Sources in Ohm's Law in M3D

SCJ 02/16/07

1. Introduction:

Consider the 3 equations that imply steady state on the ideal and resistive timescales (assuming the pressure equation has a source keeping it constant):

$$\nabla p = \vec{J} \times \vec{B} \tag{1}$$

$$\vec{E} + \vec{V} \times \vec{B} = \eta \left(\vec{J} - J_{\parallel}^{0} \frac{\vec{B}}{B} - \vec{J}_{\perp}^{0} \right) \tag{2}$$

$$\frac{\partial \vec{B}}{\partial t} = -\nabla \times \vec{E} = 0 \tag{3}$$

We introduce the standard axisymmetric flux coordinate system (ψ, θ, φ) so that the magnetic field (in equilibrium) and Jacobian can be written as:

$$\vec{B} = \nabla \varphi \times \nabla \psi + g(\psi) \nabla \varphi \tag{4}$$

and

$$J \equiv \left[\nabla \psi \times \nabla \theta \bullet \nabla \varphi \right]^{-1} \tag{5}$$

2. Driven Current:

Note that $\left| \nabla \varphi \right|^2 = 1 / R^2$ where R is the standard cylindrical coordinate. Combine (3) and (2) to rewrite (2) in the form:

$$-\nabla \Phi + \frac{1}{2\pi} V_L \nabla \varphi + \vec{V} \times \vec{B} = \eta \left(\vec{J} - J_{\parallel}^{0} \frac{\vec{B}}{B} - \vec{J}_{\perp}^{0} \right) \tag{2'}$$

The first term in (2)' is the gradient of a single valued scalar potential, and the second term represents the electric field due to an applied loop voltage (it is necessary in order to make Φ single valued).

Equation (1) gives the perpendicular part of the current density, but it is not divergence free. Straightforward calculation using (1), (4), and (5) gives:

$$\nabla \bullet \left(\frac{\vec{B} \times \nabla p}{B^2} \right) = \frac{g p'}{J} \frac{\partial}{\partial \theta} \left(\frac{1}{B^2} \right) \quad . \tag{6}$$

Here, $p' = \partial p / \partial \psi$. Taking the dot product of $\vec{B}$ with Eq. (2)' gives:

$$-\vec{B} \bullet \nabla \Phi + \frac{1}{2\pi} V_L \vec{B} \bullet \nabla \varphi = \eta \left(\vec{B} \bullet \vec{J} - J_{\parallel}^{0} B \right) \tag{7}$$

If we represent the total current as:

$$\vec{J} = \left(\frac{\vec{B}\times\nabla p}{B^2}\right) + J_{\parallel}\frac{\vec{B}}{B} \tag{8}$$

Then the condition $\nabla\bullet\vec{J} = 0$ applied to (8) gives:

$$\nabla\bullet\left(J_{\parallel}\frac{\vec{B}}{B}\right) = -\nabla\bullet\left(\frac{\vec{B}\times\nabla p}{B^2}\right) \tag{9}$$

Or, using (6):

$$\frac{\partial}{\partial\theta}\left(\frac{J_{\parallel}}{B} + \frac{gp'}{B^2}\right) = 0 \tag{10}$$

Thus,

$$\left(\frac{J_{\parallel}}{B} + \frac{gp'}{B^2}\right) = f(\psi) \tag{11}$$

for some $f(\psi)$. Use Eq. (11) and (8) to eliminate $\vec{B}\bullet\vec{J}$ from (7) to obtain:

$$-\vec{B}\bullet\nabla\Phi + \frac{1}{2\pi}V_L\vec{B}\bullet\nabla\varphi = \eta\left[B^2 f(\psi) - gp' - J_{\parallel}^0 B\right] \tag{12}$$

Multiply (12) by the Jacocbian *J*, and integrate from 0 to 2π in the angle θ:

$$f(\psi) = \frac{1}{\left\langle B^2\right\rangle}\left[\frac{V_L}{2\pi\eta}\left\langle\vec{B}\bullet\nabla\varphi\right\rangle + gp' + \left\langle J_{\parallel}^0 B\right\rangle\right] \tag{13}$$

Here, the brackets denote the flux surface average:

$$\left\langle a\right\rangle \equiv \int_0^{2\pi} Ja\,d\theta \Big/ \int_0^{2\pi} J\,d\theta \tag{14}$$

