} \bullet \nabla p + \gamma p \nabla \bullet \mathbf{V} = -[U, p] + (\chi, p) + \frac{v}{R^2} p' + \gamma p \left(\nabla_\perp^2 \chi + \frac{v'}{R^2} \right)$$

$$\nabla \times \mathbf{V} = -\Delta^* U \nabla \varphi + \frac{1}{R^2} \nabla_\perp U' - \nabla_\perp \chi' \times \nabla \varphi + \nabla_\perp v \times \nabla \varphi$$

$$\mathbf{V} \times (\nabla \times \mathbf{V}) = \frac{\Delta^* U}{R^2} \nabla_\perp U - \frac{v}{R^2} \nabla_\perp \chi' + \frac{v}{R^2} \nabla_\perp v + \frac{1}{R^2} \nabla_\perp \chi \times \nabla_\perp U' - \Delta^* U \nabla_\perp \chi \times \nabla \varphi$$

$$- \frac{v}{R^2} \nabla_\perp U' \times \nabla \varphi + \nabla \varphi \left[\frac{1}{R^2} \nabla_\perp U \bullet \nabla_\perp U' + [\chi', U] - [v, U] + (\chi', \chi) - (v, \chi) \right]$$

$$\mathbf{V} \bullet \nabla \mathbf{V} = -\frac{\Delta^* U}{R^2} \nabla_\perp U + \frac{v}{R^2} \nabla_\perp \chi' + \Delta^* U \nabla_\perp \chi \times \nabla \varphi + \frac{v}{R^2} \nabla_\perp U' \times \nabla \varphi + \frac{1}{2} \nabla_\perp \frac{1}{R^2} (U, U)$$

$$+ \frac{1}{2} \nabla_\perp (\chi, \chi) + \nabla_\perp [\chi, U] - \frac{v^2}{R^3} \hat{R} + \nabla \varphi \left[[v, U] + (v, \chi) + \frac{1}{R^2} v v' \right]$$

$$\mathbf{J} \times \mathbf{B} - \nabla p = \left[-[\psi, F^*] - \frac{1}{R} (F^*, f') - \frac{1}{R^2} (\psi, \psi') - [\psi', f'] - p' \nabla \varphi \right] \nabla \varphi - \frac{1}{R^2} \Delta^* \psi \nabla_\perp \psi$$

$$- \frac{F}{R^2} \nabla_\perp F^* - \nabla_\perp p + \left[\frac{F}{R^2} \nabla_\perp \psi' - \Delta^* \psi \nabla_\perp f' \right] \times \nabla \varphi$$

Define the vector $\mathbf{Q}$ as follows:

$$\mathbf{Q} \equiv \nabla \times (\mathbf{V} \times \mathbf{B})$$

$$= \nabla_\perp \left\{ [U, \psi] - (U, f') - (\chi, \psi) + R^2 [f', \chi] \right\} \times \nabla \varphi$$

$$+ \left[\frac{F'}{R^2} \nabla_\perp U + \frac{F}{R^2} \nabla_\perp U' - \frac{v'}{R^2} \nabla_\perp \psi - \frac{v}{R^2} \nabla_\perp \psi' \right] \times \nabla \varphi$$

$$+ \left(\frac{F'}{R^2} \nabla_\perp \chi + \frac{F}{R^2} \nabla_\perp \chi' + \frac{v'}{R^2} \nabla_\perp f' + \frac{v}{R^2} \nabla_\perp f'' \right)$$

$$+ R^2 \left\{ -\left[\left(\frac{F}{R^2} \right), U \right] + \left[\left(\frac{v}{R^2} \right), \psi \right] - \nabla \bullet \frac{F}{R^2} \nabla_\perp \chi - \nabla \bullet \frac{v}{R^2} \nabla_\perp f' \right\} \nabla \varphi$$

Appendix B: Some Vector Identities used in Momentum Equations

All integrals are of the form:

$$\iint RdRdZ\ldots \equiv \iint dV_{\perp}\ldots \tag{B-1}$$

Gauss's Theorem takes the form, for an arbitrary vector $\mathbf{A}$:

$$\begin{aligned}\iint dV_{\perp}\nabla\bullet\mathbf{A} &= \iint dV_{\perp}\left[\nabla_{\perp}\bullet\mathbf{A}+\nabla_{\parallel}\bullet\mathbf{A}\right] = \oint dS\bullet\mathbf{A}+\iint dV_{\perp}\left(\mathbf{A}\bullet\nabla\varphi\right)' \\ &= \iint dV_{\perp}\left(\mathbf{A}\bullet\nabla\varphi\right)'\end{aligned} \tag{B-2}$$

Here we assumed that the surface term vanishes.

Consider now the first projection of the MHD operator: Let $\hat{\nu}_i \equiv R^2\nu_i$

$$\begin{aligned}
&-\iint dV_{\perp}\hat{\nu}_i\nabla\varphi\bullet\nabla\times\left[\left(\nabla\times\mathbf{Q}\right)\times\mathbf{B}+\left(\nabla\times\mathbf{B}\right)\times\mathbf{Q}\right] \\
&= \iint dV_{\perp}\nabla\varphi\bullet\nabla\,\hat{\nu}_i\times\left[\left(\nabla\times\mathbf{Q}\right)\times\mathbf{B}+\left(\nabla\times\mathbf{B}\right)\times\mathbf{Q}\right] \\
&= \iint dV_{\perp}\nabla\varphi\bullet\begin{bmatrix}\left(\nabla\hat{\nu}_i\bullet\mathbf{B}\right)\left(\nabla\times\mathbf{Q}\right)-\nabla\hat{\nu}_i\bullet\left(\nabla\times\mathbf{Q}\right)\mathbf{B} \\ +\left(\nabla\hat{\nu}_i\bullet\mathbf{Q}\right)\left(\nabla\times\mathbf{B}\right)-\left(\nabla\hat{\nu}_i\bullet\nabla\times\mathbf{B}\right)\mathbf{Q}\end{bmatrix} \\
&= \iint dV_{\perp}\begin{bmatrix}+\left(\nabla\hat{\nu}_i\bullet\mathbf{B}\right)\left(\nabla\varphi\bullet\nabla\times\mathbf{Q}\right)-\nabla\hat{\nu}_i\bullet\left(\nabla\times\mathbf{Q}\right)\nabla\varphi\bullet\mathbf{B} \\ +\nabla\hat{\nu}_i\bullet\mathbf{Q}\left(\nabla\varphi\bullet\nabla\times\mathbf{B}\right)-\left(\nabla\hat{\nu}_i\bullet\nabla\times\mathbf{B}\right)\nabla\varphi\bullet\mathbf{Q}\end{bmatrix} \\
&= \iint dV_{\perp}\begin{bmatrix}+\left(\nabla\hat{\nu}_i\bullet\mathbf{B}\right)\nabla\bullet\left(\mathbf{Q}\times\nabla\varphi\right)-\left(\nabla\varphi\bullet\mathbf{B}\right)\nabla\bullet\left(\mathbf{Q}\times\nabla\hat{\nu}_i\right) \\ +\nabla\hat{\nu}_i\bullet\mathbf{Q}\left(\nabla\varphi\bullet\nabla\times\mathbf{B}\right)-\left(\nabla\hat{\nu}_i\bullet\nabla\times\mathbf{B}\right)\nabla\varphi\bullet\mathbf{Q}\end{bmatrix} \\
&= \iint dV_{\perp}\begin{bmatrix}-\left(\mathbf{Q}\times\nabla\varphi\right)\bullet\nabla_{\perp}\left(\nabla\hat{\nu}_i\bullet\mathbf{B}\right)+\left(\mathbf{Q}\times\nabla\hat{\nu}_i\right)\bullet\nabla\left(\nabla\varphi\bullet\mathbf{B}\right)-\left[\left(\nabla\varphi\bullet\mathbf{B}\right)\left(\mathbf{Q}\times\nabla\hat{\nu}_i\bullet\nabla\varphi\right)\right]' \\ +\nabla\hat{\nu}_i\bullet\mathbf{Q}\left(\nabla\varphi\bullet\nabla\times\mathbf{B}\right)-\left(\nabla\hat{\nu}_i\bullet\nabla\times\mathbf{B}\right)\nabla\varphi\bullet\mathbf{Q}\end{bmatrix} \\
&= \iint dV_{\perp}\begin{bmatrix}-\left(\mathbf{Q}\times\nabla\varphi\right)\bullet\nabla_{\perp}\left(\nabla\hat{\nu}_i\bullet\mathbf{B}\right)+\left(\mathbf{Q}\times\nabla\hat{\nu}_i\right)\bullet\nabla_{\perp}\left(\nabla\varphi\bullet\mathbf{B}\right)-\left(\nabla\varphi\bullet\mathbf{B}\right)\left(\mathbf{Q}\times\nabla\hat{\nu}_i\bullet\nabla\varphi\right)' \\ +\nabla\hat{\nu}_i\bullet\mathbf{Q}\left(\nabla\varphi\bullet\nabla\times\mathbf{B}\right)-\left(\nabla\hat{\nu}_i\bullet\nabla\times\mathbf{B}\right)\nabla\varphi\bullet\mathbf{Q}\end{bmatrix}
\end{aligned}$$

Second Projection of the MHD operator:

$$
\begin{aligned}
&\iint dV_\perp \nu_i R^2 \nabla\varphi \bullet \left[\left(\nabla\times\mathbf{Q}\right)\times\mathbf{B} + \left(\nabla\times\mathbf{B}\right)\times\mathbf{Q} \right] \\
&= \iint dV_\perp \left[\nu_i R^2 \left(\mathbf{B}\times\nabla\varphi\right)\bullet\left(\nabla\times\mathbf{Q}\right) + \nu_i R^2 \left(\mathbf{Q}\times\nabla\varphi\right)\bullet\left(\nabla\times\mathbf{B}\right) \right] \\
&= \iint dV_\perp \left\{ \nabla\bullet\left[\mathbf{Q}\times\left(\nu_i R^2 \mathbf{B}\times\nabla\varphi\right)\right] + \mathbf{Q}\bullet\nabla\times\left(\nu_i R^2\mathbf{B}\times\nabla\varphi\right) + \nu_i R^2 \mathbf{Q}\bullet\left(\nabla\varphi\times\mathbf{J}\right) \right\} \\
&= \iint dV_\perp \left\{ \left[\nu_i R^2 \mathbf{Q}\times\left(\mathbf{B}\times\nabla\varphi\right)\bullet\nabla\varphi \right]' + \mathbf{Q}\bullet\nabla\times\left(\nu_i R^2\mathbf{B}\times\nabla\varphi\right) + \nu_i R^2 \mathbf{Q}\bullet\left(\nabla\varphi\times\mathbf{J}\right) \right\} \\
&= \iint dV_\perp \left\{ \nu_i R^2 \left[\left(\nabla\varphi\bullet\mathbf{B}\right)\left(\nabla\varphi\bullet\mathbf{Q}\right) - \frac{1}{R^2}\left(\mathbf{Q}\bullet\mathbf{B}\right) \right]' + \mathbf{Q}\bullet\left[\nabla\times\left(\nu_i R^2\mathbf{B}\times\nabla\varphi\right) + \nu_i R^2 \left(\nabla\varphi\times\mathbf{J}\right) \right] \right\} \\
&= \iint dV_\perp \left\{ -\nu_i \left(\mathbf{Q}_\perp\bullet\mathbf{B}\right)' + \mathbf{Q}\bullet\left[\nu_i \nabla_\perp \left(R^2\mathbf{B}\bullet\nabla\varphi\right) - R^2 \left(\nabla_\perp\bullet\nu_i\mathbf{B}_\perp\right)\nabla\varphi \right] \right\} \\
&= \iint dV_\perp \left\{ \begin{array}{l} \nu_i \left[-\mathbf{Q}\bullet\nabla_\perp\psi\times\nabla\varphi + \mathbf{Q}\bullet\nabla_\perp f' \right]' \\ +\mathbf{Q}\bullet\left[-R^2\left[\nu_i,\psi\right]\nabla\varphi + R^2\nabla\bullet\left(\nu_i\nabla_\perp f'\right)\nabla\varphi + \nu_i\nabla_\perp F \right] \end{array} \right\} \\
&= \iint dV_\perp \left\{ \begin{array}{l} \nu_i \left[\mathbf{Q}\bullet\nabla\varphi\times\nabla_\perp\psi + \mathbf{Q}\bullet\nabla_\perp f' \right]' \\ +\mathbf{Q}\bullet\nabla\varphi\left[-R^2\left[\nu_i,\psi\right] + R^2\nabla\bullet\left(\nu_i\nabla_\perp f'\right) \right] + \nu_i\mathbf{Q}\bullet\nabla_\perp F \end{array} \right\}
\end{aligned}
$$