Substitution of (13) into (11) and using (8) gives the final expression for the current:

$$\vec{J} = \frac{\vec{B}\times\nabla p}{B^2} + \left[\frac{V_L\left\langle\vec{B}\bullet\nabla\varphi\right\rangle}{2\pi\eta\left\langle B^2\right\rangle} + gp'\left(\frac{1}{\left\langle B^2\right\rangle} - \frac{1}{B^2}\right) + \frac{\left\langle J_{\parallel}^0 B\right\rangle}{\left\langle B^2\right\rangle}\right]\vec{B} \tag{15}$$

Thus, the steady state current is uniquely determined from the quantities in (15), and does not depend on $\vec{J}_{\perp}^0$ in (2)'

3. Steady state velocity:

Now, we can use Equations (2)' and (15) to solve for the steady state perpendicular velocity. First, take the $\nabla\psi\times\vec{B}\bullet$ projection of Equation (2)' and rearrange:

$$\nabla\psi\times\vec{B}\bullet\left[-\nabla\Phi + \frac{1}{2\pi}V_L\nabla\varphi + \vec{V}\times\vec{B} = \eta\left(\vec{J} - J_{\parallel}^0\frac{\vec{B}}{B} - \vec{J}_{\perp}^0\right)\right] \tag{16}$$

or

$$\frac{g}{J}\Phi_\theta + \frac{1}{2\pi}V_L\frac{|\nabla\psi|^2}{R^2} + B^2\vec{V}\bullet\nabla\psi = \eta\left(-p'|\nabla\psi|^2 - \nabla\psi\times\vec{B}\bullet\vec{J}_\perp^0\right) \tag{17}$$

Solving (17) for the velocity gives:

$$J\vec{V}\bullet\nabla\psi = \frac{J}{B^2}\eta\left(-p'|\nabla\psi|^2 - \nabla\psi\times\vec{B}\bullet\vec{J}_\perp^0\right) - \frac{1}{2\pi}V_L\frac{J|\nabla\psi|^2}{B^2R^2} - \frac{g}{B^2}\Phi_\theta \tag{16}$$

Now we need to eliminate the last term. This is done by taking the parallel projection of (2)':

$$\vec{B}\bullet\left[-\nabla\Phi + \frac{1}{2\pi}V_L\nabla\varphi + \vec{V}\times\vec{B} = \eta\left(\vec{J} - J_\parallel^0\frac{\vec{B}}{B} - \vec{J}_\perp^0\right)\right]$$

or, using (15)

$$-\vec{B}\bullet\nabla\Phi + \frac{1}{2\pi}V_L\vec{B}\bullet\nabla\varphi = \eta\left[\frac{V_LB^2\left\langle\vec{B}\bullet\nabla\varphi\right\rangle}{2\pi\eta\left\langle B^2\right\rangle} + gp'\left(\frac{B^2}{\left\langle B^2\right\rangle} - 1\right) + B^2\frac{\left\langle J_\parallel^0B\right\rangle}{\left\langle B^2\right\rangle}\right] - \eta J_\parallel^0B$$

or

$$\begin{aligned}-\frac{g}{B^2}\Phi_\theta &= -\frac{1}{2\pi}\frac{g}{B^2}JV_L\vec{B}\bullet\nabla\varphi \\ &\quad +\eta\left[g\frac{V_LJ\left\langle\vec{B}\bullet\nabla\varphi\right\rangle}{2\pi\eta\left\langle B^2\right\rangle} + g^2p'J\left(\frac{1}{\left\langle B^2\right\rangle} - \frac{1}{B^2}\right) + gJ\frac{\left\langle J_\parallel^0B\right\rangle}{\left\langle B^2\right\rangle} - \frac{g}{B}JJ_\parallel^0\right]\end{aligned} \tag{17}$$

Thus, combining (16) and (17) gives

$$\begin{aligned}J\vec{V}\bullet\nabla\psi &= -\frac{J}{B^2}\eta\left(p'|\nabla\psi|^2 + \nabla\psi\times\vec{B}\bullet\vec{J}_\perp^0\right) - \frac{1}{2\pi}V_LJ\left(1 - \frac{\left\langle B_T^2\right\rangle}{\left\langle B^2\right\rangle}\right) \\ &\quad -\eta g^2p'J\left(\frac{1}{B^2} - \frac{1}{\left\langle B^2\right\rangle}\right) + \eta gJ\left(\frac{\left\langle J_\parallel^0B\right\rangle}{\left\langle B^2\right\rangle} - \frac{1}{B}J_\parallel^0\right)\end{aligned}$$