Third Projection of the MHD operator:

$$
\begin{aligned}
&-\iint dV_\perp \nabla\nu_i \bullet \left[\left(\nabla\times\mathbf{Q}\right)\times\mathbf{B} + \left(\nabla\times\mathbf{B}\right)\times\mathbf{Q} \right] \\
&= \iint dV_\perp \left[\left(\nabla\nu_i\times\mathbf{B}\right)\bullet\nabla\times\mathbf{Q} - \mathbf{Q}\bullet\nabla\nu_i\times\left(\nabla\times\mathbf{B}\right) \right] \\
&= \iint dV_\perp \left[\nabla\bullet\left[\mathbf{Q}\times\left(\nabla\nu_i\times\mathbf{B}\right)\right] + \mathbf{Q}\bullet\nabla\times\left(\nabla\nu_i\times\mathbf{B}\right) - \mathbf{Q}\bullet\nabla\nu_i\times\left(\nabla\times\mathbf{B}\right) \right] \\
&= \iint dV_\perp \left\{ \left[\mathbf{Q}\times\left(\nabla\nu_i\times\mathbf{B}\right)\bullet\nabla\varphi \right]' + \mathbf{Q}\bullet\left[\nabla\times\left(\nabla\nu_i\times\mathbf{B}\right) - \nabla\nu_i\times\mathbf{J} \right] \right\} \\
&= \iint dV_\perp \left\{ \left[-\mathbf{B}\bullet\nabla\varphi\left(\mathbf{Q}\bullet\nabla\nu_i\right) \right]' + \mathbf{Q}\bullet\left[\nabla\times\left(\nabla\nu_i\times\mathbf{B}\right) - \nabla\nu_i\times\mathbf{J} \right] \right\}
\end{aligned}
$$

$$
= \iint dV_{\perp} \left\{ \begin{array}{l} -\left[\dfrac{F}{R^2} \left(\mathbf{Q} \bullet \nabla \nu_i \right) \right]' \\ + \mathbf{Q} \bullet \left[\begin{array}{l} -\nabla \left(\nu_i, \psi \right) \times \nabla \varphi + \nabla R^2 \left[f', \nu_i \right] \times \nabla \varphi - R^2 \nabla \bullet \left[\dfrac{F}{R^2} \nabla \nu_i \right] \nabla \varphi + \dfrac{F'}{R^2} \nabla \nu_i \\ + \nabla \varphi \left(\nu_i, F^* \right) - \dfrac{1}{R^2} \nabla \nu_i \times \nabla_{\perp} \psi' + \left(\Delta^* \psi \right) \nabla \nu_i \times \nabla \varphi \end{array} \right] \end{array} \right\}
$$

$$
= \iint dV_{\perp} \left\{ \begin{array}{l} -\dfrac{F}{R^2} \left(\mathbf{Q} \bullet \nabla \nu_i \right)' \\ + \mathbf{Q} \bullet \left[\begin{array}{l} \left[-\nabla \left(\nu_i, \psi \right) + \nabla R^2 \left[f', \nu_i \right] + \Delta^* \psi \nabla \nu_i \right] \times \nabla \varphi \\ + \left[-R^2 \nabla \bullet \left[\dfrac{F}{R^2} \nabla \nu_i \right] + \left(\nu_i, F^* \right) - \left[\nu_i, \psi' \right] \right] \nabla \varphi \end{array} \right] \end{array} \right\}
$$

$$
= \iint dV_{\perp} \left\{ \begin{array}{l} -\dfrac{F}{R^2} \left(\mathbf{Q} \bullet \nabla \nu_i \right)' + \left(\mathbf{Q} \bullet \nabla \varphi \right) \left[-F \Delta^* \nu_i + \left(\nu_i, f'' \right) - \left[\nu_i, \psi' \right] \right] \\ + \left(\nabla \varphi \times \mathbf{Q} \right) \bullet \left[-\nabla \left(\nu_i, \psi \right) + \nabla R^2 \left[f', \nu_i \right] + \Delta^* \psi \nabla \nu_i \right] \end{array} \right\}
$$

Appendix C: Momentum Eqn. First Projection Algebra

Consider the first integral needed for the finite element discretization. Multiply by the 2D test function $\hat{v}_i \equiv R^2 v_i(R,Z)$ and integrate over R and Z:

$$
\begin{aligned}
-\iint dRdZ(R^2 v_i)R\nabla\varphi\bullet\nabla\times L\{\mathbf{V}\} &= \iint dRdZR\nabla\varphi\bullet\nabla(R^2 v_i)\times L\{\mathbf{V}\} \\
&= \iint dV_\perp \begin{bmatrix} -(\mathbf{Q}\times\nabla\varphi)\bullet\nabla_\perp(\nabla\hat{v}_i\bullet\mathbf{B}) + (\mathbf{Q}\times\nabla\hat{v}_i)\bullet\nabla_\perp(\nabla\varphi\bullet\mathbf{B}) \\ -(\nabla\varphi\bullet\mathbf{B})(\mathbf{Q}\times\nabla\hat{v}_i\bullet\nabla\varphi)' \\ +\nabla\hat{v}_i\bullet\mathbf{Q}(\nabla\varphi\bullet\nabla\times\mathbf{B}) - (\nabla\hat{v}_i\bullet\nabla\times\mathbf{B})\nabla\varphi\bullet\mathbf{Q} \end{bmatrix} \\
&= \iint dV_\perp\left[L_{11}\{U\} + L_{12}\{v\} + L_{13}\{\chi\}\right]
\end{aligned}
$$

This integral is the sum of 5 terms:

$$
\begin{aligned}
(\mathbf{Q}\times\nabla\varphi) = &-\frac{1}{R^2}\nabla_\perp\left\{[U,\psi] - (U,f') - (\chi,\psi) + R^2[f',\chi]\right\} - \frac{F'}{R^4}\nabla_\perp U - \frac{F}{R^4}\nabla_\perp U' \\
&+\frac{v'}{R^4}\nabla_\perp\psi + \frac{v}{R^4}\nabla_\perp\psi' + \frac{F'}{R^2}\nabla_\perp\chi\times\nabla\varphi + \frac{F}{R^2}\nabla_\perp\chi'\times\nabla\varphi + \frac{v'}{R^2}\nabla_\perp f'\times\nabla\varphi + \frac{v}{R^2}\nabla_\perp f''\times\nabla\varphi
\end{aligned}
$$

$$\nabla_\perp(\mathbf{B}\bullet\nabla v_i) = \nabla_\perp[v_i,\psi] - \nabla_\perp(v_i, f')$$

Term1:

$$
\begin{aligned}
&-(\mathbf{Q}\times\nabla\varphi)\bullet\nabla_\perp(\mathbf{B}\bullet\nabla v_i) \\
&= +\frac{1}{R^2}([U,\psi],[\hat{v}_i,\psi]) - \frac{1}{R^2}((U,f'),[\hat{v}_i,\psi]) - \frac{1}{R^2}((\chi,\psi),[\hat{v}_i,\psi]) \\
&+\frac{1}{R^2}(R^2[f',\chi],[\hat{v}_i,\psi]) + \frac{F}{R^4}(U',[\hat{v}_i,\psi]) + \frac{F'}{R^4}(U,[\hat{v}_i,\psi]) \\
&-\frac{v'}{R^4}(\psi,[\hat{v}_i,\psi]) - \frac{v}{R^4}(\psi',[\hat{v}_i,\psi]) - \frac{F'}{R^2}[[\hat{v}_i,\psi],\chi] - \frac{F}{R^2}[[\hat{v}_i,\psi],\chi'] \\
&-\frac{v'}{R^2}[[\hat{v}_i,\psi],f'] - \frac{v}{R^2}[[\hat{v}_i,\psi],f''] \\
&-\frac{1}{R^2}([U,\psi],(\hat{v}_i,f')) + \frac{1}{R^2}((U,f'),(\hat{v}_i,f')) + \frac{1}{R^2}((\chi,\psi),(\hat{v}_i,f')) \\
&-\frac{1}{R^2}(R^2[f',\chi],(\hat{v}_i,f')) - \frac{F'}{R^4}(U,(\hat{v}_i,f')) - \frac{F}{R^4}(U',(\hat{v}_i,f')) \\
&+\frac{v'}{R^4}(\psi,(\hat{v}_i,f')) + \frac{v}{R^4}(\psi',(\hat{v}_i,f')) + \frac{F'}{R^2}[(\hat{v}_i,f'),\chi] + \frac{F}{R^2}[(\hat{v}_i,f'),\chi'] \\
&+\frac{v'}{R^2}[(\hat{v}_i,f'),f'] + \frac{v}{R^2}[(\hat{v}_i,f'),f'']
\end{aligned}
$$

$$\left(\mathbf{Q}\times\nabla\hat{v}_i\right)=\nabla\varphi\left\{\begin{array}{l}\left(\left[U,\psi\right],\hat{v}_i\right)-\left(\left(U,f'\right),\hat{v}_i\right)-\left(\left(\chi,\psi\right),\hat{v}_i\right)+\dfrac{F'}{R^2}\left(U,\hat{v}_i\right)\\+\dfrac{F}{R^2}\left(U',\hat{v}_i\right)-\dfrac{v'}{R^2}\left(\psi,\hat{v}_i\right)-\dfrac{v}{R^2}\left(\psi',\hat{v}_i\right)+\left(R^2\left[f',\chi\right],\hat{v}_i\right)\\\dfrac{F'}{R^2}\left[\chi,\hat{v}_i\right]+\dfrac{F}{R^2}\left[\chi',\hat{v}_i\right]+\dfrac{v'}{R^2}\left[f',\hat{v}_i\right]+\dfrac{v}{R^2}\left[f'',\hat{v}_i\right]\end{array}\right\}$$

$$+R^2\left\{\left[\left(\frac{F}{R^2}\right),U\right]-\left[\left(\frac{v}{R^2}\right),\psi\right]+\nabla\bullet\frac{F}{R^2}\nabla_\perp\chi+\nabla\bullet\frac{v}{R^2}\nabla_\perp f'\right\}\nabla\hat{v}_i\times\nabla\varphi$$

$$\nabla_\perp\left(\mathbf{B}\bullet\nabla\varphi\right)=\nabla_\perp\left(\frac{F}{R^2}\right)$$

Term 2:

$$\left(\mathbf{Q}\times\nabla v_i\right)\bullet\nabla_\perp\left(\mathbf{B}\bullet\nabla\varphi\right)$$

$$=-R^2\left[U,\frac{F}{R^2}\right]\left[\frac{F}{R^2},v_i\right]$$

$$+R^2\left[\psi,\frac{v}{R^2}\right]\left[\frac{F}{R^2},v_i\right]-\left\{R^2\nabla\bullet\frac{v}{R^2}\nabla_\perp f'\right\}\left[v_i,\frac{F}{R^2}\right]$$

$$-\left\{R^2\nabla\bullet\frac{F}{R^2}\nabla_\perp\chi\right\}\left[v_i,\frac{F}{R^2}\right]$$

$$\left(\mathbf{B}\bullet\nabla\varphi\right)=\left(\frac{F}{R^2}\right)$$

Term3:

$$-(\mathbf{B}\bullet\nabla\varphi)\left(\mathbf{Q}\times\nabla\hat{v}_i\bullet\nabla\varphi\right)'$$

$$=\frac{F}{R^4}\left\{\begin{array}{l}-\left(\left[U,\psi\right],\hat{v}_i\right)'+\left(\left(U,f'\right),\hat{v}_i\right)'-\left(\dfrac{F}{R^2}\left(U',\hat{v}_i\right)+\dfrac{F'}{R^2}\left(U,\hat{v}_i\right)\right)'\\+\left(\left(\chi,\psi\right),\hat{v}_i\right)'+\left(\dfrac{v}{R^2}\left(\psi',\hat{v}_i\right)+\dfrac{v'}{R^2}\left(\psi,\hat{v}_i\right)\right)'-\left(R^2\left[f',\chi\right],\hat{v}_i\right)'\\-\left(F\left[\chi,\hat{v}_i\right]+v\left[f',\hat{v}_i\right]\right)''\end{array}\right\}$$

$$\left(\mathbf{Q}\bullet\nabla\varphi\right)=-\left[\frac{F}{R^2},U\right]+\left[\frac{v}{R^2},\psi\right]-\left\{\nabla\bullet\frac{F}{R^2}\nabla_\perp\chi+\nabla\bullet\frac{v}{R^2}\nabla_\perp f'\right\}$$

$$\nabla v_i\bullet\nabla\times\mathbf{B}=\left[\hat{v}_i,F\right]+\left[\hat{v}_i,f''\right]+\frac{1}{R^2}\left(\hat{v}_i,\psi'\right)$$