or, upon rearranging

$$\begin{aligned}J\vec{V}\bullet\nabla\psi &= -\eta p'J\left[\frac{|\nabla\psi|^2}{B^2} + g^2\left(\frac{1}{B^2} - \frac{1}{\left\langle B^2\right\rangle}\right)\right] - \frac{1}{2\pi}V_LJ\left(1 - \frac{\left\langle B_T^2\right\rangle}{\left\langle B^2\right\rangle}\right) \\ &\quad +\eta J\left(g\frac{\left\langle J_\parallel^0B\right\rangle}{\left\langle B^2\right\rangle} - \frac{g}{B}J_\parallel^0 - \frac{\nabla\psi\times\vec{B}\bullet\vec{J}_\perp^0}{B^2}\right)\end{aligned} \tag{18}$$

4. Ohmic Limit

Equation (18) is the general result. If there were no source terms, we could average over flux surfaces to get the standard result for an Ohmic tokamak:

$$\left\langle \vec{V}\bullet\nabla\psi \right\rangle = -\eta\, p' \left\langle \frac{\left|\nabla\psi\right|^2}{B^2} \right\rangle \left[1 + g^2 \left\langle \frac{\left|\nabla\psi\right|^2}{B^2} \right\rangle^{-1} \left(\left\langle B^{-2} \right\rangle - \left\langle B^2 \right\rangle^{-1} \right) \right] - \frac{1}{2\pi} V_L \left[1 - \frac{\left\langle B_T^2 \right\rangle}{\left\langle B^2 \right\rangle} \right], \quad (19)$$

where $B_T^2 \equiv g^2 / R^2$. The first term in brackets in (19) is the Pfirsch-Schlüter diffusion term, and the second is the classical pinch. Note that we can define:

$$q_*^2 \equiv g^2 \left\langle \frac{\left|\nabla\psi\right|^2}{B^2} \right\rangle^{-1} \left(\left\langle B^{-2} \right\rangle - \left\langle B^2 \right\rangle^{-1} \right) \quad (20)$$

In the large aspect ratio, circular limit, q* reduces to the safety factor. Then (19) becomes:

$$\left\langle \vec{V}\bullet\nabla\psi \right\rangle = -\eta\, p' \left\langle \frac{\left|\nabla\psi\right|^2}{B^2} \right\rangle \left[1 + 2q_*^2 \right] - \frac{1}{2\pi} V_L \left[1 - \frac{\left\langle B_T^2 \right\rangle}{\left\langle B^2 \right\rangle} \right] \quad (21)$$

5. Current Drive Limit

If V_L=0, then (18) gives:

$$J\vec{V}\bullet\nabla\psi = -\eta\, p' J \left[\frac{\left|\nabla\psi\right|^2}{B^2} + g^2 \left(\frac{1}{B^2} - \frac{1}{\left\langle B^2 \right\rangle} \right) \right] + \eta J \left(g \frac{\left\langle J_\parallel^0 B \right\rangle}{\left\langle B^2 \right\rangle} - \frac{g}{B} J_\parallel^0 - \frac{\nabla\psi \times \vec{B}\bullet\vec{J}_\perp^0}{B^2} \right) \quad (22)$$

rewrite as:

$$\vec{V}\bullet\nabla\psi = -\eta \left[p' \frac{\left|\nabla\psi\right|^2}{B^2} + \frac{\nabla\psi \times \vec{B}\bullet\vec{J}_\perp^0}{B^2} \right] + \eta \left(g \frac{\left\langle J_\parallel^0 B \right\rangle}{\left\langle B^2 \right\rangle} - \frac{g}{B} J_\parallel^0 - p' g^2 \left(\frac{1}{B^2} - \frac{1}{\left\langle B^2 \right\rangle} \right) \right) \quad (23)$$

The first bracket will vanish if the perpendicular current source is defined as:

$$\vec{J}_\perp^0 = \frac{\vec{B} \times \nabla p}{B^2} \quad (24)$$

and the second bracket will vanish if the parallel current source has the property:

$$J_\parallel^0 B = \left[g p' \left(\frac{B^2}{\left\langle B^2 \right\rangle} - 1 \right) + B^2 \frac{\left\langle J_\parallel^0 B \right\rangle}{\left\langle B^2 \right\rangle} \right] \quad (25)$$