Term 4:

$$\begin{aligned}-\left(\mathbf{Q}\bullet\nabla\varphi\right)\nabla\hat{v}_i\bullet\nabla\times\mathbf{B}=&+\left[\frac{F}{R^2},U\right]\left\{\left[\hat{v}_i,F\right]+\left[\hat{v}_i,f''\right]+\frac{1}{R^2}\left(\hat{v}_i,\psi'\right)\right\}\\&-\left[\frac{v}{R^2},\psi\right]\left\{\left[\hat{v}_i,F\right]+\left[\hat{v}_i,f''\right]+\frac{1}{R^2}\left(\hat{v}_i,\psi'\right)\right\}\\&+\nabla\bullet\frac{v}{R^2}\nabla_\perp f'\left\{\left[\hat{v}_i,F\right]+\left[\hat{v}_i,f''\right]+\frac{1}{R^2}\left(\hat{v}_i,\psi'\right)\right\}\\&+\nabla\bullet\frac{F}{R^2}\nabla_\perp\chi\left\{\left[\hat{v}_i,F\right]+\left[\hat{v}_i,f''\right]+\frac{1}{R^2}\left(\hat{v}_i,\psi'\right)\right\}\end{aligned}$$

$$\left(\nabla\varphi\bullet\nabla\times\mathbf{B}\right)=-\frac{1}{R^2}\Delta^*\psi$$

$$\begin{aligned}\mathbf{Q}\bullet\nabla\hat{v}_i=&\left[\hat{v}_i,\left[U,\psi\right]\right]-\left[\hat{v}_i,\left(U,f'\right)\right]+\frac{F}{R^2}\left[\hat{v}_i,U'\right]+\frac{F'}{R^2}\left[\hat{v}_i,U\right]-\left[\hat{v}_i,\left(\chi,\psi\right)\right]+\left[\hat{v}_i,R^2\left[f',\chi\right]\right]\\&-\frac{v}{R^2}\left[\hat{v}_i,\psi'\right]-\frac{v'}{R^2}\left[\hat{v}_i,\psi\right]+\left(\frac{F'}{R^2}\left(\chi,\hat{v}_i\right)+\frac{F}{R^2}\left(\chi',\hat{v}_i\right)+\frac{v'}{R^2}\left(f',\hat{v}_i\right)+\frac{v}{R^2}\left(f'',\hat{v}_i\right)\right)\end{aligned}$$

Term 5:

$$\begin{aligned}\left(\nabla\varphi\bullet\nabla\times\mathbf{B}\right)\mathbf{Q}\bullet\nabla\hat{v}_i=&\frac{1}{R^2}\Delta^*\psi\left\{-\left[\hat{v}_i,\left[U,\psi\right]\right]+\left[\hat{v}_i,\left(U,f'\right)\right]-\left(\frac{F}{R^2}\left[\hat{v}_i,U\right]\right)'\right\}\\&+\frac{1}{R^2}\Delta^*\psi\left\{+\frac{v}{R^2}\left[\hat{v}_i,\psi'\right]+\left(\frac{v}{R^2}\right)'\left[\hat{v}_i,\psi\right]-\left(\frac{v}{R^2}\left(f',\hat{v}_i\right)\right)'\right\}\\&+\frac{1}{R^2}\Delta^*\psi\left\{+\left[\hat{v}_i,\left(\chi,\psi\right)\right]-\left[\hat{v}_i,R^2\left[f',\chi\right]\right]-\left(\frac{F}{R^2}\left(\chi,\hat{v}_i\right)\right)'\right\}\end{aligned}$$

Sumarize:

$$\begin{aligned}&\iint dRdZR\nabla\varphi\bullet\nabla\hat{v}_i\times\left[\left(\nabla\times\mathbf{Q}\right)\times\mathbf{B}+\left(\nabla\times\mathbf{B}\right)\times\mathbf{Q}\right]\\&=\iint dRdZR\left[L_{11}\left\{U\right\}+L_{12}\left\{v\right\}+L_{13}\left\{\chi\right\}\right]\end{aligned}$$

Appendix D: Momentum Eqn. 2nd Projection Algebra

$$
\begin{aligned}
&\iint dRdZR\nu_i R^2 \nabla\varphi \bullet L\{\mathbf{V}\} \\
&= \iint dV_\perp R^2 \nu_i \nabla\varphi \bullet \left[(\nabla\times\mathbf{Q})\times\mathbf{B} + (\nabla\times\mathbf{B})\times\mathbf{Q} + \nabla(\mathbf{V}\bullet\nabla p + \gamma p \nabla\bullet\mathbf{V}) \right] \\
&= \iint dV_\perp \left\{ \begin{array}{l} \nu_i \left[\mathbf{Q}\bullet\nabla\varphi\times\nabla_\perp\psi + \mathbf{Q}\bullet\nabla_\perp f' \right]' + \mathbf{Q}\bullet\nabla\varphi\left[-R^2[\nu_i,\psi] + R^2\nabla\bullet(\nu_i\nabla_\perp f') \right] \\ +\nu_i \mathbf{Q}\bullet\nabla_\perp F + \nu_i \left(-[U,p] + (\chi,p) + \frac{\nu}{R^2}p' + \gamma p\left(\nabla_\perp^2\chi + \frac{\nu'}{R^2} \right) \right)' \end{array} \right\} \\
&= \iint dV_\perp \left[L_{21}\{U\} + L_{22}\{\nu\} + L_{23}\{\chi\} \right]
\end{aligned}
$$

Some intermediate results:

$$
\mathbf{Q}\bullet\nabla\varphi = -\left[\frac{F}{R^2}, U \right] + \left[\frac{\nu}{R^2}, \psi \right] - \left\{ \nabla\bullet\frac{F}{R^2}\nabla_\perp\chi + \nabla\bullet\frac{\nu}{R^2}\nabla_\perp f' \right\}
$$

$$
\begin{aligned}
\mathbf{Q}\bullet\nabla_\perp F = &\left[F,[U,\psi] \right] - \left[F,(U,f') \right] - \left[F,(\chi,\psi) \right] + \left[F, R^2[f',\chi] \right] \\
&+ \frac{F'}{R^2}[F,U] + \frac{F}{R^2}[F,U'] - \frac{\nu'}{R^2}[F,\psi] - \frac{\nu}{R^2}[F,\psi'] \\
&+ \frac{F'}{R^2}(F,\chi) + \frac{F}{R^2}(F,\chi') + \frac{\nu'}{R^2}(F,f') + \frac{\nu}{R^2}(F,f'')
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{Q}\bullet\nabla_\perp f' = &\left[f',[U,\psi] \right] - \left[f',(U,f') \right] - \left[f',(\chi,\psi) \right] + \left[f', R^2[f',\chi] \right] \\
&+ \frac{F'}{R^2}[f',U] + \frac{F}{R^2}[f',U'] - \frac{\nu'}{R^2}[f',\psi] - \frac{\nu}{R^2}[f',\psi'] \\
&+ \frac{F'}{R^2}(f',\chi) + \frac{F}{R^2}(f',\chi') + \frac{\nu'}{R^2}(f',f') + \frac{\nu}{R^2}(f',f'')
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{Q}\bullet\nabla\varphi\times\nabla_\perp\psi = &-\frac{1}{R^2}\left([U,\psi],\psi \right) + \frac{1}{R^2}\left((U,f'),\psi \right) + \frac{1}{R^2}\left((\chi,\psi),\psi \right) - \frac{1}{R^2}\left(R^2[f',\chi],\psi \right) \\
&- \frac{F'}{R^4}(U,\psi) - \frac{F}{R^4}(U',\psi) + \frac{\nu'}{R^4}(\psi,\psi) + \frac{\nu}{R^4}(\psi',\psi) \\
&- \left(\frac{F'}{R^2}[\chi,\psi] + \frac{F}{R^2}[\chi',\psi] + \frac{\nu'}{R^2}[f',\psi] + \frac{\nu}{R^2}[f'',\psi] \right)
\end{aligned}
$$

First Term:

$$\nu_i\left[\mathbf{Q}\bullet\nabla\varphi\times\nabla_\perp\psi+\mathbf{Q}\bullet\nabla_\perp f'\right]'$$

$$=\nu_i\begin{pmatrix}-\frac{1}{R^2}\left(\left[U,\psi\right],\psi\right)+\frac{1}{R^2}\left(\left(U,f'\right),\psi\right)-\frac{F'}{R^4}\left(U,\psi\right)-\frac{F}{R^4}\left(U',\psi\right)\\+\left[f',\left[U,\psi\right]\right]-\left[f',\frac{1}{R}\left(U,f'\right)\right]+\frac{F'}{R^2}\left[f',U\right]+\frac{F}{R^2}\left[f',U'\right]\end{pmatrix}'$$

$$+\nu_i\begin{pmatrix}\frac{\nu'}{R^4}\left(\psi,\psi\right)+\frac{\nu}{R^4}\left(\psi',\psi\right)-\left(\frac{\nu'}{R^2}\left[f',\psi\right]+\frac{\nu}{R^2}\left[f'',\psi\right]\right)\\-\frac{\nu'}{R^2}\left[f',\psi\right]-\frac{\nu}{R^2}\left[f',\psi'\right]+\frac{\nu'}{R^2}\left(f',f'\right)+\frac{\nu}{R^2}\left(f',f''\right)\end{pmatrix}'$$

$$+\nu_i\begin{pmatrix}\frac{1}{R^2}\left(\left(\chi,\psi\right),\psi\right)-\frac{1}{R^2}\left(R^2\left[f',\chi\right],\psi\right)-\left(\frac{F'}{R^2}\left[\chi,\psi\right]+\frac{F}{R^2}\left[\chi',\psi\right]\right)\\-\left[f',\left(\chi,\psi\right)\right]+\left[f',R\left[f',\chi\right]\right]+\frac{F'}{R^2}\left(f',\chi\right)+\frac{F}{R^2}\left(f',\chi'\right)\end{pmatrix}'$$

Second Term:

$$\mathbf{Q}\bullet\nabla\varphi\left[-R^2\left[\nu_i,\psi\right]+R^2\nabla\bullet\left(\nu_i\nabla_\perp f'\right)\right]$$

$$=-\left[\frac{F}{R^2},U\right]\left[-R^2\left[\nu_i,\psi\right]+R^2\nabla\bullet\left(\nu_i\nabla_\perp f'\right)\right]$$

$$+\left\{\left[\frac{\nu}{R^2},\psi\right]-\nabla\bullet\frac{\nu}{R^2}\nabla_\perp f'\right\}\left[-R^2\left[\nu_i,\psi\right]+R^2\nabla\bullet\left(\nu_i\nabla_\perp f'\right)\right]$$

$$-\nabla\bullet\frac{F}{R^2}\nabla_\perp\chi\left[-R^2\left[\nu_i,\psi\right]+R^2\nabla\bullet\left(\nu_i\nabla_\perp f'\right)\right]$$

Third Term:

$$\nu_i\mathbf{Q}\bullet\nabla_\perp F=\nu_i\left[F,\left[U,\psi\right]\right]-\nu_i\left[F,\left(U,f'\right)\right]+\nu_i\frac{F'}{R^2}\left[F,U\right]+\nu_i\frac{F}{R^2}\left[F,U'\right]$$

$$-\nu_i\frac{\nu'}{R^2}\left[F,\psi\right]-\nu_i\frac{\nu}{R^2}\left[F,\psi'\right]+\nu_i\frac{\nu'}{R^2}\left(F,f'\right)+\nu_i\frac{\nu}{R^2}\left(F,f''\right)$$

$$+\nu_i\frac{F'}{R^2}\left(F,\chi\right)+\nu_i\frac{F}{R^2}\left(F,\chi'\right)-\nu_i\left[F,\left(\chi,\psi\right)\right]+\nu_i\left[F,R^2\left[f',\chi\right]\right]$$

Fourth Term:

$$=-\nu_i\left[U,p\right]'$$

$$+\left(\nu_i\frac{\nu}{R^2}p'+\nu_i\gamma p\frac{\nu'}{R^2}\right)'$$

$$+\left(\nu_i\left(\chi,p\right)+\nu_i\gamma p\nabla_\perp^2\chi\right)'$$

$$\iint dRdZR\nu_i R^2 \nabla\varphi \bullet L\left\{\mathbf{V}\right\}$$
$$= \iint dV_\perp R^2 \nu_i \nabla\varphi \bullet \left[\left(\nabla\times\mathbf{Q}\right)\times\mathbf{B} + \left(\nabla\times\mathbf{B}\right)\times\mathbf{Q} + \nabla\left(\mathbf{V}\bullet\nabla p + \gamma p \nabla\bullet\mathbf{V}\right)\right]$$
$$= \iint dV_\perp \left[L_{21}\left\{U\right\} + L_{22}\left\{v\right\} + L_{23}\left\{\chi\right\}\right]$$
.

Appendix E: Momentum Eqn. 3rd Projection Algebra:

$$\iint dRdZR v_i \nabla_\perp \bullet \left[(\nabla \times \mathbf{Q}) \times \mathbf{B} + (\nabla \times \mathbf{B}) \times \mathbf{Q} + \nabla (\mathbf{V} \bullet \nabla p) \right]$$

$$= -\iint dRdZR \nabla_\perp v_i \bullet \left[(\nabla \times \mathbf{Q}) \times \mathbf{B} + (\nabla \times \mathbf{B}) \times \mathbf{Q} + \nabla_\perp (\mathbf{V} \bullet \nabla p) \right]$$

$$= \iint RdRdZ \left\{ \begin{array}{l} -\frac{F}{R} (\mathbf{Q} \bullet \nabla v_i)' + \left(-F \Delta^* v_i + (v_i, f'') - [v_i, \psi'] \right) \mathbf{Q} \bullet \nabla \varphi \\ -\mathbf{Q} \bullet \nabla \left((v_i, \psi) + R^2 [v_i, f'] \right) \times \nabla \varphi + \left(\Delta^* \psi \right) \mathbf{Q} \bullet \nabla v_i \times \nabla \varphi \\ -\nabla_\perp^2 v_i \left(-[U, p] + (\chi, p) + \frac{v}{R^2} p' + \gamma p \left(\nabla_\perp^2 \chi + \frac{v'}{R^2} \right) \right) \end{array} \right\}$$

$$= \int RdRdZ \left[L_{31} \{U\} + L_{32} \{v\} + L_{33} \{\chi\} \right]$$

Some intermediate results:

$$\mathbf{Q} \bullet \nabla v_i = [v_i, [U, \psi]] - [v_i, (U, f')] + \frac{F}{R^2} [v_i, U'] + \frac{F'}{R^2} [v_i, U] - [v_i, (\chi, \psi)]$$

$$+ \left[v_i, R^2 [f', \chi] \right] - \frac{v}{R^2} [v_i, \psi'] - \frac{v'}{R^2} [v_i, \psi] + \left(\frac{F}{R^2} (\chi, v_i) + \frac{v}{R^2} (f', v_i) \right)'$$

$$\mathbf{Q} \bullet \nabla \varphi = -\left[\frac{F}{R^2}, U \right] + \left[\frac{v}{R^2}, \psi \right] - \left\{ \nabla \bullet \frac{F}{R^2} \nabla_\perp \chi + \nabla \bullet \frac{v}{R^2} \nabla_\perp f' \right\}$$

$$\mathbf{Q} \bullet \nabla v_i \times \nabla \varphi = \frac{1}{R^2} ([U, \psi], v_i) - \frac{1}{R^2} ((U, f'), v_i) + \left(\frac{F}{R^4} (U, v_i) \right)' - \left(\frac{v}{R^4} (\psi, v_i) \right)' + \left(\frac{v}{R^2} [f', v_i] \right)'$$

$$- \frac{1}{R^2} ((\chi, \psi), v_i) + \frac{1}{R^2} \left(R^2 [f', \chi], v_i \right) + \left(\frac{1}{R^2} F [\chi, v_i] \right)'$$

$$
\begin{aligned}
&-\mathbf{Q}\bullet\nabla\left(\left(\nu_i,\psi\right)+R^2\left[\nu_i,f'\right]\right)\times\nabla\varphi= \\
&-\frac{1}{R^2}\left(\left[U,\psi\right],\left(\nu_i,\psi\right)\right)+\frac{1}{R^2}\left(\left(U,f'\right),\left(\nu_i,\psi\right)\right)-\frac{1}{R^2}\left(\left[U,\psi\right],R^2\left[\nu_i,f'\right]\right)+\frac{1}{R^2}\left(\left(U,f'\right),R^2\left[\nu_i,f'\right]\right) \\
&-\frac{F'}{R^4}\left(U,\left(\nu_i,\psi\right)\right)-\frac{F}{R^4}\left(U',\left(\nu_i,\psi\right)\right)-\frac{F'}{R^4}\left(U,R^2\left[\nu_i,f'\right]\right)-\frac{F}{R^4}\left(U',R^2\left[\nu_i,f'\right]\right) \\
&+\frac{\nu'}{R^4}\left(\psi,\left(\nu_i,\psi\right)\right)+\frac{\nu}{R^4}\left(\psi',\left(\nu_i,\psi\right)\right)+\frac{\nu'}{R^4}\left(\psi,R^2\left[\nu_i,f'\right]\right)+\frac{\nu}{R^4}\left(\psi',R^2\left[\nu_i,f'\right]\right) \\
&-\frac{\nu'}{R^2}\left[f',\left(\nu_i,\psi\right)\right]-\frac{\nu}{R^2}\left[f'',\left(\nu_i,\psi\right)\right]-\frac{\nu'}{R^2}\left[f',R\left[\nu_i,f'\right]\right]-\frac{\nu}{R^2}\left[f'',R\left[\nu_i,f'\right]\right] \\
&+\frac{1}{R^2}\left(\left(\chi,\psi\right),\left(\nu_i,\psi\right)\right)-\frac{1}{R^2}\left(R^2\left[f',\chi\right],\left(\nu_i,\psi\right)\right) \\
&+\frac{1}{R^2}\left(\left(\chi,\psi\right),R^2\left[\nu_i,f'\right]\right)-\frac{1}{R^2}\left(R^2\left[f',\chi\right],R^2\left[\nu_i,f'\right]\right) \\
&-\frac{F'}{R^2}\left[\chi,\left(\nu_i,\psi\right)\right]-\frac{F}{R^2}\left[\chi',\left(\nu_i,\psi\right)\right]-\frac{F'}{R^2}\left[\chi,R^2\left[\nu_i,f'\right]\right]-\frac{F}{R^2}\left[\chi',R^2\left[\nu_i,f'\right]\right]
\end{aligned}
$$

$$
\begin{aligned}
-\frac{F}{R}\left(\mathbf{Q}\bullet\nabla\nu_i\right)' &= -\frac{F}{R}\left[\nu_i,\left[U,\psi\right]\right]'+\frac{F}{R}\left[\nu_i,\left(U,f'\right)\right]'-\frac{F}{R}\left(\frac{F}{R^2}\left[\nu_i,U\right]\right)'' \\
&+\frac{F}{R}\left(\frac{\nu}{R^2}\left[\nu_i,\psi\right]\right)''-\frac{F}{R}\left(\frac{\nu}{R^2}\left(f',\nu_i\right)\right)'' \\
&+\frac{F}{R}\left[\nu_i,\left(\chi,\psi\right)\right]'-\frac{F}{R}\left[\nu_i,R^2\left[f',\chi\right]\right]'-\frac{F}{R}\left(\frac{F}{R^2}\left(\chi,\nu_i\right)\right)''
\end{aligned}
$$

$$
\begin{aligned}
&\left(-\left[\nu_i,\psi'\right]-F\Delta^*\nu_i+\left(\nu_i,f''\right)\right)\mathbf{Q}\bullet\nabla\varphi \\
&=-\left[\frac{F}{R^2},U\right]\left(-\left[\nu_i,\psi'\right]-F\Delta^*\nu_i+\left(\nu_i,f''\right)\right) \\
&+\left[\frac{\nu}{R^2},\psi\right]\left(-\left[\nu_i,\psi'\right]-F\Delta^*\nu_i+\left(\nu_i,f''\right)\right) \\
&-\left\{\nabla\bullet\frac{\nu}{R^2}\nabla_\perp f'\right\}\left(-\left[\nu_i,\psi'\right]-F\Delta^*\nu_i+\left(\nu_i,f''\right)\right) \\
&-\left\{\nabla\bullet\frac{F}{R^2}\nabla_\perp\chi\right\}\left(-\left[\nu_i,\psi'\right]-F\Delta^*\nu_i+\left(\nu_i,f''\right)\right)
\end{aligned}
$$

$$\Delta^*\psi \mathbf{Q}\bullet\nabla \nu_i \times \nabla\varphi$$

$$= +\frac{1}{R^2}\left([U,\psi],\nu_i\right)\Delta^*\psi - \frac{1}{R^2}\left((U,f'),\nu_i\right)\Delta^*\psi + \left(\frac{F}{R^4}(U,\nu_i)\right)'\Delta^*\psi$$

$$-\left(\frac{v}{R^4}(\psi,\nu_i)\right)'\Delta^*\psi + \left(\frac{v}{R^2}[f',\nu_i]\right)'\Delta^*\psi$$

$$-\frac{1}{R^2}\left((\chi,\psi),\nu_i\right)\Delta^*\psi + \frac{1}{R^2}\left(R^2[f',\chi],\nu_i\right)\Delta^*\psi + \left(\frac{1}{R^2}F[\chi,\nu_i]\right)'\Delta^*\psi$$

Combine and summarize:

$$\iint dRdZR\nu_i\nabla_\perp\bullet\left[(\nabla\times\mathbf{Q})\times\mathbf{B}+(\nabla\times\mathbf{B})\times\mathbf{Q}+\nabla\left(\mathbf{V}\bullet\nabla p+\gamma p\nabla\bullet\mathbf{V}\right)\right]$$

$$= -\iint dRdZR\nabla_\perp\nu_i\bullet\left[(\nabla\times\mathbf{Q})\times\mathbf{B}+(\nabla\times\mathbf{B})\times\mathbf{Q}+\nabla_\perp\left(\mathbf{V}\bullet\nabla p+\gamma p\nabla\bullet\mathbf{V}\right)\right]$$

$$= \iint RdRdZ\left\{\begin{array}{l} -\frac{F}{R}\left(\mathbf{Q}\bullet\nabla\nu_i\right)' + \left(-F\Delta^*\nu_i + (\nu_i, f'') - [\nu_i,\psi']\right)\mathbf{Q}\bullet\nabla\varphi \\ -\mathbf{Q}\bullet\nabla\left((\nu_i,\psi) + R^2[\nu_i, f']\right)\times\nabla\varphi + \Delta^*\psi\mathbf{Q}\bullet\nabla\nu_i\times\nabla\varphi \\ +\nabla_\perp^2\nu_i\left([p,U] + (\chi,p) + \frac{v}{R^2}p' + \gamma p\left(\nabla_\perp^2\chi + \frac{v'}{R^2}\right)\right) \end{array}\right\}$$

$$= \int RdRdZ\left[L_{31}\{U\} + L_{32}\{v\} + L_{33}\{\chi\}\right]$$

Appendix F: Magnetic Field Evolution Equation Algebra:

Applying the operator $\hat{\varphi}\cdot$ to (2.1) results in

$$\nabla_\perp^2 \dot{f} = R\left[U, \frac{F}{R^2}\right] - \left(\frac{F}{R}, \chi\right) - \left(\frac{F}{R}\right)\nabla_\perp^2 \chi + R\left[\frac{v_\varphi}{R^2}, \psi\right] - \frac{1}{R}\left(\frac{v_\varphi}{R}, f'\right) - \frac{v_\varphi}{R^2}\Delta^* f'$$
$$+\frac{\eta}{R}\left[\Delta^*\left(R\nabla_\perp^2 f\right) + \frac{\Delta^* f''}{R} - \frac{1}{R^2}\frac{\partial f''}{\partial R} - \frac{2}{R^2}\frac{\partial \psi'}{\partial z}\right] + \frac{1}{R}\left[\left(\eta, R\nabla_\perp^2 f\right) - \left[\eta, \psi'\right] + \frac{1}{R}\left(\eta, f''\right)\right]$$
$$+R\left[\frac{1}{R^2\rho}\left(\Delta^*\psi + \frac{1}{R}\frac{\partial f'}{\partial z}\right), \psi\right] - \frac{1}{R}\left(\frac{1}{R\rho}\left(\Delta^*\psi + \frac{1}{R}\frac{\partial f'}{\partial z}\right), f'\right) - \frac{1}{R^2\rho}\left(\Delta^*\psi + \frac{1}{R}\frac{\partial f'}{\partial z}\right)\Delta^* f'$$
$$+\frac{1}{R}\left(\frac{F}{R^2\rho}, \psi'\right) + \frac{F}{R^3\rho}\Delta^*\psi' + R\left[\frac{F}{R^3\rho}, f''\right] + R\left[\frac{F}{R^2\rho}, R\nabla_\perp^2 f\right] + R\left[\frac{1}{\rho}, p_e\right] + \text{hyper-resistive terms}$$

$$\iint (\text{RHS}) v_i(R,z)\, dRdz = -\iint v_i \nabla\varphi\cdot\nabla\times\mathbf{E} RdRdz$$
$$= -\iint \nabla\varphi\cdot\nabla\times\left(v_i\mathbf{E}\right) RdRdz + \iint \nabla\varphi\cdot\left(\nabla v_i\times\mathbf{E}\right) RdRdz$$
$$= -\oint v_i\mathbf{E}\cdot\mathbf{dl} + \iint\left(E_R\frac{\partial v_i}{\partial z} - E_z\frac{\partial v_i}{\partial R}\right) dRdz$$